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Research article

Quasi-monomiality and convergence theorem for the Boas-Buck-Sheffer polynomials

Shahid Ahmad Wani[1], **Kottakkaran Sooppy Nisar**[2,*]

[1] Department of Mathematics, Aligarh Muslim University, Aligarh, India
[2] Department of Mathematics, College of Arts and Sciences, Wadi Aldawaser, 11991, Prince Sattam bin Abdulaziz University, Saudi Arabia

* **Correspondence:** E-mail: n.sooppy@psau.edu.sa; Tel: +966563456976.

Abstract: A mixed family of polynomials, called the Boas-Buck-Sheffer family is introduced and their quasi-monomial properties are established in this article. Also, the generalizations of the Szasz operators including this mixed polynomial family are obtained and their convergence is studied.

Keywords: Boas-Buck polynomials; monomiality principle; Szasz operators
Mathematics Subject Classification: 33C45, 33C99, 33E20

1. Introduction and preliminaries

The Sheffer sequences $s_n(x),\ n \in \mathbb{N}_0$ are determined by the generating relation [1]:

$$\mathcal{A}(w)\exp(x\mathcal{J}(w)) = \sum_{n=0}^{\infty} s_n(x)\frac{w^n}{n!}, \quad x, w \in \mathbb{R}, \tag{1.1}$$

where $\mathcal{A}(w)$, $\mathcal{J}(w)$ are power series such that

$$\mathcal{A}(w) = \sum_{k=0}^{\infty} \beta_k \frac{w^k}{k!}, \qquad \beta_0 \neq 0, \tag{1.2}$$

$$\mathcal{J}(w) = \sum_{l=1}^{\infty} \mathfrak{h}_l \frac{w^l}{l!}, \qquad \mathfrak{h}_1 \neq 0. \tag{1.3}$$

The Sheffer polynomials are extensively studied with full illustration due to their importance in the field of applied sciences and mathematical physics. Recent research shows stimulated attention on those sequences and their various presentations [2]. In [3], the close relationship between these polynomial sequences and Riordan arrays was established by demonstrating the isomorphism between the Riordan and Sheffer group. Based on these results, the determinant approach has been presented in [4].

Recently, several mixed special polynomial families connected to the Sheffer sequences are investigated systematically [5–9]. In [10], Roman studied the technique of combining two sequences applying Umbral composition which is a systematic method of forming mixed special sequences. In this work, the composition of Boas-Buck and Sheffer polynomials is taken into account to introduce a vibrant and novel family of special polynomials, called the Boas-Buck-Sheffer family.

In 1956, Boas and Buck [11] studied a large class of generating functions of polynomial sets, called the Boas-Buck polynomials determined through the following generating relation:

$$A(w)\psi(xH(w)) = \sum_{n=0}^{\infty} p_n(x)\frac{w^n}{n!}, \tag{1.4}$$

where $A(w)$, $\psi(w)$, $H(w)$ are power series such that

$$A(w) = \sum_{k=0}^{\infty} a_k \frac{w^k}{k!}, \quad a_0 \neq 0; \tag{1.5}$$

$$\psi(w) = \sum_{l=0}^{\infty} \gamma_l \frac{w^l}{l!}, \quad \gamma_l \neq 0, \ \forall\, l, \tag{1.6}$$

with $\psi(w)$ not a polynomial and

$$H(w) = \sum_{m=1}^{\infty} h_m \frac{w^m}{m!}, \quad h_1 \neq 0. \tag{1.7}$$

These sets involve general classes of polynomials, like Brenke polynomials $Y_n(x)$ [12] (for $H(w) = w$), Sheffer polynomials $s_n(x)$ [13] (for $\psi(w) = \exp(w)$), Appell polynomials $\mathscr{A}_n(x)$ [14] (for $H(w) = w$, $\psi(w) = \exp(w)$) and those determined sets like certain constant multiples of the Laguerre, Hermite and Jacobi polynomials.

The idea of monomiality appeared in 1941 when Steffenson [15] developed the concept of poweroid and later on this method was re-modified by Dattoli [16]. According to the hypothesis of monomiality, the Φ^+ and Φ^- operators occur and perform as the multiplicative and derivative operators for $\{r_n(x)\}_{n\in\mathbb{N}}$ polynomial set:

$$\Phi^+\{r_n(x)\} = r_{n+1}(x), \tag{1.8}$$

$$\Phi^-\{r_n(x)\} = n\ r_{n-1}(x). \tag{1.9}$$

The operators Φ^+ and Φ^- exhibit the commutation expression

$$[\Phi^-, \Phi^+] = \hat{1} \tag{1.10}$$

and therefore represents the structure of Weyl group.

If the underlying set $\{r_n(x)\}_{n\in\mathbb{N}}$ is quasi-monomial and its properties can be achieved from Φ^+ and Φ^-, operators. Thus, following properties holds:

$$\Phi^+\Phi^-\{r_n(x)\} = n\ r_n(x). \tag{1.11}$$

Throughout this paper, we assume that $r_0(x) = 1$, then $r_n(x)$ is given by

$$r_n(x) = \Phi^{+^n}\{1\} \tag{1.12}$$

and thus the generating expression of $r_n(x)$ can be determined in the following way

$$G(x, \mathcal{J}(w)) = \exp(\mathcal{J}(w)\Phi^+)\{1\} = \sum_{n=0}^{\infty} r_n(x)\frac{(\mathcal{J}(w))^n}{n!}. \tag{1.13}$$

The Boas-Buck polynomial set defined in Eq 1.4 is quasi-monomial with the operations of the following multiplicative and derivative expressions [17]:

$$\Phi_p^+ = \frac{A'(H^{-1}(\varsigma))}{A(H^{-1}(\varsigma))} + xD_xH'(H^{-1}(\varsigma))\varsigma^{-1}, \tag{1.14}$$

$$\Phi_p^- = H^{-1}(\varsigma), \tag{1.15}$$

where $\varsigma \in \Lambda^{(-1)}$ is given by

$$\varsigma(1) = 0 \text{ and } \varsigma(x^n) = \frac{\gamma_{n-1}}{\gamma_n}x^{n-1}, \quad n = 1, 2, \cdots \tag{1.16}$$

and $\Lambda^{(j)}, \quad j \in \mathbb{Z}$ represents the space of operators operating on analytic functions which increase the degree of each polynomial exactly by j for $j \geq 0$ or $j \leq 0$, respectively.

The following linear positive operators was developed by Szasz [18]:

$$S_m(g; u) := e^{-mu}\sum_{k=0}^{\infty}\frac{(mu)^k}{k!}g\left(\frac{k}{m}\right), \tag{1.17}$$

where $u \geq 0$ and $g \in C[0, \infty)$, whenever the preceding sum converges.

Later, various extensions of the Szasz operators are obtained such as Appell, Sheffer, Brenke, Boas-Buck, and mixed special polynomials, see, for example, [19–21].

In this paper, the Boas-Buck-Sheffer family is proposed and studied through different aspects. In Section 2, an important property quasi-monomiality of this family is established. The extension of Szasz operators is established and their approximation properties are studied in Section 3.

2. The Boas-Buck-Sheffer polynomials

The Boas-Buck-Sheffer polynomials are proposed here by the means of generating expression. Further, quasi-monomial and approximation properties of these polynomials are demonstrated.

First, the generating expression for the Boas-Buck-Sheffer polynomials is demonstrated by the following result:

Theorem 1. *For the Boas-Buck-Sheffer polynomials, the following generating expression:*

$$\mathcal{A}(w)A(\mathcal{J}(w))\psi(xH(\mathcal{J}(w))) = \sum_{n=0}^{\infty} {}_pS_n(x)\frac{w^n}{n!}. \tag{2.1}$$

holds true.

Proof. Changing w in Eq 1.1 by Φ_p^+, that is the multiplicative operator of the polynomials $p_n(x)$, we have

$$\mathcal{A}(w)\exp\left(\Phi_p^+\mathcal{J}(w)\right)=\sum_{n=0}^{\infty}s_n\left(\Phi_p^+\right)\frac{w^n}{n!},\tag{2.2}$$

which by virtue of Eq 1.13, becomes

$$\mathcal{A}(w)\sum_{n=0}^{\infty}p_n(x)\frac{(\mathcal{J}(w))^n}{n!}=\sum_{n=0}^{\infty}s_n\left(\Phi_p^+\right)\frac{w^n}{n!}.\tag{2.3}$$

Using Eq 1.4 with w replaced by $\mathcal{J}(w)$ in the l.h.s of expression (2.3) and representing $s_n\left(\Phi_p^+\right)$ in the right hand side by the Boas-Buck-Sheffer polynomials ${}_ps_n(x)$, that is

$$s_n\left(\Phi_p^+\right)=s_n\left(\frac{A'(H^{-1}(\varsigma))}{A(H^{-1}(\varsigma))}+xD_xH'(H^{-1}(\varsigma))\varsigma^{-1}\right)={}_ps_n(x),\tag{2.4}$$

assertion (2.1) follows. □

Next, we derive the quasi-monomial properties of the Boas-Buck-Sheffer polynomials ${}_ps_n(x)$.

The following result is established to frame the ${}_ps_n(x)$ in the context of monomiality hypothesis:

Theorem 2. *The following multiplicative and derivative operators for the Boas-Buck-Sheffer polynomials ${}_ps_n(x)$, hold true:*

$$\Phi_{ps}^+=xH'(H^{-1}(\varsigma))\,\mathcal{J}'(\mathcal{J}^{-1}(H^{-1}(\varsigma)))D_x\varsigma^{-1}+\frac{A'(H^{-1}(\varsigma))}{A(H^{-1}(\varsigma))}\mathcal{J}'(\mathcal{J}^{-1}(H^{-1}(\varsigma)))+\frac{\mathcal{A}'(\mathcal{J}^{-1}(H^{-1}(\varsigma)))}{\mathcal{A}(\mathcal{J}^{-1}(H^{-1}(\varsigma)))}\tag{2.5}$$

$$\Phi_{pS}^-=\mathcal{J}^{-1}(H^{-1}(\varsigma)).\tag{2.6}$$

Proof. Notice that Eq 1.16 is equivalent to the relation [17].

$$\varsigma\psi(xw)=w\psi(xw),\tag{2.7}$$

so that, we can write

$$\mathcal{J}^{-1}(H^{-1}(\varsigma))\mathcal{A}(w)A(\mathcal{J}(w))\psi(xH(\mathcal{J}(w)))=w\mathcal{A}(w)A(\mathcal{J}(w))\psi(xH(\mathcal{J}(w))).\tag{2.8}$$

Differentiating (2.1) with respect to (w.r.t) w partially, we obtain the following expression.

$$\left(x\frac{H'(\mathcal{J}(w))}{H(\mathcal{J}(w))}\mathcal{J}'(w)D_x+\frac{A'(\mathcal{J}(w))}{A(\mathcal{J}(w))}\mathcal{J}'(w)+\frac{\mathcal{A}'(w)}{\mathcal{A}(w)}\right)\sum_{n=0}^{\infty}{}_ps_n(x)\frac{w^n}{n!}=\sum_{n=0}^{\infty}{}_ps_{n+1}(x)\frac{w^n}{n!}.\tag{2.9}$$

Using Eq 2.8 and equating the like powers of w on both sides of Eq 2.9, the following expression is obtained:

$$\Big(xH'(H^{-1}(\varsigma))\,\mathcal{J}'(\mathcal{J}^{-1}(H^{-1}(\varsigma)))D_x\varsigma^{-1}+\frac{A'(H^{-1}(\varsigma))}{A(H^{-1}(\varsigma))}\mathcal{J}'(\mathcal{J}^{-1}(H^{-1}(\varsigma)))$$

$$+\frac{\mathcal{A}'(\mathcal{J}^{-1}(H^{-1}(\varsigma)))}{\mathcal{A}(\mathcal{J}^{-1}(H^{-1}(\varsigma)))}\Big)\,{}_p s_n(x)={}_p s_{n+1}(x). \tag{2.10}$$

In view of Eq 1.8, we get assertion (2.5).

Again, from expression (2.8), we have

$$\mathcal{J}^{-1}(H^{-1}(\varsigma))\sum_{n=0}^{\infty}{}_p s_n(x)\frac{w^n}{n!}=\sum_{n=0}^{\infty}{}_p s_n(x)\frac{w^{n+1}}{n!},$$

Simplifying and then equating the coefficients of like powers of t, we have assertion (2.6). □

Theorem 3. *For the Boas-Buck-Sheffer polynomials ${}_p s_n(x)$, the following differential equation holds true:*

$$\Big(xH'(H^{-1}(\varsigma))\,\mathcal{J}'(\mathcal{J}^{-1}(H^{-1}(\varsigma)))D_x\varsigma^{-1}\mathcal{J}^{-1}(H^{-1}(\varsigma))$$

$$+\frac{A'(H^{-1}(\varsigma))}{A(H^{-1}(\varsigma))}\mathcal{J}'(\mathcal{J}^{-1}(H^{-1}(\varsigma)))\mathcal{J}^{-1}(H^{-1}(\varsigma))+\frac{\mathcal{A}'(\mathcal{J}^{-1}(H^{-1}(\varsigma)))}{\mathcal{A}(\mathcal{J}^{-1}(H^{-1}(\varsigma)))}\mathcal{J}^{-1}(H^{-1}(\varsigma))-n\Big)\,{}_p s_n(x)=0. \tag{2.11}$$

Proof. Inserting expressions (2.5) and (2.6) in the monomiality expression (1.11), assertion (2.11) follows. □

Remark 2.1. *For $H(w)=w$, the Boas-Buck polynomials $p_n(x)$ reduce to Brenke polynomials $Y_n(x)$ [12], thus in consideration of $H(w)=w$, the Boas-Buck-Sheffer polynomials ${}_p s_n(x)$ reduce to the Brenke-Sheffer polynomials ${}_Y s_n(x)$. Accordingly, taking $H(w)=w$ in Eqs 2.1, 2.5, 2.6 and 2.11, the following consequences of results 2.1, 2.2 and 2.3 are derived:*

Corollary 2.1. *The following generating expression for the Brenke-Sheffer polynomials holds true:*

$$\mathcal{A}(w)A(\mathcal{J}(w))\psi(x\mathcal{J}(w))=\sum_{n=0}^{\infty}{}_Y s_n(x)\frac{w^n}{n!}. \tag{2.12}$$

Corollary 2.2. *The Brenke-Sheffer polynomials ${}_Y s_n(x)$ are quasi-monomial w.r.t the following Φ^+_{Ys} and Φ^-_{Ys} operators:*

$$\Phi^+_{Ys}=x\mathcal{J}'(\mathcal{J}^{-1}(\varsigma))D_x\varsigma^{-1}+\frac{A'(\varsigma)}{A(\varsigma)}\mathcal{J}'(\mathcal{J}^{-1}(\varsigma))$$

$$+\frac{\mathcal{A}'(\mathcal{J}^{-1}(\varsigma))}{\mathcal{A}(\mathcal{J}^{-1}(\varsigma))} \tag{2.13}$$

$$\Phi^-_{YS}=\mathcal{J}^{-1}(\varsigma). \tag{2.14}$$

Corollary 2.3. *The Brenke-Sheffer polynomials ${}_{Y}s_n(x)$ satisfy the following differential equation:*

$$\Bigg(x\mathcal{J}'(\mathcal{J}^{-1}(\varsigma))D_x\varsigma^{-1}\mathcal{J}^{-1}(\varsigma)+\frac{A'(\varsigma)}{A(\varsigma)}\mathcal{J}'(\mathcal{J}^{-1}(\varsigma))\mathcal{J}^{-1}(\varsigma)$$
$$+\frac{\mathcal{A}'(\mathcal{J}^{-1}(\varsigma))}{\mathcal{A}(\mathcal{J}^{-1}(\varsigma))}\mathcal{J}^{-1}(\varsigma)-n\Bigg){}_{Y}s_n(x)=0. \tag{2.15}$$

Remark 2.2. *In consideration of $\psi(w) = \exp(w)$, the Boas-Buck polynomials $p_n(x)$ reduce to the Sheffer polynomials $s_n(x)$ [13]. Accordingly, taking $\psi(w) = \exp(w)$ (for this case $\varsigma = D_x$) in Eqs 2.1, 2.5, 2.6 and 2.11, the analogous results for the two-iterated Sheffer polynomials $s_n^{[2]}(x)$ established in [22] are obtained.*

Remark 2.3. *In consideration of $\mathcal{J}(w) = w$, the polynomials $s_n(x)$ reduce to $A_n(x)$ [14]. Accordingly, taking $\mathcal{J}(w) = w$ (for this case $\varsigma = D_x$) in Eqs 2.1, 2.5, 2.6 and 2.11, the analogous results for the Boas-Buck-Appell polynomials ${}_{p}A_n(x)$ can be obtained easily.*

3. Generalization of the Szasz operators involving the Boas-Buck-Sheffer polynomials and approximation properties of S_n operators

The Boas-Buck-Sheffer polynomials determined by the generating expression (2.1) holding the conditions given by expressions (1.1)–(1.2) and (1.5)–(1.7). Thus restricting to the ${}_{p}s_n(x)$, satisfying the assumptions:

$$(i)\ \mathcal{A}(1)\neq 0, A(\mathcal{J}(1))\neq 0, \mathcal{J}'(1)=1, H'(\mathcal{J}(1))=1, {}_{p}s_n(x)\geq 0, k=0,1,2,....$$

$$(ii)\ \psi : R\rightarrow(0,\infty),$$

$$(iii)\ (2.1)\ and\ (1.1)-(1.7)\ converges\ for\ |t|<R\ (R>1). \tag{3.1}$$

With the above assumptions, we give a new formulation of positive linear operators with ${}_{p}s_n(x)$ as:

$$S_n(f;x)=\frac{1}{\mathcal{A}(1)A(\mathcal{J}(1))\psi(nxH(\mathcal{J}(1)))}\sum_{n=0}^{\infty}{}_{p}s_n(nx)g(\frac{k}{n}), \tag{3.2}$$

where $x\geq 0$ and $n\in N$.

Remark 3.1. *For $\mathcal{J}(w) = w$, the operator given by (3.2) gives the operators of the Boas-Buck-Appell polynomials.*

Remark 3.2. *For $\mathcal{A}(w) = 1$ and $\mathcal{J}(w) = w$, the operator given by (3.2) (resp. (2.1)) meets the operators of the Boas-Buck polynomials [21].*

Remark 3.3. *For $\mathcal{A}(w) = 1$, $\mathcal{J}(w) = w$ and $H(w) = w$ the operator given by (3.2) (resp. (2.1)) meets the operators of the Brenke-type polynomials [12].*

Remark 3.4. *For $\mathcal{A}(w) = 1$, $\mathcal{J}(w) = w$ and $\psi(w) = e^w$, the operator given by (3.2) (resp. (2.1)) meets the operators of the Sheffer polynomials [20].*

Remark 3.5. *For $\mathcal{A}(w) = 1$, $\mathcal{J}(w) = w$, $H(w) = w$ and $\psi(w) = e^w$ the operator given by (3.2) (resp. (2.1)) meets the operators of the Appell polynomials [19].*

Remark 3.6. *For $\mathcal{A}(w) = 1$, $A(w) = 1$, $\mathcal{J}(w) = w$, $H(w) = w$ and $\psi(w) = e^w$ the operator given by (3.2) (resp. (2.1)) meets the Szasz operators [18].*

Next, we give the approximation results by means of $\mathcal{S}_n$ positive linear operators using Krovokin's theorem.

Lemma 4. *For the operators $\mathcal{S}_n(f; x)$ and $x \in [0, \infty)$, we have*

$$\mathcal{S}_n(1; x) = 1 \tag{3.3}$$

$$\mathcal{S}_n(s; x) = \frac{\psi'\Big(nxH(\mathcal{J}(1))\Big)}{\psi\Big(nxH(\mathcal{J}(1))\Big)}x + \frac{1}{n}\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(\mathcal{J}(1))}{A(\mathcal{J}(1))}\right) \tag{3.4}$$

$$\begin{aligned}\mathcal{S}_n(s^2; x) = {} & \frac{\psi''\Big(nxH(\mathcal{J}(1))\Big)}{\psi\Big(nxH(\mathcal{J}(1))\Big)}\,x^2 + \left(2\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + 2\frac{A'(\mathcal{J}(1))}{A(\mathcal{J}(1))} + 1 + \mathcal{J}''(1) + H''(\mathcal{J}(1))\right) \\ & \times \frac{\psi'\Big(nxH(\mathcal{J}(1))\Big)}{n\psi\Big(nxH(\mathcal{J}(1))\Big)}\,x \\ & + \frac{1}{n^2}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(\mathcal{J}(1))}{A(\mathcal{J}(1))}(1 + \mathcal{J}''(1)) + 2\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}\frac{A'(\mathcal{J}(1))}{A(\mathcal{J}(1))} + \frac{A''(\mathcal{J}(1))}{A(\mathcal{J}(1))}\right).\end{aligned} \tag{3.5}$$

proof. From the generating function of the Boas-Buck-Sheffer polynomials given by (2.1), we have

$$\sum_{k=0}^{\infty} {}_sA_k(nx) = \mathcal{A}(w)A(\mathcal{J}(w))\psi\Big(nxH(\mathcal{J}(w))\Big). \tag{3.6}$$

Differentiating above equation w.r.t. w successively on both sides and then putting $w = 1$ and using $\mathcal{J}'(1) = 1$ and $H'(\mathcal{J}(1))$ we have

$$\sum_{k=0}^{\infty} {}_ps_k(nx) = \mathcal{A}(1)A(\mathcal{J}(1))\psi\Big(nxH(\mathcal{J}(1))\Big), \tag{3.7}$$

$$\begin{aligned}\sum_{k=0}^{\infty} k\,{}_ps_k(nx) = {} & \Big(\mathcal{A}'(1)A(\mathcal{J}(1)) + \mathcal{A}(1)A'(\mathcal{J}(1))\Big)\psi\Big(nxH(\mathcal{J}(1))\Big) \\ & + \mathcal{A}(1)A(\mathcal{J}(1))\psi'\Big(nxH(\mathcal{J}(1))\Big)nx,\end{aligned} \tag{3.8}$$

$$\begin{aligned}\sum_{k=0}^{\infty} k^2\,{}_ps_k(nx) = {} & \Big(2\mathcal{A}'(1)A(\mathcal{J}(1)) + \mathcal{A}''(1)A(\mathcal{J}(1)) + \mathcal{A}(1)A''(\mathcal{J}(1)) \\ & + \mathcal{A}'(1)A(\mathcal{J}(1)) + \mathcal{A}(1)A'(\mathcal{J}(1))(\mathcal{J}''(1) + H''(\mathcal{J}(1)))\Big)\psi\Big(nxH(\mathcal{J}(1))\Big) \\ & + \Big(\mathcal{A}(1)A(\mathcal{J}(1))(1 + (\mathcal{J}''(1)) + H''(\mathcal{J}(1))) + 2\mathcal{A}'(1)A(1) + 2\mathcal{A}'(1)A(\mathcal{J}(1)) + \mathcal{A}(1)A'(\mathcal{J}(1))\Big) \\ & \times \psi'\Big(nxH(\mathcal{J}(1))\Big)nx + \mathcal{A}(1)A(\mathcal{J}(1))\psi''\Big(nxH(\mathcal{J}(1))\Big)(nx)^2.\end{aligned} \tag{3.9}$$

In view of the equalities (3.7)–(3.9), assertions (3.3)–(3.5) are obtained.

Next, we define the class of $\mathcal{E}$ as follows:

$$\mathcal{E} := \left\{ g : x \in [0,\infty), \frac{g(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\}. \tag{3.10}$$

Theorem 5. *Let $g \in C[0,\infty) \cap \mathcal{E}$ and suppose*

$$\lim_{y\to\infty} \frac{G'(y)}{G(y)} = 1, \qquad \lim_{y\to\infty} \frac{G''(y)}{G(y)} = 1. \tag{3.11}$$

Then,

$$\lim_{n\to\infty} \mathcal{S}_n(g; x) = g(x) \tag{3.12}$$

on each compact subset of $[0,\infty)$*, uniformly.*

Proof. In view of Lemma 3.1 and and assumption given by expression (3.10), we find

$$\lim_{n\to\infty} \mathcal{S}_n(s^i; x) = x^i, \quad i = 0, 1, 2. \tag{3.13}$$

The above-mentioned convergences are satisfied uniformly in each compact subset of $[0,\infty)$. The use of universal Krovokin-type property (vi) of Theorem 4.1.4 from [23], desired result is established.

In order to estimate the rate of convergence, we will give some definitions and lemmas.

Definition 3.1. Let $g \in \hat{C}[0,\infty)$ and $\delta > 0$. Then $w(g;\delta)$, that is, modulus of continuity of the function g is determined by

$$w(g;\delta) := \sup_{\substack{u,v\in[0,\infty)\\ |u-v|}} |g(u) - g(v)|, \tag{3.14}$$

where $\hat{C}[0,\infty)$ is the space of uniformly continuous functions on $[0,\infty)$.

Lemma 6. *For $x \in [0,\infty)$, the following identities:*

$$\mathcal{S}_n(s-x; x) := \left(\frac{\psi'\big(nxH(\mathcal{J}(1))\big)}{\psi\big(nxH(\mathcal{J}(1))\big)} - 1 \right) x + \frac{1}{n} \left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(\mathcal{J}(1))}{A(\mathcal{J}(1))} \right) \tag{3.15}$$

$$\begin{aligned} \mathcal{S}_n((s-x)^2; x) :=\ & \frac{\psi''\big(nxH(\mathcal{J}(1))\big) - 2\psi'\big(nxH(\mathcal{J}(1))\big) + \psi\big(nxH(\mathcal{J}(1))\big)}{\psi\big(nxH(\mathcal{J}(1))\big)} x^2 \\ & + \frac{\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(\mathcal{J}(1))}{A(\mathcal{J}(1))}\right)\Big(2\psi(nxH(\mathcal{J}(1))) - \psi(nxH(\mathcal{J}(1)))\Big) + \Big(1 + \mathcal{J}''(1) + H''(\mathcal{J}(1))\Big)\psi\big(nxH(\mathcal{J}(1))\big)}{n\psi\big(nxH(\mathcal{J}(1))\big)} x \\ & + \frac{1}{n^2} \left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(\mathcal{J}(1))}{A(\mathcal{J}(1))}(1 + \mathcal{J}''(1)) + 2\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}\frac{A'(\mathcal{J}(1))}{A(\mathcal{J}(1))} + \frac{A''(\mathcal{J}(1))}{A(\mathcal{J}(1))} \right) \end{aligned} \tag{3.16}$$

are satisfied.

Proof. In view of linearity property of $\mathcal{S}_n$ operators:

$$\mathcal{S}_n((s-x)^2;x) = \mathcal{S}_n(s^2;x) - 2x\mathcal{S}_n(s;x) + x^2\mathcal{S}_n(1;x) \tag{3.17}$$

and applying Lemma 3.1, equalities (3.15) and (3.16) are obtained.

Theorem 7. *Let $g \in \hat{C}[0,\infty) \cap E, \mathcal{S}_n$ operators satisfy the following inequality:*

$$|\mathcal{S}_n(g;x) - g(x)| \leqslant 2w(g; \sqrt{\mu_n(x)}), \tag{3.18}$$

where,

$$\mu := \mu_n(x) = \mathcal{S}_n((s-x)^2;x). \tag{3.19}$$

Proof. Using Lemma 3.1 and the modulus of continuity property, we get

$$|\mathcal{S}_n(g;x) - g(x)| = \left| \frac{1}{\mathcal{A}(1)A(\mathcal{J}(1))\psi(nxH\mathcal{J}(1))} \sum_{k=0}^{\infty} {}_p s_k(nx) g\left(\frac{k}{n}\right) - g(x) \right|$$

$$\leqslant \frac{1}{\mathcal{A}(1)A(\mathcal{J}(1))\psi(nxH\mathcal{J}(1))} \sum_{k=0}^{\infty} {}_p s_k(nx) \left| g\left(\frac{k}{n}\right) - g(x) \right|$$

$$\leqslant w(g;\delta)\left(1 + \frac{1}{\mathcal{A}(1)A(\mathcal{J}(1))\psi(nxH\mathcal{J}(1))} \frac{1}{\delta} \sum_{k=0}^{\infty} {}_p s_k(nx) \left|\frac{k}{n} - x\right| \right). \tag{3.20}$$

The use of Cauchy-Schwarz inequality, and taking into account the Lemma 3.2, we have

$$\sum_{k=0}^{\infty} {}_p s_k(nx)\left|\frac{k}{n} - x\right| \leqslant \sqrt{\mathcal{A}(1)A(\mathcal{J}(1))\psi(nxH\mathcal{J}(1))}\left\{ \sum_{k=0}^{\infty} {}_p s_k(nx)\left|\frac{k}{n} - x\right|^2 \right\}^{\frac{1}{2}}$$

$$= \frac{\psi''\Big(nxH(\mathcal{J}(1))\Big) - 2\psi'\Big(nxH(\mathcal{J}(1))\Big) + \psi\Big(nxH(\mathcal{J}(1))\Big)}{\psi\Big(nxH(\mathcal{J}(1))\Big)} x^2$$

$$+ \frac{\left(\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(\mathcal{J}(1))}{A(\mathcal{J}(1))}\right)\Big(2\psi(nxH(\mathcal{J}(1))) - \psi(nxH(\mathcal{J}(1)))\Big) + \Big(1 + \mathcal{J}''(1) + H''(\mathcal{J}(1))\Big)\psi\Big(nxH(\mathcal{J}(1))\Big)}{n\psi\Big(nxH(\mathcal{J}(1))\Big)} x$$

$$+ \frac{1}{n^2}\left(\frac{\mathcal{A}''(1)}{\mathcal{A}(1)} + \frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + \frac{A'(\mathcal{J}(1))}{A(\mathcal{J}(1))}(1 + \mathcal{J}''(1)) + 2\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}\frac{A'(\mathcal{J}(1))}{A(\mathcal{J}(1))} + \frac{A''(\mathcal{J}(1))}{A(\mathcal{J}(1))}\right). \tag{3.21}$$

Using Eq 3.21, Eq 3.20 becomes

$$|\mathcal{S}_n(g;x) - g(x)| \leqslant \left\{1 + \frac{1}{\delta}\sqrt{\mu_n(x)}\right\} w(g;\delta). \tag{3.22}$$

On choosing $\delta = \sqrt{\mu_n(x)}$, we get the required result.

Definition 3.2. Let $g \in C_B[a,b]$, then the second modulus of continuity g is

$$w_2(g;\delta) := \sup_{0<t\leq\delta} \|g(.+2t) - 2g(.+t) + g(.)\|_{C_B}, \tag{3.23}$$

where $C_B[0,\infty)$ is the class of real valued functions on $[0,\infty)$, bounded and uniformly continuous with the norm

$$\|g\|_{C_B} = \sup_{x\in[0,\infty)} |g(x)|. \tag{3.24}$$

Lemma 8. *(Gavrea and Raşa [24]). Let $g \in C^2[0,a]$ and $(K_n)_{n\geq 0}$ be a sequence of linear positive operators with the property $K_n(1;x) = 1$. Then,*

$$|K_n(g;x) - g(x)| \leq \|g'\| \sqrt{K_n((s-x)^2;x)} + \frac{1}{2}\|g''\|K_n((s-x)^2;x). \tag{3.25}$$

Lemma 9. *(Zhuk [25]). Let $f \in C[a,b]$ and $h \in (0, \frac{a-b}{2})$. Let g_h be the second-order Steklov function to the function g. Then, the subsequent inequalities hold true*

$$(i)\ \|g_h - g\| \leq \frac{3}{4} w_2(g;h), \tag{3.26}$$

$$(ii)\ \||g_h''| \leq \frac{3}{2h^2} w_2(g;h). \tag{3.27}$$

Theorem 10. *For $g \in C[0,a]$, the following estimate*

$$|\mathcal{S}_n(g;x) - g(x)| \leq \frac{2}{\beta}\|g\|l^2 + \frac{3}{4}(\beta + 2 + l^2) w_2(g;l) \tag{3.28}$$

holds, where

$$l := l_n(x) = \sqrt[4]{\mathcal{S}_n((s-x;x)^2;x)}. \tag{3.29}$$

Proof. Let g_l be the second-order Steklov function linked to g. Considering the identity $\mathcal{Z}_n(1;x) = 1$, we have

$$\begin{aligned} |\mathcal{S}_n(g;x) - g(x)| &\leq |\mathcal{S}_n(g - g_l;x)| + |\mathcal{S}_n(g_l;x) - g(x)| + |(g_l;x) - g(x)|, \\ &\leq 2\|g_l - g\| + |\mathcal{S}_n(g_l;x) - g_l(x)|. \end{aligned} \tag{3.30}$$

Taking into the fact that $g_l \in C^2[0,\beta]$, it follows from Lemma 3.5

$$|\mathcal{S}_n(g_l;x) - g_l(x)| \leq \|g_l'\| \sqrt{S_n((s-x)^2;x)} + \frac{1}{2}\|g_l''\|S_n((s-x)^2;x). \tag{3.31}$$

On combining the Landau inequality with Lemma 3.6, we have

$$\begin{aligned} \|g_l''\| &\leq \frac{2}{\beta}\|g_l\| + \frac{\beta}{2}\|g_l''\|, \\ &\leq \frac{2}{\beta}\|g\| + \frac{3\beta}{4l^2} w_2(g;l). \end{aligned} \tag{3.32}$$

Using the last inequality and taking $l = \sqrt[4]{\mathcal{S}_n((s-x)^2);x}$, we have

$$|\mathcal{S}_n(g_l;x) - g_l(x)| \leq \frac{2}{\beta}\|g\|l^2 + \frac{3\beta}{4} w_2(g;l) + \frac{3}{4}l^2 w_2(g;l). \tag{3.33}$$

Substituting (3.33) in (3.30), hence Lemma (3.6) confers the proof of the theorem.

Remark 3.7. *In Theorem 3.7, for the special case $\psi(w) = e^w$, $H(\mathcal{J}(w)) = w$, $\mathcal{A}(w) = 1$, $A(\mathcal{J}(w)) = 1$ and $x = 0$, one can get $l = 0$ from the equality $l := l_n(x) = \sqrt[4]{S_n((s-x)^2; x)}$. The inequality obtained in Theorem 3.7, still holds true when $l = 0$.*

4. Conclusions

Several mixed special polynomial families related to the Sheffer sequences are available in the literature. In this line, we established a mixed family of polynomials called the Boas-Buck-Sheffer family and studied their properties. Also, we studied the extension of the Szasz operators involving the Boas-Buck-Sheffer polynomials and gave their approximation properties.

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Conflict of interest

We declare that we have no conflict of interests.

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