AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(5): 4399–4411.
DOI:10.3934/math.2020280
Received: 15 October 2019
Accepted: 07 May 2020
Published: 11 May 2020

Research article

Construction of partially degenerate Laguerre-Genocchi polynomials with their applications

Talha Usman[1], Mohd Aman[2], Owais Khan[3], Kottakkaran Sooppy Nisar[4] and Serkan Araci[5,*]

[1] Department of Mathematics, School of Basic and Applied Sciences, Lingaya's Vidyapeeth, Faridabad-121002, Haryana, India

[2] Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh-202002, India

[3] Department of Mathematics, Integral University, Lucknow-226026, India

[4] Department of Mathematics, College of Arts and Sciences, Wadi Aldawaser, 11991, Prince Sattam bin Abdulaziz University, Saudi Arabia

[5] Department of Economics, Faculty of Economics, Administrative and Social Science, Hasan Kalyoncu University, TR-27410 Gaziantep, Turkey

* **Correspondence:** Email: mtsrkn@hotmail.com.

Abstract: Various applications of degenerate polynomials in different areas call for the thoughtful study and research, and many extensions and variants can be found in the literature. In this paper, we introduce partially degenerate Laguerre-Genocchi polynomials and investigate their properties and identities. Furthermore, we introduce a generalized form of partially degenerate Laguerre-Genocchi polynomials and derive some interesting properties and identities. The results obtained are of general character and can be reduced to yield formulas and identities for relatively simple polynomials and numbers.

Keywords: Laguerre polynomials; partially degenerate Laguerre-Genocchi polynomials; summation formula; symmetric identities
Mathematics Subject Classification: 05A10, 05A15, 33C45

1. Introduction and preliminaries

We begin with the following definitions of notations:

$$\mathbb{N} = \{1, 2, 3, \cdots\} \text{ and } \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Also, as usual, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The two variable Laguerre polynomials $L_n(u, v)$ [1] are defined by the Taylor expansion about $\tau = 0$ (also popularly known as generating function) as follows:

$$\sum_{p=0}^{\infty} L_p(u, v)\frac{\tau^p}{p!} = e^{v\tau}C_0(u\tau),$$

where is the 0-th order Tricomi function [19] given by

$$C_0(u) = \sum_{p=0}^{\infty} \frac{(-1)^p u^p}{(p!)^2}$$

and has the series representation

$$L_p(u, v) = \sum_{s=0}^{p} \frac{p!(-1)^s v^{p-s} u^s}{(p-s)!(s!)^2}.$$

The classical Euler polynomials $E_p(u)$, Genocchi polynomials $G_p(u)$ and the Bernoulli polynomials $B_p(u)$ are usually defined by the generating functions (see, for details and further work, [1, 2, 4–7, 9, 11, 12, 20]):

$$\sum_{p=0}^{\infty} E_p(u)\frac{\tau^p}{p!} = \frac{2}{e^\tau + 1}e^{u\tau} \quad (|\tau| < \pi),$$

$$\sum_{p=0}^{\infty} G_p(u)\frac{\tau^p}{p!} = \frac{2\tau}{e^\tau + 1}e^{u\tau} \quad (|\tau| < \pi)$$

and

$$\sum_{p=0}^{\infty} B_p(u)\frac{\tau^p}{p!} = \frac{\tau}{e^\tau - 1}e^{u\tau} \quad (|\tau| < 2\pi).$$

The Daehee polynomials, recently originally defined by Kim *et al.* [9], are defined as follows

$$\sum_{p=0}^{\infty} D_p(u)\frac{\tau^p}{p!} = \frac{\log(1+\tau)}{\tau}(1+\tau)^u, \tag{1.1}$$

where, for $u = 0$, $D_p(0) = D_p$ stands for Daehee numbers given by

$$\sum_{p=0}^{\infty} D_p \frac{\tau^p}{p!} = \frac{\log(1+\tau)}{\tau}. \tag{1.2}$$

Due to Kim *et al.*'s idea [9], Jang *et al.* [3] gave the partially degenarate Genocchi polynomials as follows:

$$\frac{2\log(1+\tau\lambda)^{\frac{1}{\lambda}}}{e^\tau + 1}e^{u\tau} = \sum_{p=0}^{\infty} G_{p,\lambda}(u)\frac{\tau^p}{p!}, \tag{1.3}$$

which, for the case $u = 0$, yields the partially degenerate Genocchi numbers $G_{n,\lambda} := G_{n,\lambda}(0)$.

Pathan *et al.* [17] considered the generalization of Hermite-Bernoulli polynomials of two variables ${}_HB_p^{(\alpha)}(u,v)$ as follows

$$\left(\frac{\tau}{e^\tau - 1}\right)^\alpha e^{u\tau+v\tau^2} = \sum_{p=0}^{\infty} {}_HB_p^{(\alpha)}(u,v)\frac{\tau^p}{p!}. \tag{1.4}$$

On taking $\alpha = 1$ in (1.4) yields a well known result of [2, p. 386 (1.6)] given by

$$\left(\frac{\tau}{e^\tau - 1}\right) e^{u\tau+v\tau^2} = \sum_{p=0}^{\infty} {}_HB_p(u,v)\frac{\tau^p}{p!}. \tag{1.5}$$

The two variable Laguerre-Euler polynomials (see [7, 8]) are defined as

$$\sum_{p=0}^{\infty} {}_LE_p(u,v)\frac{\tau^p}{p!} = \frac{2}{e^\tau + 1} e^{v\tau} C_0(u\tau). \tag{1.6}$$

The alternating sum $T_k(p)$, where $k \in \mathbb{N}_0$, (see [14]) is given as

$$T_k(p) = \sum_{j=0}^{p} (-1)^j j^k$$

and possess the generating function

$$\sum_{r=0}^{\infty} T_k(p)\frac{\tau^r}{r!} = \frac{1 - (-e^\tau)^{(p+1)}}{e^\tau + 1}. \tag{1.7}$$

The idea of degenerate numbers and polynomials found existence with the study related to Bernoulli and Euler numbers and polynomials. Lately, many researchers have begun to study the degenerate versions of the classical and special polynomials (see [3, 10–16, 18], for a systematic work). Influenced by their works, we introduce partially degenerate Laguerre-Genocchi polynomials and also a new generalization of partially degenerate Laguerre-Genocchi polynomials and then give some of their applications. We also derive some implicit summation formula and general symmetry identities.

2. Partially degenerate Laguerre-Genocchi polynomials

Let $\lambda, \tau \in \mathbb{C}$ with $|\tau\lambda| \leq 1$ and $\tau\lambda \neq -1$. We introduce and investigate the partially degenerate Laguerre-Genocchi polynomials as follows:

$$\sum_{p=0}^{\infty} {}_LG_{p,\lambda}(u,v)\frac{\tau^p}{p!} = \frac{2\log(1+\lambda\tau)^{\frac{1}{\lambda}}}{e^\tau + 1} e^{v\tau} C_0(u\tau). \tag{2.1}$$

In particular, when $\lambda \to 0$, ${}_LG_{p,\lambda}(u,v) \to {}_LG_p(u,v)$ and they have the closed form given as

$${}_LG_{p,\lambda}(u,v) = \sum_{q=0}^{p} \binom{p}{q} G_{q,\lambda} L_{p-q}(u,v).$$

Clearly, $u = 0$ in (2.1) gives ${}_LG_{p,\lambda}(0,0) := G_{p,\lambda}$ that stands for the partially degenerate Genocchi polynomials [3].

Theorem 1. *For $p \in \mathbb{N}_o$, the undermentioned relation holds:*

$$ {}_LG_{p,\lambda}(u,v) = \sum_{q=0}^{p} \binom{p}{q+1} q!(-\lambda)^q {}_LG_{p-q-1}(u,v). \tag{2.2} $$

Proof. With the help of (2.1), one can write

$$ \begin{aligned} \sum_{p=0}^{\infty} {}_LG_{p,\lambda}(u,v)\frac{\tau^p}{p!} =& \frac{2\log(1+\lambda\tau)^{\frac{1}{\lambda}}}{e^\tau+1} e^{v\tau} C_0(u\tau) \\ =& \tau \left\{ \sum_{q=0}^{\infty} \frac{(-1)^q}{q+1}(\lambda\tau)^q \right\} \left\{ \sum_{p=0}^{\infty} {}_LG_p(u,v)\frac{\tau^p}{p!} \right\} \\ =& \sum_{p=0}^{\infty} \left\{ \sum_{q=0}^{p} \binom{p}{q} \frac{(-\lambda)^q}{q+1} q! {}_LG_{p-q}(u,v) \right\} \frac{\tau^{p+1}}{p!}, \end{aligned} $$

where, ${}_LG_{p-q}(u,v)$ are the Laguerre-Genocchi polynomials (see [8]). Finally, the assertion easily follows by equating the coefficients $\frac{\tau^p}{p!}$. □

Theorem 2. *For $p \in \mathbb{N}_o$, the undermentioned relation holds:*

$$ {}_LG_{p+1,\lambda}(u,v) = \sum_{q=0}^{p} \binom{p}{q} \lambda^q (p+1) {}_LG_{p-q+1}(u,v) D_q. \tag{2.3} $$

Proof. We first consider

$$ \begin{aligned} I_1 = \frac{1}{\tau}\frac{2\log(1+\lambda\tau)^{\frac{1}{\lambda}}}{e^\tau+1} e^{v\tau} C_0(u\tau) =& \left\{ \sum_{q=0}^{\infty} D_q \frac{(\lambda\tau)^q}{q!} \right\} \left\{ \sum_{p=0}^{\infty} {}_LG_p(u,v)\frac{\tau^p}{p!} \right\} \\ =& \sum_{p=1}^{\infty} \left\{ \sum_{q=0}^{p} \binom{p}{q} (\lambda)^q D_q {}_LG_{p-q}(u,v) \right\} \frac{\tau^p}{p!}. \end{aligned} $$

Next we have,

$$ \begin{aligned} I_2 = \frac{1}{\tau}\frac{2\log(1+\lambda\tau)^{\frac{1}{\lambda}}}{e^\tau+1} e^{v\tau} C_o(u\tau) =& \frac{1}{\tau} \sum_{p=0}^{\infty} {}_LG_{p,\lambda}(u,v)\frac{\tau^p}{p!} \\ =& \sum_{p=0}^{\infty} \frac{{}_LG_{p+1,\lambda}(u,v)}{p+1} \frac{\tau^p}{p!}. \end{aligned} $$

Since $I_1 = I_2$, we conclude the assertion (2.3) of Theorem 2. □

Theorem 3. *For $p \in \mathbb{N}_0$, the undermentioned relation holds:*

$$ {}_LG_{p,\lambda}(u,v) = p \sum_{q=0}^{p-1} \binom{p-1}{q} (\lambda)^q {}_LE_{p-q-1}(u,v) D_q. \tag{2.4} $$

Proof. With the help of (2.1), one can write

$$\begin{aligned}\sum_{p=0}^{\infty} {}_{L}G_{p,\lambda}(x,y)\frac{\tau^p}{p!} =& \left\{\frac{\tau\log(1+\lambda\tau)}{\lambda\tau}\right\}\left\{\frac{2}{e^{\tau}+1}e^{v\tau}C_0(u\tau)\right\}\\ =&\tau\left\{\sum_{q=0}^{\infty} D_q\frac{(\lambda\tau)^q}{q!}\right\}\left\{\sum_{p=0}^{\infty} {}_{L}E_p(u,v)\frac{\tau^p}{p!}\right\}\\ =&\sum_{p=0}^{\infty}\left\{\sum_{q=0}^{p}\binom{p}{q}(\lambda)^q D_q\, {}_{L}E_{p-q}(u,v)\right\}\frac{\tau^{p+1}}{p!}.\end{aligned}$$

Finally, the assertion (2.4) straightforwardly follows by equating the coefficients of same powers of τ above. □

Theorem 4. *For $p \in \mathbb{N}_o$, the following relation holds:*

$$ {}_{L}G_{p,\lambda}(u,v+1) = \sum_{q=0}^{p}\binom{p}{q}\, {}_{L}G_{p-q,\lambda}(u,v). \tag{2.5}$$

Proof. Using (2.1), we find

$$\begin{aligned}&\sum_{p=0}^{\infty}\left\{{}_{L}G_{p,\lambda}(u,v+1) - {}_{L}G_{p,\lambda}(u,v)\right\}\frac{\tau^p}{p!} = \frac{2\log(1+\lambda\tau)^{\frac{1}{\lambda}}}{e^{\tau}+1}\\ &\times e^{(v+1)\tau}C_0(u\tau) - \frac{2\log(1+\lambda\tau)^{\frac{1}{\lambda}}}{e^{\tau}+1}e^{v\tau}C_0(u\tau)\\ &=\sum_{p=0}^{\infty} {}_{L}G_{p,\lambda}(u,v)\frac{\tau^p}{p!}\sum_{q=0}^{\infty}\frac{\tau^q}{q!} - \sum_{p=0}^{\infty} {}_{L}G_{p,\lambda}(u,v)\frac{\tau^p}{p!}\\ &=\sum_{p=0}^{\infty}\left\{\sum_{q=0}^{p}\binom{p}{q}\, {}_{L}G_{p-q,\lambda}(u,v) - {}_{L}G_{p,\lambda}(u,v)\right\}\frac{\tau^p}{p!}.\end{aligned}$$

Hence, the assertion (2.5) straightforwardly follows by equating the coefficients of τ^p above. □

Theorem 5. *For $p \in \mathbb{N}_o$, the undermentioned relation holds:*

$$ {}_{L}G_{p,\lambda}(u,v) = \sum_{q=0}^{p}\sum_{l=0}^{q}\binom{p}{q}\binom{q}{l}G_{p-q}D_{q-l}\,\lambda^{q-l}L_l(u,v). \tag{2.6}$$

Proof. Since

$$\begin{aligned}\sum_{p=0}^{\infty} {}_{L}G_{p,\lambda}(u,v)\frac{\tau^p}{p!} =& \frac{2\log(1+\lambda\tau)^{\frac{1}{\lambda}}}{e^{\tau}+1}e^{v\tau}C_0(u\tau)\\ =&\left\{\frac{2\tau}{e^{\tau}+1}\right\}\left\{\frac{2\log(1+\lambda\tau)}{\lambda\tau}\right\}e^{v\tau}C_0(u\tau)\end{aligned}$$

$$= \left\{\sum_{p=0}^{\infty} G_p \frac{\tau^p}{p!}\right\}\left\{\sum_{q=0}^{\infty} D_q \frac{(\lambda\tau)^q}{q!}\right\}\left\{\sum_{l=0}^{\infty} L_l(u,v)\frac{\tau^l}{l!}\right\},$$

we have

$$\sum_{p=0}^{\infty} {}_LG_{p,\lambda}(u,v)\frac{\tau^p}{p!} = \sum_{p=0}^{\infty}\left\{\sum_{q=0}^{p}\sum_{l=0}^{q}\binom{p}{q}\binom{q}{l} G_{p-q}D_{q-l}\lambda^{q-l}L_l(u,v)\right\}\frac{\tau^p}{p!}.$$

We thus complete the proof of Theorem 5. □

Theorem 6. *(Multiplication formula). For $p \in \mathbb{N}_o$, the undermentioned relation holds:*

$$ {}_LG_{p,\lambda}(u,v) = f^{p-1}\sum_{a=0}^{f-1} {}_LG_{p,\frac{\lambda}{f}}\left(u, \frac{v+a}{f}\right). \tag{2.7}$$

Proof. With the help of (2.1), we obtain

$$\begin{aligned}\sum_{p=0}^{\infty} {}_LG_{p,\lambda}(u,v)\frac{\tau^p}{p!} &= \frac{2\log(1+\lambda\tau)^{\frac{1}{\lambda}}}{e^{\tau}+1}e^{v\tau}C_0(u\tau)\\ &= \frac{2\log(1+\lambda\tau)^{\frac{1}{\lambda}}}{e^{\tau}+1}C_0(u\tau)\sum_{a=0}^{f-1}e^{(a+v)\tau}\\ &= \sum_{p=0}^{\infty}\left\{f^{p-1}\sum_{a=0}^{f-1} {}_LG_{p,\frac{\lambda}{f}}\left(u,\frac{v+a}{f}\right)\right\}\frac{\tau^p}{p!}.\end{aligned}$$

Thus, the result in (2.7) straightforwardly follows by comparing the coefficients of τ^p above. □

3. Generalized partially degenerate Laguerre-Genocchi polynomials

Consider a Dirichlet character χ and let d $(d \in \mathbb{N})$ be the conductor connected with it such that $d \equiv 1(\mod 2)$ (see [22]). Now we present a generalization of partially degenerate Laguerre-Genocchi polynomials attached to χ as follows:

$$\sum_{p=0}^{\infty} {}_LG_{p,\chi,\lambda}(u,v)\frac{\tau^p}{p!} = \frac{2\log(1+\lambda\tau)^{\frac{1}{\lambda}}}{e^{f\tau}+1}\sum_{a=0}^{f-1}(-1)^a\chi(a)e^{(v+a)\tau}C_0(u\tau). \tag{3.1}$$

Here, $G_{p,\chi,\lambda} = {}_LG_{p,\chi,\lambda}(0,0)$ are in fact, the generalized partially degenerate Genocchi numbers attached to the Drichlet character χ. We also notice that

$$\lim_{\substack{\lambda\to 0\\ v=0}} {}_LG_{p,\chi,\lambda}(u,v) = G_{p,\chi}(u),$$

is the familiar looking generalized Genocchi polynomial (see [20]).

Theorem 7. *For $p \in \mathbb{N}_0$, the following relation holds:*

$$ {}_LG_{p,\chi,\lambda}(u,v) = \sum_{q=0}^{p} \binom{p}{q} \lambda^q D_q \, {}_LG_{p-q,\chi}(u,v). \tag{3.2}$$

Proof. In view of (3.1), we can write

$$\begin{aligned}
\sum_{p=0}^{\infty} {}_LG_{p,\chi,\lambda}(u,v)\frac{\tau^p}{p!} &= \frac{2\log(1+\lambda\tau)^{\frac{1}{\lambda}}}{e^{f\tau}+1}\sum_{a=0}^{f-1}(-1)^a\chi(a)e^{(v+a)\tau}C_0(u\tau) \\
&= \left\{\frac{\log(1+\lambda\tau)}{\lambda\tau}\right\}\left\{\frac{2\tau}{e^{f\tau}+1}\sum_{a=0}^{f-1}(-1)^a\chi(a)e^{(v+a)\tau}C_0(u\tau)\right\} \\
&= \left\{\sum_{q=0}^{\infty} D_q\frac{\lambda^q}{\tau^q}q!\right\}\left\{\sum_{p=0}^{\infty} {}_LG_{p,\chi}(u,v)\frac{\tau^p}{p!}\right\}.
\end{aligned}$$

Finally, the assertion (3.2) of Theorem 7 can be achieved by equating the coefficients of same powers of τ. □

Theorem 8. *The undermentioned formula holds true:*

$$ {}_LG_{p,\chi,\lambda}(u,v) = f^{p-1}\sum_{a=0}^{f-1}(-1)^a\chi(a) {}_LG_{p,\frac{\lambda}{f}}\left(u,\frac{a+v}{f}\right). \tag{3.3}$$

Proof. We first evaluate

$$\begin{aligned}
\sum_{p=0}^{\infty} {}_LG_{p,\chi,\lambda}(u,v)\frac{\tau^p}{p!} &= \frac{2\log(1+\lambda\tau)^{\frac{1}{\lambda}}}{e^{f\tau}+1}\sum_{a=0}^{f-1}(-1)^a\chi(a)e^{(v+a)\tau}C_0(u\tau) \\
&= \frac{1}{f}\sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{2\log(1+\lambda\tau)^{\frac{f}{\lambda}}}{e^{f\tau}+1}e^{\left(\frac{a+v}{f}\right)f\tau}C_0(u\tau) \\
&= \sum_{p=0}^{\infty}\left\{f^{p-1}\sum_{a=0}^{f-1}(-1)^a\chi(a) {}_LG_{p,\frac{\lambda}{f}}\left(u,\frac{a+v}{f}\right)\right\}\frac{\tau^p}{p!}.
\end{aligned}$$

Now, the Theorem 8 can easily be concluded by equating the coefficients $\frac{\tau^p}{p!}$ above. □

Using the result in (3.1) and with a similar approach used just as in above theorems, we provide some more theorems given below. The proofs are being omitted.

Theorem 9. *The undermentioned formula holds true:*

$$ {}_LG_{p,\chi,\lambda}(u,v) = \sum_{q=0}^{p}\frac{G_{p-q,\chi,\lambda}(v)\,(-u)^q\,p!}{(q!)^2\,(p-q)!}. \tag{3.4}$$

Theorem 10. *The undermentioned formula holds true:*

$$ {}_LG_{p,\chi,\lambda}(u,v) = \sum_{q=0}^{p,l}\frac{G_{p-q-l,\chi,\lambda}\,(v)^q\,(-u)^l\,p!}{(p-q-l)!\,(q)!\,(l!)^2}. \tag{3.5}$$

4. Implicit summation formulae

Theorem 11. *The undermentioned formula holds true:*

$$ {}_LG_{l+h,\lambda}(u,v) = \sum_{p,n=0}^{l,h} \binom{l}{p}\binom{h}{n}(u-v)^{p+n}\,{}_LG_{l+h-n-p,\lambda}(u,v). \tag{4.1}$$

Proof. On changing τ by $\tau+\mu$ and rewriting (2.1), we evaluate

$$e^{-v(\tau+\mu)}\sum_{l,h=0}^{\infty} {}_LG_{l+h,\lambda}(u,v)\frac{\tau^l\mu^h}{l!h!} = \frac{2\log(1+\lambda(\tau+\mu))^{\frac{1}{\lambda}}}{e^{\tau+\mu}+1}C_o(u(\tau+\mu)),$$

which, upon replacing v by u and solving further, gives

$$e^{(u-v)(\tau+\mu)}\sum_{l,h=0}^{\infty} {}_LG_{l+h,\lambda}(u,v)\frac{\tau^l\mu^h}{l!h!} = \sum_{l,h=0}^{\infty} {}_LG_{l+h,\lambda}(u,v)\frac{\tau^l\mu^h}{l!h!},$$

and also

$$\sum_{P=0}^{\infty}\frac{(u-v)^P(\tau+u)^P}{P!}\sum_{l,h=0}^{\infty} {}_LG_{l+h,\lambda}(u,v)\frac{\tau^l\mu^h}{l!h!} = \sum_{l,h=0}^{\infty} {}_LG_{l+h,\lambda}(u,v)\frac{\tau^l\mu^h}{l!h!}. \tag{4.2}$$

Now applying the formula [21, p.52(2)]

$$\sum_{P=0}^{\infty} f(P)\frac{(u+v)^P}{P!} = \sum_{p,q=0}^{\infty} f(p+q)\frac{u^p}{p!}\frac{v^q}{q!},$$

in conjunction with (4.2), it becomes

$$\sum_{p,n=0}^{\infty}\frac{(u-v)^{p+n}\tau^p\mu^n}{p!n!}\sum_{l,h=0}^{\infty} {}_LG_{l+h,\lambda}(u,v)\frac{\tau^l\mu^h}{l!h!} = \sum_{l,h=0}^{\infty} {}_LG_{l+h,\lambda}(u,v)\frac{\tau^l\mu^h}{l!h!}. \tag{4.3}$$

Further, upon replacing l by $l-p$, h by $h-n$, and using the result in [21, p.100 (1)], in the left of (4.3), we obtain

$$\sum_{p,n=0}^{\infty}\sum_{l,h=0}^{\infty}\frac{(u-v)^{p+n}}{p!\,n!}\,{}_LG_{l+h-p-n,\lambda}(u,v)\frac{\tau^l\mu^h}{(l-p)!(h-n)!} = \sum_{l,h=0}^{\infty} {}_LG_{l+h,\lambda}(u,v)\frac{\tau^l\mu^h}{l!\,h!}.$$

Finally, the required result can be concluded by equating the coefficients of the identical powers of τ^l and μ^h above. □

Corollary 4.1. *For $h=0$ in (4.1), we get*

$$ {}_LG_{l,\lambda}(u,v) = \sum_{\rho=0}^{l}\binom{l}{\rho}(u-v)^p\,{}_LG_{l-\rho,\lambda}(u,v).$$

Some identities of Genocchi polynomials for special values of the parameters u and v in Theorem 11 can also be obtained. Now, using the result in (2.1) and with a similar approach, we provide some more theorems given below. The proofs are being omitted.

Theorem 12. *The undermentioned formula holds good:*

$$_LG_{p,\lambda}(u, v+\mu) = \sum_{q=0}^{p} \binom{p}{q} \mu^q {}_LG_{p-q,\lambda}(u, v)$$

Theorem 13. *The undermentioned implicit holds true:*

$$\sum_{p=0}^{\infty} {}_LG_{p,\lambda}(u, v)\frac{\tau^p}{p!} = \frac{2\log(1+\lambda\tau)^{\frac{1}{\lambda}}}{e^\tau + 1} e^{v\tau} C_o(u\tau) = \sum_{q=0}^{p} \binom{p}{q} G_{p-q,\lambda}\; L_p(u, v)$$

and

$$_LG_{p,\lambda}(u, v) = \sum_{q=0}^{p} \binom{p}{q} G_{p-q,\lambda}(u, v)\; L_p(u, v).$$

Theorem 14. *The undermentioned implicit summation formula holds:*

$$_LG_{p,\lambda}(u, v+1) + {}_LG_{p,\lambda}(u, v) = 2p \sum_{q=0}^{p-1} \binom{p-1}{q} \frac{(-\lambda)^q q!}{q+1} L_{p-q-1}(u, v).$$

Theorem 15. *The undermentioned formula holds true:*

$$_LG_{p,\lambda}(u, v+1) = \sum_{q=0}^{p} {}_LG_{p-q,\lambda}(u, v).$$

5. Symmetry identities

Symmetry identities involving various polynomials have been discussed (e.g., [7, 9–11, 17]). As in above-cited work, here, in view of the generating functions (1.3) and (2.1), we obtain symmetry identities for the partially degenerate Laguerre-Genocchi polynomials $_LG_{n,\lambda}(u, v)$.

Theorem 16. *Let $\alpha, \beta \in \mathbb{Z}$ and $p \in \mathbb{N}_0$, we have*

$$\sum_{q=0}^{p} \binom{p}{q} \beta^q \alpha^{p-q} {}_LG_{p-q,\lambda}(u\beta, v\beta) {}_LG_{q,\lambda}(u\alpha, v\alpha)$$

$$= \sum_{q=0}^{p} \binom{p}{q} \alpha^q \beta^{p-q} {}_LG_{p-q,\lambda}(u\alpha, v\alpha) {}_LG_{q,\lambda}(u\beta, v\beta).$$

Proof. We first consider

$$g(\tau) = \frac{\left\{2\log(1+\lambda)^{\frac{\beta}{\lambda}}\right\}}{(e^{\alpha\tau}+1)} \frac{\left\{2\log(1+\lambda)^{\frac{\alpha}{\lambda}}\right\}}{(e^{\beta\tau}+1)} e^{(\alpha+\beta)v\tau} C_0(u\alpha\tau) C_0(u\beta\tau).$$

Now we can have two series expansion of $g(\tau)$ in the following ways:
On one hand, we have

$$\begin{aligned} g(\tau) &= \left(\sum_{p=0}^{\infty} {}_LG_{p,\lambda}(u\beta, v\beta)\frac{(\alpha\tau)^p}{p!}\right)\left(\sum_{q=0}^{\infty} {}_LG_{q,\lambda}(u\alpha, v\alpha)\frac{(\beta\tau)^q}{q!}\right) \\ &= \sum_{p=0}^{\infty}\left(\sum_{q=0}^{p}\binom{p}{q}\beta^q\alpha^{p-q}\ {}_LG_{p-q,\lambda}(u\beta, v\beta)\ {}_LG_{q,\lambda}(u\alpha, v\alpha)\right)\frac{\tau^p}{p!}. \end{aligned} \tag{5.1}$$

and on the other, we can write

$$\begin{aligned} g(\tau) &= \left(\sum_{p=0}^{\infty} {}_LG_{p,\lambda}(u\alpha, v\alpha)\frac{(\beta\tau)^p}{p!}\right)\left(\sum_{q=0}^{\infty} {}_LG_{q,\lambda}(u\beta, v\beta)\frac{(\alpha\tau)^q}{q!}\right) \\ &= \sum_{p=0}^{\infty}\left(\sum_{q=0}^{p}\binom{p}{q}\alpha^q\beta^{p-q}\ {}_LG_{p-q,\lambda}(u\alpha, v\alpha)\ {}_LG_{q,\lambda}(u\beta, v\beta)\right)\frac{\tau^p}{p!}. \end{aligned} \tag{5.2}$$

Finally, the result easily follows by equating the coefficients of τ^p on the right-hand side of Eqs (5.1) and (5.2). □

Theorem 17. *Let $\alpha,\beta \in \mathbb{Z}$ with $p \in \mathbb{N}_0$, Then,*

$$\sum_{q=0}^{p}\binom{p}{q}\beta^q\alpha^{p-q}\sum_{\sigma=0}^{\alpha-1}\sum_{\rho=0}^{\beta-1}(-1)^{\sigma+\rho}{}_LG_{p-q,\lambda}(u, v\beta + \frac{\beta}{\alpha}\sigma + \rho)G_{q,\lambda}(z\alpha)$$

$$= \sum_{q=0}^{p}\binom{p}{q}\alpha^p\beta^{p-q}\sum_{\sigma=0}^{\beta-1}\sum_{\rho=0}^{\alpha-1}(-1)^{\sigma+\rho}{}_LG_{p-q,\lambda}(u, v\alpha + \frac{\beta}{\alpha}\sigma + \rho)G_{q,\lambda}(z\beta).$$

Proof. Let

$$g(\tau) = \frac{\left\{2\log(1+\lambda)^{\frac{\alpha}{\lambda}}\right\}}{(e^{\alpha\tau}+1)^2}\frac{\left\{2\log(1+\lambda)^{\frac{\beta}{\lambda}}\right\}}{(e^{\beta\tau}+1)^2}e^{(\alpha\beta\tau+1)^2}e^{(\alpha\beta)(v+z)\tau}[Cs_0(u\tau)].$$

Considering $g(\tau)$ in two forms. Firstly,

$$\begin{aligned} g(\tau) = &\frac{\left\{2\log(1+\lambda)^{\frac{\alpha}{\lambda}}\right\}}{e^{\alpha\tau}+1}e^{\alpha\beta v\tau}C_o(u\tau)\left(\frac{e^{\alpha\beta\tau}+1}{e^{\beta\tau}+1}\right) \\ &\times\frac{\left\{2\log(1+\lambda)^{\frac{\beta}{\lambda}}\right\}}{e^{\beta\tau}+1}e^{\alpha\beta z\tau}\left(\frac{e^{\alpha\beta\tau}+1}{e^{\alpha\tau}+1}\right) \end{aligned}$$

$$\begin{aligned} = &\frac{\left\{2\log(1+\lambda)^{\frac{\alpha}{\lambda}}\right\}}{e^{\alpha\tau}+1}e^{\alpha\beta v\tau}C_0(u\tau)\left(\sum_{\sigma=0}^{\alpha-1}(-1)^{\sigma}e^{\beta\tau\sigma}\right) \\ \times &\frac{\left\{2\log(1+\lambda)^{\frac{\beta}{\lambda}}\right\}}{e^{\beta\tau}+1}e^{\alpha\beta\tau z}C_0(u\tau)\left(\sum_{\rho=0}^{\beta-1}(-1)^{\rho}e^{\alpha\tau\rho}\right), \end{aligned} \tag{5.3}$$

Secondly,

$$
\begin{aligned}
& g(\tau) \\
&= \sum_{p=0}^{\infty}\left\{\sum_{q=0}^{p}\binom{p}{q}\beta^{q}\alpha^{p-q}\sum_{\sigma=0}^{\alpha-1}\sum_{\rho=0}^{\beta-1}(-1)^{\sigma+\rho}{}_{L}G_{p-q,\lambda}\left(u\alpha, v\beta+\frac{\beta}{\alpha}\sigma+\rho\right)G_{q,\lambda}(\alpha z)\right\}\frac{\tau^{p}}{p!} \\
&= \sum_{p=0}^{\infty}\left\{\sum_{q=0}^{p}\binom{p}{q}\alpha^{q}\beta^{p-q}\sum_{\sigma=0}^{\alpha-1}\sum_{\rho=0}^{\beta-1}(-1)^{\sigma+\rho}{}_{L}G_{\sigma-\rho,\lambda}\left(u, v\alpha+\frac{\alpha}{\beta}\sigma+\rho\right)G_{q,\lambda}(z\beta)\right\}\frac{\tau^{p}}{p!}.
\end{aligned}
\tag{5.4}
$$

Finally, the result straightforwardly follows by equating the coefficients of τ^p in Eqs (5.3) and (5.4). □

We now give the following two Theorems. We omit their proofs since they follow the same technique as in the Theorems 16 and 17.

Theorem 18. *Let $\alpha,\beta \in \mathbb{Z}$ and $p \in \mathbb{N}_0$, Then,*

$$
\begin{aligned}
& \sum_{q=0}^{p}\binom{p}{q}\beta^{q}\alpha^{p-q}\sum_{\sigma=0}^{\alpha-1}\sum_{\rho=0}^{\beta-1}(-1)^{\sigma+\rho}{}_{L}G_{p-q,\lambda}\left(u, v\beta+\frac{\beta}{\alpha}\sigma\right)G_{q,\lambda}\left(z\alpha+\frac{\alpha}{\beta}\rho\right) \\
= \; & \sum_{q=0}^{p}\binom{p}{q}\alpha^{q}\beta^{p-q}\sum_{\sigma=0}^{\beta-1}\sum_{\rho=0}^{\alpha-1}(-1)^{\sigma+\rho}{}_{L}G_{p-q,\lambda}\left(u, v\alpha+\frac{\alpha}{\beta}\sigma+\rho\right){}_{L}G_{q,\lambda}\left(z\beta+\frac{\beta}{\alpha}\rho\right).
\end{aligned}
$$

Theorem 19. *Let $\alpha,\beta \in \mathbb{Z}$ and $p \in \mathbb{N}_0$, Then,*

$$
\begin{aligned}
& \sum_{q=0}^{p}\binom{p}{q}\beta^{q}\alpha^{p-q}{}_{L}G_{p-q,\lambda}(u\beta, v\beta)\sum_{\sigma=0}^{q}\binom{q}{\sigma}T_{\sigma}(\alpha-1)G_{q-\sigma,\lambda}(u\alpha) \\
= \; & \sum_{q=0}^{p}\binom{p}{q}\beta^{p-q}\alpha^{q}{}_{L}G_{p-q,\lambda}(u\alpha, v\alpha)\sum_{\sigma=0}^{q}\binom{q}{\sigma}T_{\sigma}(\beta-1)G_{q-\sigma,\lambda}(u\beta).
\end{aligned}
$$

6. Concluding remark and observation

Motivated by importance and potential for applications in certain problems in number theory, combinatorics, classical and numerical analysis and other fields of applied mathematics, various special numbers and polynomials, and their variants and generalizations have been extensively investigated (for example, see the references here and those cited therein). The results presented here, being very general, can be specialized to yield a large number of identities involving known or new simpler numbers and polynomials. For example, the case $u = 0$ of the results presented here give the corresponding ones for the generalized partially degenerate Genocchi polynomials [3].

Acknowledgment

The authors express their thanks to the anonymous reviewers for their valuable comments and suggestions, which help to improve the paper in the current form.

Conflict of interest

We declare that we have no conflict of interests.

References

1. G. Dattoli, A. Torre, *Operational methods and two-variable Laguerre polynomials*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur., **132** (1998), 1–7.
2. G. Dattoli, S. Lorenzutta, C. Cesarano, *Finite sums and generalized forms of Bernoulli polynomials*, Rend. Mat. Appl., **19** (1999), 385–391.
3. L. C. Jang, H. I. Kwon, J. G. Lee, et al. *On the generalized partially degenerate Genocchi polynomials*, Global J. Pure Appl. Math., **11** (2015), 4789–4799.
4. N. U. Khan, T. Usman, M. Aman, *Certain generating function of generalized Apostol type Legendre-based polynomials*, Note Mat., **37** (2017), 21–43.
5. N. U. Khan, T. Usman, J. Choi, *A New generalization of Apostol type Laguerre-Genocchi polynomials*, C. R. Math., **355** (2017), 607–617.
6. N. U. Khan, T. Usman, J. Choi, *A new class of generalized polynomials*, Turkish J. Math., **42** (2018), 1366–1379.
7. N. U. Khan, T. Usman, J. Choi, *A new class of generalized Laguerre-Euler polynomials*, RACSAM, **113** (2019), 861–873.
8. S. Khan, M. W. Al-Saad, R. Khan, *Laguerre–based Appell polynomials: Properties and applications*, Math. Comput. Model., **52** (2010), 247–259.
9. D. S. Kim, T. Kim, *Daehee numbers and polynomials*, Appl. Math. Sci., **7** (2013), 5969–5976.
10. D. S. Kim, T. Kim, *Some identities of degenerate special polynomials*, Open Math., **13** (2015), 380–389.
11. D. S. Kim, T. Kim, S. H. Lee, et al. *A note on the lambda-Daehee polynomials*, Int. J. Math. Anal., **7** (2013), 3069–3080.
12. D. S. Kim, S. H. Lee, T. Mansour, et al. *A note on q-Daehee polynomials and numbers*, Adv. Stud. Contemp. Math., **24** (2014), 155–160.
13. T. Kim, J. J. Seo, *A note on partially degenerate Bernoulli numbers and polynomials*, J. Math. Anal., **6** (2015), 1–6.
14. D. Lim, *Degenerate, partially degenerate and totally degenerate Daehee numbers and polynomials*, Adv. Differ. Equ., **2015** (2015), 287.
15. D. Lim, *Some identities of Degenerate Genocchi polynomials*, Bull. Korean Math. Soc., **53** (2016), 569–579.
16. J. W. Park, J. Kwon, *A note on the degenerate high order Daehee polynomials*, Global J. Appl. Math. Sci., **9** (2015), 4635–4642.
17. M. A. Pathan, W. A. Khan, *Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials*, Mediterr. J. Math., **12** (2015), 679–695.

18. F. Qi, D. V. Dolgy, T. Kim, et al. *On the partially degenerate Bernoulli polynomials of the first kind*, Global J. Pure Appl. Math., **11** (2015), 2407–2412.

19. E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960.

20. C. S. Ryoo, T. Kim, J. Choi, et al. *On the generalized q-Genocchi numbers and polynomials of higher-order*, Adv. Differ. Equ., **2011** (2011), 1–8.

21. H. M. Srivastava, H. L. Manocha, *Treatise on Generating Functions*, Ellis Horwood Limited, New York, 1984.

22. W. P. Zang, Z. F. Cao, *Another generalization of Menon's identity*, Int. J. Number Theory, **13** (2017), 2373–2379.