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Research article

Shifted Legendre polynomials-based single and double integral inequalities with arbitrary approximation order: Application to stability of linear systems with time-varying delays

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Abstract: This paper proposes novel single and double integral inequalities with arbitrary approximation order by employing shifted Legendre polynomials and Cholesky decomposition, and these inequalities could significantly reduce the conservativeness in stability analysis of linear systems with interval time-varying delays. The coefficients of the proposed single and double integral inequalities are determined by using the weighted least-squares method. Also former well-known integral inequities, such as Jensen inequality, Wirtinger-based inequality, auxiliary function-based integral inequalities, are all included in the proposed integral inequalities as special cases with lower-order approximation. Stability criterions with less conservatism are then developed for both constant and time-varying delay systems. Several numerical examples are given to demonstrate the effectiveness and benefit of the proposed method.

Keywords: shifted Legendre polynomials; Cholesky decomposition; arbitrary approximation order; time-delayed systems; stability analysis
Mathematics Subject Classification: 93D99

1. Introduction

Time-delay systems exist in many practical situations as industry process, biological, ecological groups, telecommunication, economy, mechanical engineering, and so on. A time-delay in a system often induces oscillation and instability, which motivated a huge number of researchers to study the stability analysis with various criteria [1–3]. Evaluation of system stability with a constant delay has been studied extensively and lots of theoretical tools have been presented like characteristic equation and eigenvalues analysis [4, 5]. Those methods have been well established currently which can derive effective criteria smoothly with numerical efficiency. However, this type of criteria cannot be applied to a time-varying delay system and some other methodologies have been employed.

Generally, two different methodologies have been employed: the first one is so called input-output method that treats a delay as an uncertain operator, and transforms the original time-varying delay system into a closed loop between a nominal LTI system and a perturbation depending on the delay. The stability criteria of which have been well developed by using conventional robustness tools like Small Gain Theorem [6, 7], Integral Quadratic Constraint or Quadratic Separation [8, 9]. The conservativeness is small for a slowly varying delay, but large for a quickly one because it depending on the upper bound on the derivative of the delay. Another technique is based on the proper construction of Lyapunov-Krasovskii functions. The conservativeness of this method comes from two aspects: the choice of functional and the bound on its derivative. It is not easy to find an appropriate Lyapunov-Krasovskii functional (LFK) to obtain less conservative criteria since it contains both the delay and its bounds.

In earlier research, only a single integral term was employed as a part of LFK to analysis and handle the time delay in systems [10–12]. Up to now, double, triple, even quadruple integral terms has been developed which usually bring more effective stability criteria [13–15]. And also an augmented and a delay-partitioning LKF method were proposed to reduce the conservativeness, and the difficulty now lies in the bounds of the integrals that appear in the derivative of the functional for a stability condition [16, 17].

Previously, The Jensen inequality and Wirtinger-based integral inequality were reported as the integral inequality method that yields less conservative stability criteria [2, 18]. Delay-dependent strategy and delay-independent approach under time-varying delays, uncertainties and disturbance are employed to stability analysis. Delay-dependent strategy has been received many attentions as a result of its less conservatism than delay-independent [19–27]. Later, the first- and second-order reciprocally convex approach were proposed based on a new kind of linear combination of positive functions weighted by the inverses of squared convex parameters emerges when the Jensen inequality was applied to partitioned double integral terms in the derivation of LMI conditions [28, 29]. And the optimal divided method and the secondary partitioning method were provided for stability criteria in double integral terms in LPF [30, 31].

Recently, the integral term with higher order approximation has been proposed, such as Wirtinger-based double integral inequality [32], free-matrix-based integral inequality [33], auxiliary function-based integral inequality [34]. These inequalities provided less conservation of stability criteria that those of the Jensen or Wirtinger-based single integral inequities. Especially, a novel integral inequality which called Bessel-Legendre (B-L) inequality has only been applied to the system with constant delays [35–38]. And also multiple-integral inequalities were newly developed to give high-order approximation to the original integral, the associated integral terms in LPF are also increased [39, 40].

In this study, a new single integral inequality is proposed through using shifted Legendre polynomials, and then the double integral inequality is developed with the utilization of Cholesky decomposition. Both single and double integral inequalities are with arbitrary approximation order, which encompasses the well-known Jensen and Wirtinger-based inequalities, auxiliary function-based integral inequalities, and even the B-L inequality. The proposed two inequalities yield improved stability criteria with less conservativeness.

This paper is organized as follows. Section 2 introduces the relevant theories of shifted Legendre polynomials-based single and double integral inequalities, and section 3 and 4 provide application of proposed methods to systems with constant and time-varying delays, including numerical examples.

2. Shifted Legendre polynomials-based single and double integral inequalities

2.1. Shifted Legendre polynomials for single integral

The classical shifted Legendre polynomials are a set of functions analogous to the Legendre polynomials, but defined on the interval [0, 1] as follows

$$p_i(s) = \sum_{j=0}^{i} w_{i,j} s^j, \quad j = 0, 1, \cdots, i \tag{2.1}$$

where $p_i(s)$ denotes the i-order shifted Legendre polynomial, $w_{i,j}$ denotes the jth coefficient of $p_i(s)$.

We here call classical shifted Legendre polynomials as the shifted Legendre polynomials for single integral with the following coefficient

$$w_{i,j} = (-1)^i C_{i+j}^i C_i^j \tag{2.2}$$

where C_i^j denotes the combination which can be written using factorials as

$$C_i^j = \frac{i!}{j!(i-j)!} \tag{2.3}$$

Shifted Legendre polynomials obey the orthogonality relationship, i. e.

$$\begin{aligned} \int_0^1 p_l(s) p_m(s) ds &= \sum_{i=0}^{l} \sum_{j=0}^{m} (-1)^{i+j} C_{l+i}^l C_l^i C_{m+j}^m C_m^j \int_0^1 s^{i+j} ds \\ &= \sum_{i=0}^{l} \sum_{j=0}^{m} (-1)^{i+j} C_{l+i}^l C_l^i C_{m+j}^m C_m^j \frac{1}{i+j+1} \\ &= \frac{1}{2m+1} \delta_{lm} \end{aligned} \tag{2.4}$$

where δ_{nm} denotes the Kronecker delta.

Also we can represent shifted Legendre polynomials for single integral in the matrix form as follows

$$U_m(s) = \begin{bmatrix} 1 \\ s \\ \vdots \\ s^m \end{bmatrix}, \quad L_m(s) = \begin{bmatrix} p_0(s) \\ p_1(s) \\ \vdots \\ p_m(s) \end{bmatrix} \tag{2.5}$$

The relationship between $L_m(s)$ and $U_m(s)$ is obtained

$$L_m(s) = W_m U_m(s) \tag{2.6}$$

where W_m is the coefficient matrix with the following form

$$W_m = \underbrace{[(-1)^j C_{i+j}^i C_i^j]}_{i \geq j} = \begin{bmatrix} 1 & & & \cdots & \\ 1 & -2 & & \cdots & \\ 1 & -6 & 6 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & -m(m+1) & C_{m+2}^m C_m^2 & \cdots & (-1)^m C_{2m}^m \end{bmatrix} \tag{2.7}$$

It's obvious that W_n is a lower triangular matrix.

With similar formulation, (2.4) can be rewritten as

$$G_m = \int_0^1 L_m(s) L_m^{\mathrm{T}}(s) du = \left[g_{ij}\right] = \begin{bmatrix} 1 & & & \cdots & \\ & \frac{1}{3} & & \cdots & \\ & & \frac{1}{5} & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & \frac{1}{2m+1} \end{bmatrix} \tag{2.8}$$

2.2. *Shifted Legendre polynomials for double integral*

The interest of shifted Legendre polynomials for double integral is that the orthogonality relationship exists if we use double integral instead of single integral.

The double integral of the product of two classical shifted Legendre polynomials can be obtained as follows

$$\begin{aligned} h_{lm} &= \int_0^1 \int_s^1 p_l(u) p_m(u) du\mathrm{d}s \\ &= \sum_{i=0}^{l} \sum_{j=0}^{m} (-1)^{i+j} C_{l+i}^l C_l^i C_{m+j}^m C_m^j \int_0^1 \int_s^1 u^{i+j} du\mathrm{d}s \\ &= \sum_{i=0}^{n} \sum_{j=0}^{m} (-1)^{i+j} C_{l+i}^l C_l^i C_{m+j}^m C_m^j \frac{1}{i+j+2} \\ &= \begin{cases} \frac{1}{2(2m+1)}, & l = m \\ -\frac{m}{2(2m-1)(2m+1)}, & l = m-1 \\ -\frac{l}{2(2l-1)(2l+1)}, & l = m+1 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{2.9}$$

which can also be extended using the form of matrix

$$\begin{aligned} H_m &= \int_0^1 \int_s^1 L_m(u) L_m^{\mathrm{T}}(u) duds \\ &= \frac{1}{2} \begin{bmatrix} \frac{1}{1} & -\frac{1}{1\times 3} & & & & \\ -\frac{1}{1\times 3} & \frac{1}{3} & -\frac{2}{3\times 5} & & & \\ & -\frac{2}{3\times 5} & \frac{1}{5} & -\frac{3}{5\times 7} & & \\ & & -\frac{3}{5\times 7} & \ddots & \ddots & \\ & & & \ddots & \frac{1}{2m-1} & -\frac{m}{(2m-1)(2m+1)} \\ & & & & -\frac{m}{(2m-1)(2m+1)} & \frac{1}{2m+1} \end{bmatrix} \end{aligned} \tag{2.10}$$

Considering that H_m is a real-valued symmetric positive semi-definite matrix, we can gain the associated lower triangular matrix using Cholesky decomposition

$$H_m = B_m B_m^{\mathrm{T}} \tag{2.11}$$

where

$$B_m = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & & & & \\ -\frac{1}{3} & \frac{\sqrt{2}}{3} & & & \\ & -\frac{\sqrt{2}}{5} & \frac{\sqrt{3}}{5} & & \\ & & \ddots & \ddots & \\ & & & -\frac{\sqrt{m}}{2m+1} & \frac{\sqrt{m+1}}{2m+1} \end{bmatrix} \tag{2.12}$$

Since $B_m > 0$, H_m has the unique Cholesky decomposition. Unfortunately, (2.10) shows that $L_m(u)$ is not a proper set of basic functions when the double integral is employed instead of single integral. Thus, we need to find new ones. We introduce the linear combination of $\{p_j(s)\}$ as follows

$$\bar{p}_i(s) = \sum_{j=0}^{i} d_{i,j} p_j(s) \tag{2.13}$$

i.e.

$$\bar{L}_m(u) = \begin{bmatrix} \bar{p}_0(u) \\ \bar{p}_1(u) \\ \vdots \\ \bar{p}_m(u) \end{bmatrix} = D_m L_m(u) \tag{2.14}$$

where D_m denotes the transition matrix from $L_m(u)$ to $\bar{L}_m(u)$ with the form

$$D_m = \underbrace{[d_{ij}]}_{i\geq j} = \begin{bmatrix} d_{00} & & \cdots & \\ d_{10} & d_{11} & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ d_{m0} & d_{m1} & \cdots & d_{mm} \end{bmatrix} \tag{2.15}$$

In order to obtain the proper shifted Legendre polynomials for double integral, the following equation should be solved.

$$\bar{H}_m = \int_0^1 \int_s^1 \bar{L}_m(u)\bar{L}_m^{\mathrm{T}}(u) du ds = D_m H_m D_m^{\mathrm{T}} = \begin{bmatrix} \bar{h}_{11} & & & \\ & \bar{h}_{22} & & \\ & & \ddots & \\ & & & \bar{h}_{mm} \end{bmatrix} \tag{2.16}$$

where $\{d_{ij}\}$ and $\{h_{ij}\}$ are coefficients to be determined.

Substituting (2.11) into (2.16) yields

$$D_m V_m = \sqrt{\bar{H}_m} = \begin{bmatrix} \sqrt{\bar{h}_{00}} & & & \\ & \sqrt{\bar{h}_{11}} & & \\ & & \ddots & \\ & & & \sqrt{\bar{h}_{mm}} \end{bmatrix} \tag{2.17}$$

By solving a serial of linear equations of (2.17), the matrices D_m and $\bar{H}_m$ are achieved as following

$$D_m = \underbrace{[d_{ij} = \frac{2j+1}{i+1}]}_{i \geq j} = \begin{bmatrix} 1 & & & \\ \frac{1}{2} & \frac{3}{2} & & \\ \vdots & \vdots & \ddots & \\ \frac{1}{m+1} & \frac{3}{m+1} & \cdots & \frac{2m+1}{m+1} \end{bmatrix} \tag{2.18}$$

$$\bar{H}_m = \underbrace{[\bar{h}_{ii} = \frac{1}{2i+2}]}_{i=j} = \mathrm{diag}\{\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2m+2}\} \tag{2.19}$$

Thus the vector of shifted Legendre polynomials are achieved

$$\bar{L}_m(u) = D_m L_m(u) = D_m W_m U_m(u) = \bar{W}_m U_m(u) \tag{2.20}$$

where, by (2.6),

$$\bar{W}_m = \underbrace{\left[(-1)^j \sum_{k=j}^{i} \frac{2k+1}{i+1} C_{k+j}^{k} C_{k}^{j} \right]}_{i \geq j}$$
$$= \begin{bmatrix} 1 & & & & \\ 2 & -3 & & & \\ 3 & -12 & 10 & & \\ \vdots & \vdots & \vdots & \ddots & \\ m & \sum\limits_{k=1}^{m} \frac{2k+1}{m+1} k(k+1) & \sum\limits_{k=2}^{m} \frac{2k+1}{m+1} C_{k+2}^{k} C_{k}^{2} & \cdots & (-1)^m \frac{2m+1}{m+1} C_{2m}^{m} \end{bmatrix} \tag{2.21}$$

2.3. Shifted Legendre polynomials-based single integral inequality

For continuously vector function $\dot{x}(\tau) : [a,b] \rightarrow \mathbf{R}^n$, the associated function $\dot{\tilde{x}}(s) : [0,1] \rightarrow \mathbf{R}^n$ is defined as follows

$$\dot{\tilde{x}}(s) = \dot{x}(\tau) = \dot{x}((b-a)s + a) \tag{2.22}$$

where $\tau = (b-a)s + a$.

We can develop the relationships between the single integrals of $\dot{x}(\tau)$ and $\dot{\tilde{x}}(s)$

$$(b-a) \int_0^1 s^k \dot{\tilde{x}}(s) ds = \frac{1}{(b-a)^k} \int_a^b (\tau - a)^k \dot{x}(\tau) d\tau, \quad k = 0,1,2,\cdots \tag{2.23}$$

The best weighted square approximation can be obtained with minimizing the following cost function

$$
\begin{aligned}
J_s &= \int_a^b (f(\tau) - \dot{x}(\tau))^{\mathrm{T}} R (f(\tau) - \dot{x}(\tau)) d\tau \\
&= (b-a) \int_0^1 (\tilde{f}(s) - \dot{\tilde{x}}(s))^{\mathrm{T}} R (\tilde{f}(s) - \dot{\tilde{x}}(s)) ds
\end{aligned} \tag{2.24}
$$

where $R > 0$ denotes a symmetric positive-defined matrix with proper dimensions, $\tilde{f}(s)$ denotes the approximation function defined as follows

$$
\tilde{f}(s) = \sum_{i=0}^{m} \beta_i p_i(s) \tag{2.25}
$$

where $\beta_i \in \mathbf{R}^n$ denotes the weight corresponding to the shifted Legendre polynomial $p_i(s)$ for single integral.

Substituting (2.25) into (2.24) yields

$$
\begin{aligned}
J_s &= (b-a) \int_0^1 (\sum_{i=0}^{m} \beta_i p_i(s) - \dot{\tilde{x}}(s))^{\mathrm{T}} R (\sum_{i=0}^{m} \beta_i p_i(s) - \dot{\tilde{x}}(s)) ds \\
&= (b-a) \begin{bmatrix} \sum_{i=0}^{m} \sum_{j=0}^{m} \beta_i^{\mathrm{T}} R \beta_j \int_0^1 p_i(s) p_j(s) \mathrm{d}s \\ -\mathrm{sym}\left(\sum_{j=0}^{m} \beta_i^{\mathrm{T}} R \int_0^1 \dot{\tilde{x}}(s) p_i(s) \mathrm{d}s \right) \end{bmatrix} + \int_a^b \dot{x}^{\mathrm{T}}(\tau) R \dot{x}(\tau) \mathrm{d}\tau \\
&= (b-a) \sum_{i=0}^{m} \frac{1}{2i+1} \beta_i^{\mathrm{T}} R \beta_i - \sum_{i=0}^{m} \mathrm{sym}(\beta_i^{\mathrm{T}} R \omega_i) + \int_a^b \dot{x}^{\mathrm{T}}(\tau) R \dot{x}(\tau) \mathrm{d}\tau
\end{aligned} \tag{2.26}
$$

where ω_i denotes the integral of the product of $\dot{\tilde{x}}(s)$ and the i-th shifted Legendre polynomial $p_i(s)$ for single integral. sym() is defined as the sum of vector/matrix with its own transpose $\mathrm{sym}(x) = x + x^{\mathrm{T}}$.

$$
\omega_i = (b-a) \int_0^1 \dot{\tilde{x}}(s) p_i(s) \mathrm{d}s \tag{2.27}
$$

i.e.

$$
\varpi_m = \begin{bmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_m \end{bmatrix} = (b-a) \begin{bmatrix} \int_0^1 \dot{\tilde{x}}(s) p_0(s) \mathrm{d}s \\ \int_0^1 \dot{\tilde{x}}(s) p_1(s) \mathrm{d}s \\ \vdots \\ \int_0^1 \dot{\tilde{x}}(s) p_m(s) \mathrm{d}s \end{bmatrix} = (b-a) \widehat{W}_m \begin{bmatrix} \int_0^1 \dot{\tilde{x}}(s) \mathrm{d}s \\ \int_0^1 \dot{\tilde{x}}(s) s \mathrm{d}s \\ \vdots \\ \int_0^1 \dot{\tilde{x}}(s) s^m \mathrm{d}s \end{bmatrix} \tag{2.28}
$$

where $\widehat{W}_m$ denotes the extension matrix associated to W_m

$$
\widehat{W}_m = \underbrace{[(-1)^j C_{i+j}^i C_i^j I]}_{i \geq j} = \begin{bmatrix} I & & & \cdots & \\ I & -2I & & \cdots & \\ I & -6I & 6I & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \\ I & -m(m+1)I & C_{m+2}^m C_m^2 I & \cdots & (-1)^m C_{2m}^m I \end{bmatrix} \tag{2.29}
$$

where I denotes the identity matrix with proper dimensions.

Substituting (2.23) into (2.28) yields

$$\varpi_m = \begin{bmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_m \end{bmatrix} = \widehat{W}_m \begin{bmatrix} \int_a^b \dot{x}(\tau)\mathrm{d}\tau \\ \frac{1}{b-a}\int_a^b (\tau-a)\dot{x}(\tau)\mathrm{d}\tau \\ \vdots \\ \frac{1}{(b-a)^m}\int_a^b (\tau-a)^m\dot{x}(\tau)\mathrm{d}\tau \end{bmatrix} \tag{2.30}$$

According to the static condition of (2.26), we obtain

$$\frac{\partial J_s}{\partial \beta_i} = (R+R^{\mathrm{T}})\left(\frac{b-a}{2i+1}\beta_i - \omega_i\right) = 0 \tag{2.31}$$

The second condition of (2.26)

$$[\frac{\partial^2 J_s}{\partial \beta_i \partial \beta_j}] = \frac{b-a}{2i+1}(R+R^{\mathrm{T}})\delta_{ij} > 0 \tag{2.32}$$

It means that the optimal $\beta_i^* = (2i+1)\omega_i/(b-a)$ leads to the only minimum cost value

$$L_s \geq L_s^* = \int_a^b \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s - \frac{1}{b-a}\sum_{i=0}^{m}\omega_i^{\mathrm{T}}[(2i+1)R]\omega_i > 0 \tag{2.33}$$

Lemma 1 (shifted Legendre polynomials-based single integral inequality): For any symmetric positive-defined constant matrix $R \in \mathbf{R}^{n\times n}$, $R > 0$, and vector function $\dot{x}(t) : [a,b] \rightarrow \mathbf{R}^n$ such that the integrations concerned are well defined, then the following inequality exists

$$\begin{aligned} \int_a^b \dot{x}^{\mathrm{T}}(\tau)R\dot{x}(\tau)\mathrm{d}\tau &\geq \frac{1}{b-a}\varpi_m^{\mathrm{T}}\Omega_m(R)\varpi_m \\ &= \frac{1}{b-a}\begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R & 0 & 0 & \cdots & 0 \\ 0 & 3R & 0 & \cdots & 0 \\ 0 & 0 & 5R & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (2m+1)R \end{bmatrix} \begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix} \end{aligned} \tag{2.34}$$

Proof: It can be obtained from (2.33) observably.

Remark 1: The right term of the proposed single integral inequality (2.34) is approximation with arbitrary order to the left term, i.e., when $\dot{x}(t) = c_0 + c_1 t + \cdots + c_m t^m$, $c_i \in \mathbf{R}^n$, $i = 0,1,\cdots,m$, the left term is exactly equal to the right term.

Proof: The function $\dot{x}(t) = c_0 + c_1 t + \cdots + c_m t^m$ can be rewritten as

$$\begin{aligned} \dot{x}((b-a)s+a) &= c_0 + c_1[(b-a)s+a] + \cdots + c_m[(b-a)s+a]^m \\ &= \tilde{c}_0 + \tilde{c}_1 s + \cdots + \tilde{c}_m s^m \\ &= \dot{\tilde{x}}(s) \end{aligned} \tag{2.35}$$

where

$$\tilde{c}_k = (b-a)^k \sum_{i=k}^{m} a^{k-i} C_k^i \tag{2.36}$$

$\dot{\tilde{x}}(s)$ can also be expressed by serial of shifted Legendre polynomials $\{p_k(s)\}$ as follows

$$\dot{\tilde{x}}(s) = \lambda_0 p_0(s) + \lambda_1 p_1(s) + \cdots + \lambda_m p_m(s) \tag{2.37}$$

where

$$\lambda_i = \frac{\int_0^1 \dot{\tilde{x}}(s) p_i(s) \mathrm{d}s}{\int_0^1 p_i(s) p_i(s) \mathrm{d}s} = \frac{2i+1}{b-a}\omega_i \tag{2.38}$$

Thus the left term of (2.34) becomes

$$\begin{aligned}
\int_a^b \dot{x}^T(\tau) R \dot{x}(\tau) \mathrm{d}\tau &= (b-a) \int_0^1 \left(\sum_{i=0}^{m} \lambda_i p_i(s) \right) R \left(\sum_{i=0}^{m} \lambda_i p_i(s) \right) \mathrm{d}s \\
&= (b-a) \sum_{i=0}^{m} \sum_{j=0}^{m} \lambda_i^{\mathrm{T}} R \lambda_j \int_0^1 p_i(s) p_j(s) \mathrm{d}s \\
&= (b-a) \sum_{i=0}^{m} \frac{1}{2i+1} \lambda_i^{\mathrm{T}} R \lambda_i \\
&= (b-a) \sum_{i=0}^{m} \frac{1}{2i+1} \left(\frac{2i+1}{b-a}\omega_i \right)^{\mathrm{T}} R \left(\frac{2i+1}{b-a}\omega_i \right)^{\mathrm{T}} \\
&= \frac{1}{b-a} \sum_{i=0}^{m} (2i+1) \omega_i^{\mathrm{T}} R \omega_i
\end{aligned} \tag{2.39}$$

This complete the proof.

Remark 2: The integral inequality (2.34) degenerates to Jensen inequality when $m = 0$ [2].
Proof: Substituting $m = 0$ into (2.34) yields

$$\begin{aligned}
\int_a^b \dot{x}^{\mathrm{T}}(\tau) R \dot{x}(\tau) \mathrm{d}\tau \geq \frac{1}{b-a} \varpi^{\mathrm{T}} \Omega \varpi &= \frac{1}{b-a} \omega_0^{\mathrm{T}} R \omega_0 \\
&= \frac{1}{b-a} \left(\int_a^b \dot{x}(\tau) \mathrm{d}\tau \right)^{\mathrm{T}} R \left(\int_a^b \dot{x}(\tau) \mathrm{d}\tau \right) \\
&= \frac{1}{b-a} (x(b) - x(a))^{\mathrm{T}} R (x(b) - x(a))
\end{aligned} \tag{2.40}$$

This complete the proof.

Remark 3: The integral inequality (2.34) degenerates to Wirtinger-based inequality when $m = 1$ [18].
Proof: According to (2.30) we have

$$\omega_0 = \int_a^b \dot{x}(\tau) \mathrm{d}\tau = x(b) - x(a) = \omega_{\mathrm{Wirtinger},0} \tag{2.41}$$

$$\begin{aligned}\omega_1 &= \int_a^b \dot{x}(\tau)\mathrm{d}\tau - \frac{2}{b-a}\int_a^b (\tau-a)\dot{x}(\tau)\mathrm{d}\tau \\ &= x(b)-x(a)-\frac{2}{b-a}\left[(b-a)x(b)-\int_a^b x(\tau)\mathrm{d}\tau\right] \\ &= -\left[x(a)+x(b)-\frac{2}{b-a}\int_a^b x(\tau)\mathrm{d}\tau\right] \\ &= -\omega_{\text{Wirtinger},1}\end{aligned} \tag{2.42}$$

Substituting (2.41) and (2.42) into (2.34) yields

$$\begin{aligned}\int_a^b \dot{x}^{\mathrm{T}}(\tau)R\dot{x}(\tau)\mathrm{d}\tau &\geq \frac{1}{b-a}\begin{bmatrix}\omega_0\\ \omega_1\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}R & \\ & 3R\end{bmatrix}\begin{bmatrix}\omega_0\\ \omega_1\end{bmatrix} \\ &= \frac{1}{b-a}\begin{bmatrix}\omega_{\text{Wirtinger},0}\\ \omega_{\text{Wirtinger},1}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}R & \\ & 3R\end{bmatrix}\begin{bmatrix}\omega_{\text{Wirtinger},0}\\ \omega_{\text{Wirtinger},1}\end{bmatrix}\end{aligned} \tag{2.43}$$

This complete the proof.

2.4. Shifted Legendre polynomials-based double integral inequality

For continuously vector function $\dot{x}(\tau) : [a,b] \to \mathbf{R}^n$, and it's associated function $\dot{\tilde{x}}(s) : [0,1] \to \mathbf{R}^n$ defined in (2.22), we can develop the relationships between the double integrals of $\dot{x}(\tau)$ and $\dot{\tilde{x}}(s)$ as follows

$$\begin{gathered}(b-a)^2 \int_0^1 \int_s^1 u^k \dot{\tilde{x}}(u)duds = \tfrac{1}{(b-a)^k}\int_a^b \int_\theta^b (\tau-a)^k \dot{x}(\tau)d\tau\mathrm{d}\theta \\ k = 0,1,2,\cdots\end{gathered} \tag{2.44}$$

where

$$u = \frac{\tau-a}{b-a}, \quad s = \frac{\theta-a}{b-a}$$

The best weighted square approximation with double integral can be obtained with minimizing the following cost function

$$\begin{aligned}J_d &= \int_a^b \int_\theta^b (g(\tau)-\dot{x}(\tau))^{\mathrm{T}}R(g(\tau)-\dot{x}(\tau))d\tau\mathrm{d}\theta \\ &= (b-a)^2\int_0^1\int_s^1 (\tilde{g}(u)-\dot{\tilde{x}}(u))^{\mathrm{T}}R(\tilde{g}(u)-\dot{\tilde{x}}(u))duds\end{aligned} \tag{2.45}$$

where $R > 0$ denotes a positive-defined matrix with proper dimensions, $\tilde{g}(u)$ denotes the approximation function defined as follows

$$\tilde{g}(u) = \sum_{i=0}^{m} \beta_i \bar{p}_i(s) \tag{2.46}$$

where $\beta_i \in \mathbf{R}^n$ denotes the weight corresponding to the shifted Legendre polynomial $\bar{p}_i(s)$ for double integral .

Substituting (2.46) into (2.45) yields

$$
\begin{aligned}
J_d &= (b-a)^2 \int_0^1 \int_s^1 \left(\sum_{i=0}^{m} \beta_i \bar{p}_i(u) - \dot{\tilde{x}}(u) \right)^{\mathrm{T}} R \left(\sum_{i=0}^{m} \beta_i \bar{p}_i(u) - \dot{\tilde{x}}(u) \right) du ds \\
&= (b-a)^2 \begin{bmatrix} \sum_{i=0}^{m} \sum_{j=0}^{m} \beta_i^{\mathrm{T}} R \beta_j \int_0^1 \int_s^1 \bar{p}_i(s) \bar{p}_j(s) du \mathrm{d}s \\ -\mathrm{sym}\left(\sum_{j=0}^{m} \beta_i^{\mathrm{T}} R \int_0^1 \int_s^1 \dot{\tilde{x}}(s) \bar{p}_j(s) du \mathrm{d}s \right) \end{bmatrix} + \int_a^b \int_\theta^b \dot{x}^{\mathrm{T}}(\tau) R \dot{x}(\tau) \mathrm{d}\tau \mathrm{d}\theta \\
&= (b-a)^2 \sum_{i=0}^{m} \frac{1}{2i+2} \beta_i^{\mathrm{T}} R \beta_i - (b-a) \sum_{i=0}^{m} \mathrm{sym}(\beta_i^{\mathrm{T}} R \nu_i) + \int_a^b \int_\theta^b \dot{x}^{\mathrm{T}}(\tau) R \dot{x}(\tau) \mathrm{d}\tau \mathrm{d}\theta
\end{aligned}
\tag{2.47}
$$

where ν_i denotes the integral of the product of $\dot{\tilde{x}}(s)$ and the i-th shifted Legendre polynomial $p_i(s)$ for single integral

$$
\nu_i = (b-a) \int_0^1 \int_s^1 \dot{\tilde{x}}(s) \bar{p}_i(u) du \mathrm{d}s \tag{2.48}
$$

i.e.

$$
\bar{\nu}_m = \begin{bmatrix} \nu_0 \\ \nu_1 \\ \vdots \\ \nu_m \end{bmatrix} = (b-a) \begin{bmatrix} \int_0^1 \int_s^1 \dot{\tilde{x}}(s) \bar{p}_0(u) du \mathrm{d}s \\ \int_0^1 \int_s^1 \dot{\tilde{x}}(s) \bar{p}_1(u) du \mathrm{d}s \\ \vdots \\ \int_0^1 \int_s^1 \dot{\tilde{x}}(s) \bar{p}_m(u) du \mathrm{d}s \end{bmatrix} = (b-a) \widehat{\bar{W}}_m \begin{bmatrix} \int_0^1 \int_s^1 \dot{\tilde{x}}(s) du \mathrm{d}s \\ \int_0^1 \int_s^1 \dot{\tilde{x}}(s) du \mathrm{d}s \\ \vdots \\ \int_0^1 \int_s^1 \dot{\tilde{x}}(s) u^m du \mathrm{d}s \end{bmatrix} \tag{2.49}
$$

where $\widehat{\bar{W}}_m$ denotes the extension matrix associated to $\bar{W}_m$

$$
\begin{aligned}
\widehat{\bar{W}}_m &= \underbrace{[(-1)^j \sum_{k=j}^{i} \frac{2k+1}{i+1} C_{k+j}^{k} C_k^j I]}_{i \geq j} \\
&= \begin{bmatrix} I & & & & \\ 2I & -3I & & & \\ 3I & -12I & 10 & & \\ \vdots & \vdots & \vdots & \ddots & \\ mI & \sum_{k=1}^{m} \frac{2k+1}{m+1} k(k+1) I & \sum_{k=2}^{m} \frac{2k+1}{m+1} C_{k+2}^{k} C_k^2 I & \cdots & (-1)^m \frac{2m+1}{m+1} C_{2m}^{m} I \end{bmatrix}
\end{aligned}
\tag{2.50}
$$

Substituting (2.44) into (2.49) yields

$$
\bar{\nu}_m = \begin{bmatrix} \nu_0 \\ \nu_1 \\ \vdots \\ \nu_m \end{bmatrix} = \widehat{\bar{W}}_m \begin{bmatrix} \frac{1}{b-a} \int_a^b \int_\theta^b \dot{x}(\tau) d\tau \mathrm{d}\theta \\ \frac{1}{(b-a)^2} \int_a^b \int_\theta^b (\tau-a) \dot{x}(\tau) d\tau \mathrm{d}\theta \\ \vdots \\ \frac{1}{(b-a)^{m+1}} \int_a^b \int_\theta^b (\tau-a)^m \dot{x}(\tau) d\tau \mathrm{d}\theta \end{bmatrix} = \widehat{\bar{W}}_m \begin{bmatrix} \frac{1}{b-a} \int_a^b (\tau-a) \dot{x}(\tau) d\tau \\ \frac{1}{(b-a)^2} \int_a^b (\tau-a)^2 \dot{x}(\tau) d\tau \\ \vdots \\ \frac{1}{(b-a)^{m+1}} \int_a^b (\tau-a)^{m+1} \dot{x}(\tau) d\tau \end{bmatrix} \tag{2.51}
$$

According to the static condition of (2.47), we obtain

$$\frac{\partial J_d}{\partial \beta_i} = (R + R^{\mathrm{T}})\left[\frac{(b-a)^2}{2i+2}\beta_i - (b-a)\nu_i\right] = 0 \tag{2.52}$$

The second condition of (2.47)

$$[\frac{\partial^2 J_d}{\partial \beta_i \partial \beta_j}] = \frac{(b-a)^2}{2i+2}(R + R^{\mathrm{T}})\delta_{ij} > 0 \tag{2.53}$$

It means that the optimal $\beta_i^* = \frac{2i+2}{b-a}\nu_i$ leads to the only minimum cost value

$$L_d \geq L_d^* = \int_a^b \int_\theta^b \dot{x}^{\mathrm{T}}(\tau)R\dot{x}(\tau)\mathrm{d}\tau\mathrm{d}\theta - \sum_{i=0}^{m} \nu_i^{\mathrm{T}}[(2i+2)R]\nu_i > 0 \tag{2.54}$$

Lemma 2 (shifted Legendre polynomials-based double integral inequality): For any positive-defined constant matrix $R \in \mathbf{R}^{n\times n}$, $R > 0$, and vector function $\dot{x}(t) : [a,b] \to \mathbf{R}^n$ such that the integrations concerned are well defined, then the following inequality exists

$$\int_a^b \int_\theta^b \dot{x}^{\mathrm{T}}(\tau)R\dot{x}(\tau)\mathrm{d}\tau\mathrm{d}\theta \geq \bar{\nu}_m^T\bar{\Omega}_m(R)\bar{\nu}_m = \begin{bmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \\ \vdots \\ \nu_m \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 2R & 0 & 0 & \cdots & 0 \\ 0 & 4R & 0 & \cdots & 0 \\ 0 & 0 & 6R & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (2m+2)R \end{bmatrix} \begin{bmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \\ \vdots \\ \nu_m \end{bmatrix} \tag{2.55}$$

Proof: It can be obtained from (2.54) observably.

Remark 1: The right term of the proposed single integral inequality (2.34) is approximation with arbitrary order to the left term, i.e., when $\dot{x}(t) = c_0 + c_1 t + \cdots + c_m t^m$, $c_i \in \mathbf{R}^n$, $i = 0,1,\cdots,m$, the left term is exactly equal to the right term.
Proof: The function $\dot{x}(t) = c_0 + c_1 t + \cdots + c_m t^m$ can be rewritten as

$$\begin{aligned} \dot{x}((b-a)s + a) &= c_0 + c_1[(b-a)s + a] + \cdots + c_m[(b-a)s + a]^m \\ &= \tilde{c}_0 + \tilde{c}_1 s + \cdots + \tilde{c}_m s^m \\ &= \dot{\tilde{x}}(s) \end{aligned} \tag{2.56}$$

where

$$\tilde{c}_k = (b-a)^k \sum_{i=k}^{m} a^{k-i} C_k^i \tag{2.57}$$

$\dot{\tilde{x}}(s)$ can also be expressed by serial of shifted Legendre polynomials $\{\bar{p}_k(s)\}$ as follows

$$\dot{\tilde{x}}(s) = \lambda_0\bar{p}_0(s) + \lambda_1\bar{p}_1(s) + \cdots + \lambda_m\bar{p}_m(s) \tag{2.58}$$

where

$$\lambda_i = \frac{\int_0^1 \int_s^1 \dot{\bar{x}}(s)\bar{p}_i(u)du\mathrm{d}s}{\int_0^1 \int_s^1 \bar{p}_i\bar{p}_i(u)du\mathrm{d}s} = \frac{2i+2}{b-a}\nu_i \tag{2.59}$$

Thus the left term of (2.34) becomes

$$\begin{aligned}\int_a^b \int_\theta^b \dot{x}^{\mathrm{T}}(\tau)R\dot{x}(\tau)\mathrm{d}\tau\mathrm{d}\theta &= (b-a)^2 \int_0^1 \int_s^1 \dot{\bar{x}}(u)^{\mathrm{T}}R\dot{\bar{x}}(s)duds \\ &= (b-a)^2 \sum_{i=0}^{m}\sum_{j=0}^{m}\left(\frac{2i+2}{b-a}\nu_i\right)^{\mathrm{T}} R\left(\frac{2j+2}{b-a}\nu_j\right)\int_0^1\int_s^1 \bar{p}_i(s)\bar{p}_j(s)du\mathrm{d}s \\ &= \sum_{i=0}^{m}\nu_i^{\mathrm{T}}[(2i+2)R]\nu_i\end{aligned} \tag{2.60}$$

This complete the proof.

Remark 2: The integral inequality (2.34) degenerates to auxiliary function-based integral inequality when $m = 1$ [34].
Proof: According to (2.51) we have

$$\nu_0 = \frac{1}{b-a}\int_a^b \int_\theta^b \dot{x}(\tau)d\tau\mathrm{d}\theta = x(b) - \frac{1}{b-a}\int_a^b x(\tau)\mathrm{d}\tau \tag{2.61}$$

$$\begin{aligned}\nu_1 &= \frac{2}{b-a}\int_a^b \int_\theta^b \dot{x}(\tau)d\tau\mathrm{d}\theta - \frac{2}{(b-a)^2}\int_a^b \int_\theta^b (\tau-a)\dot{x}(\tau)d\tau\mathrm{d}\theta \\ &= 2\left[x(b) - \frac{1}{b-a}\int_a^b x(\tau)\mathrm{d}\tau\right] - 3\left[x(b) - \frac{2}{(b-a)^2}\int_a^b \int_\theta^b x(\tau)d\tau\mathrm{d}\theta\right] \\ &= -x(b) - \frac{2}{b-a}\int_a^b x(\tau)\mathrm{d}\tau + \frac{6}{(b-a)^2}\int_a^b \int_\theta^b x(\tau)d\tau\mathrm{d}\theta\end{aligned} \tag{2.62}$$

Note that ν_0 and ν_1 are just the coefficients of auxiliary function-based integral inequality. This complete the proof.

3. Applications to systems with constant delays

3.1. Systems with constant delays

Let us consider the following linear system with constant delay interval

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_h x(t-h) \\ x(t) &= \varphi(t), \quad t \in [-h,0]\end{aligned} \tag{3.1}$$

where $x(t) \in \mathbf{R}^n$ denotes the state vector of the system with n dimensions, A and A_h are real known constant matrices with appropriate dimensions, the continuously differentiable functions $\varphi(t)$ denote the initial condition, $h \geq 0$ denotes the system's constant delay.

Theorem 1: The system (3.1) is asymptotically stable if there exist matrices $P > 0$, $Q > 0$, $R > 0$ and $S > 0$ such that the following conditions hold [41]:

$$\begin{bmatrix} B^{\mathrm{T}}PC + C^{\mathrm{T}}PB + e_1^{\mathrm{T}}Qe_1 - e_2^{\mathrm{T}}Qe_2 + h^2A_e^{\mathrm{T}}RA_e + \frac{1}{2}h^2A_e^{\mathrm{T}}SA_e \\ -\Psi^{\mathrm{T}}\widehat{W}_m^{\mathrm{T}}\Omega_m(R)\widehat{W}_m\Psi - \bar{\Psi}^{\mathrm{T}}\widehat{\bar{W}}_m^{\mathrm{T}}\bar{\Omega}_m(S)\widehat{\bar{W}}_m\bar{\Psi} \end{bmatrix} < 0 \tag{3.2}$$

where the notations in (3.2) are intermediate variables that defined properly in previous and in the process of proof, which can be found as B in (3.10), C in (3.12), e_1, e_2 in (3.10), h in (3.1), A_e in (3.11), Ψ in (3.13), $\widehat{W}_m$ in (2.29), Ω_m in (2.34), $\bar{\Psi}$ in (3.14), $\widehat{\bar{W}}_m$ in (3.7), $\bar{\Omega}_m$ in (3.18).

Proof: We define a set of functions $\{y_k(t)\}$ as follows

$$\begin{aligned} y_k(t) &\triangleq h\int_0^1 \dot{\bar{x}}(s)u^k \mathrm{d}u = \frac{1}{h^k}\int_{t-h}^{t} \dot{x}(\tau)(\tau - t + h)^k \mathrm{d}\tau \\ k &= 0,1,2,\cdots \end{aligned} \tag{3.3}$$

The time derivatives of $y_k(t)$ can be obtained as follows

$$\begin{aligned} \dot{y}_k(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{h^k}\int_{t-h}^{t} \dot{x}(\tau)(\tau - t + h)^k \mathrm{d}\tau\right] \\ &= \dot{x}(t) - \frac{k}{h^k}\int_{t-h}^{t} \dot{x}(\tau)(\tau - t + h)^{k-1}\mathrm{d}\tau \\ &= \dot{x}(t) - \frac{k}{h}y_{k-1}(t) \\ &= Ax(t) + A_hx(t-h) - \frac{k}{h}y_{k-1}(t) \\ (k &\geq 1) \end{aligned} \tag{3.4}$$

And the initial we have

$$\begin{aligned} y_0(t) &= \int_{t-h}^{t} \dot{x}(\tau)\mathrm{d}\tau = x(t) - x(t-h) \\ \dot{y}_1(t) &= \dot{x}(t) - \frac{1}{h}y_0(t) = (A - \frac{1}{h}I)x(t) + (A_h + \frac{1}{h}I)x(t-h) \end{aligned} \tag{3.5}$$

Let $a = t - h$, $b = t$, we can obtain $\{\omega_k\}$ and $\{\nu_k\}$ for shifted Legendre polynomials-based single and double integral inequalities, respectively

$$\begin{bmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_m \end{bmatrix} = \widehat{W}_m \begin{bmatrix} \int_{t-h}^{t} \dot{x}(\tau)\mathrm{d}\tau \\ \frac{1}{h}\int_{t-h}^{t} \dot{x}(\tau)(\tau - t + h)\mathrm{d}\tau \\ \vdots \\ \frac{1}{h^m}\int_{t-h}^{t} \dot{x}(\tau)(\tau - t + h)^m \mathrm{d}\tau \end{bmatrix} = \widehat{W}_m \begin{bmatrix} y_0(t) \\ y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} \tag{3.6}$$

$$\begin{bmatrix} \nu_0 \\ \nu_1 \\ \vdots \\ \nu_{m-1} \end{bmatrix} = \widehat{\bar{W}}_m \begin{bmatrix} \frac{1}{h}\int_{t-h}^{t} \dot{x}(\tau)(\tau - t + h)\mathrm{d}\tau \\ \frac{1}{h^2}\int_{t-h}^{t} \dot{x}(\tau)(\tau - t + h)^2 \mathrm{d}\tau \\ \vdots \\ \frac{1}{h^m}\int_{t-h}^{t} \dot{x}(\tau)(\tau - t + h)^m \mathrm{d}\tau \end{bmatrix} = \widehat{\bar{W}}_m \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix} \tag{3.7}$$

We define extra-states $\chi(t)$ and $\xi(t)$ as follows

$$\chi(t) = \begin{bmatrix} x(t) \\ \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} \end{bmatrix}, \quad \xi(t) = \begin{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} \end{bmatrix} \tag{3.8}$$

The extra-states $\chi(t)$ can be expressed by $\xi(t)$

$$\chi(t) = B\xi(t) \tag{3.9}$$

where

$$B = \begin{bmatrix} e_1 \\ e_3 \\ e_4 \\ \vdots \\ e_m \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I_n & 0_n \end{bmatrix} & \\ & I_{nm} \end{bmatrix} \tag{3.10}$$

where $e_k = [\, \underbrace{0 \quad 0 \quad 0}_{k-1} \; I \; \underbrace{0 \quad 0 \quad 0}_{m+2-k} \,]$ denotes the k-th row coefficient of $\xi(t)$, I_n and 0_n denote the identity and zeros matrix with dimensions $n \times n$, respectively.

And the system (3.1) can be rewritten as

$$\dot{x}(t) = A_e\xi(t) \tag{3.11}$$

where $A_e = Ae_1 + A_he_2$.

The time derivative of $\chi(t)$ can be obtained as follows

$$\dot{\chi}(t) = C\xi(t) \tag{3.12}$$

where

$$C = \begin{bmatrix} \begin{bmatrix} A & A_h \end{bmatrix} & 0_{n\times nm} \\ M & -\frac{1}{h}\Lambda \end{bmatrix}$$

$$M = \begin{bmatrix} A - \frac{1}{h}I & A_h + \frac{1}{h}I \\ A & A_h \\ A & A_h \\ \vdots & \vdots \\ A & A_h \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0 & & & & \\ 2I & 0 & & & \\ & 3I & 0 & & \\ & & \ddots & \ddots & \\ & & & mI & 0 \end{bmatrix}$$

According to (3.6) and (3.8), we have

$$\begin{bmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_m \end{bmatrix} = \widehat{W}_m \begin{bmatrix} y_0(t) \\ y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} = \widehat{W}_m \begin{bmatrix} x(t) - x(t-h) \\ y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} = \widehat{W}_m \Psi \xi(t) \tag{3.13}$$

where

$$\Psi = \begin{bmatrix} I_n & -I_n & 0 \\ 0 & 0 & I_{nm} \end{bmatrix}$$

With similar method, we have following according to (3.7) and (3.8)

$$\begin{bmatrix} \nu_0 \\ \nu_1 \\ \vdots \\ \nu_{m-1} \end{bmatrix} = \widehat{\bar{W}}_m \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \widehat{\bar{W}}_m \bar{\Psi} \xi(t) \tag{3.14}$$

where $\bar{\Psi} = \begin{bmatrix} 0_{nm\times n} & 0_{nm\times n} & I_{nm} \end{bmatrix}$

In order to analysis the stability of the system (3.1), we consider the following Lyapunov-Krasovskii functional (LKF) candidates

$$V = \begin{bmatrix} \chi(t)^{\mathrm{T}} P \chi(t) + \int_{t-h}^{t} x^{\mathrm{T}}(\tau) Q x(\tau) \mathrm{d}\tau \\ +h \int_{t-h}^{t} \int_{\theta}^{t} \dot{x}^{\mathrm{T}}(\tau) R \dot{x}(\tau) \mathrm{d}\tau \mathrm{d}\theta + \int_{t-h}^{t} \int_{\gamma}^{t} \int_{\theta}^{t} \dot{x}^{\mathrm{T}}(\tau) S \dot{x}(\tau) \mathrm{d}\tau \mathrm{d}\theta \mathrm{d}\gamma \end{bmatrix} \tag{3.15}$$

Taking the time derivative of $V(t)$ yields

$$\begin{aligned} \dot{V}(t) &= \begin{bmatrix} \chi^{\mathrm{T}}(t) P \dot{\chi}(t) + \dot{\chi}^{\mathrm{T}}(t) P \chi(t) \\ +x^{\mathrm{T}}(t) Q x(t) - x^{\mathrm{T}}(t-h) Q x(t-h) \\ +h^2 \dot{x}^{\mathrm{T}}(t) R \dot{x}(t) - h \int_{t-h}^{t} \dot{x}^{\mathrm{T}}(\tau) R \dot{x}(\tau) \mathrm{d}\tau \\ +\frac{h^2}{2} \dot{x}^{\mathrm{T}}(t) S \dot{x}(t) - \int_{t-h}^{t} \int_{\theta}^{t} \dot{x}^{\mathrm{T}}(\tau) S \dot{x}(\tau) \mathrm{d}\tau \mathrm{d}\theta \end{bmatrix} \\ &\leq \xi^{\mathrm{T}}(t) \begin{bmatrix} B^{\mathrm{T}} P C + C^{\mathrm{T}} P B + e_1^{\mathrm{T}} Q e_1 - e_2^{\mathrm{T}} Q e_2 + h^2 A_e^{\mathrm{T}} R A_e + \frac{1}{2} h^2 A_e^{\mathrm{T}} S A_e \\ -\Psi^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_m(R) \widehat{W}_m \Psi - \bar{\Psi}^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_m(S) \widehat{\bar{W}}_m \bar{\Psi} \end{bmatrix} \xi(t) \\ &< 0 \end{aligned} \tag{3.16}$$

Recalling that (2.34) and (2.55), following inequalities are employed to yield the upper bound of $\dot{V}(t)$

$$h \int_{t-h}^{t} \dot{x}^{\mathrm{T}}(\tau) R \dot{x}(\tau) \mathrm{d}\tau \geq \varpi^{\mathrm{T}} \Omega_m(R) \varpi = \xi^{\mathrm{T}}(t) \left(\Psi^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_m(R) \widehat{W}_m \Psi \right) \xi(t) \tag{3.17}$$

$$\int_{t-h}^{t} \int_{\theta}^{t} \dot{x}^{\mathrm{T}}(\tau) S \dot{x}(\tau) \mathrm{d}\tau \mathrm{d}\theta \geq \bar{\nu}^{\mathrm{T}} \bar{\Omega}_m(S) \bar{\nu} = \xi^{\mathrm{T}}(t) \left(\bar{\Psi}^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_m(S) \widehat{\bar{W}}_m \bar{\Psi} \right) \xi(t) \tag{3.18}$$

This complete the proof.

3.2. Examples

Example 1: We consider the well-known delay dependent stable system (3.1) with following coefficient matrices as given in [29]:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_h = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

Using delay sweeping techniques the maximum allowable delay $h_{\max} = 6.1725$ can be obtained. Also many recent papers provide different results using Jensen inequality, Wirtinger-based inequality, and so on. The allowable maximum delays are shown in Table 1. We observe that the upper bounds obtained by our proposed inequalities are significantly better than those in other literatures.

Table 1. The maximum allowable delay.

Theorems	$h_{\max}$	Number of variables
Sun et al. (2010) [24]	4.47	$1.5n^2 + 1.5n$
Park, Ko, and Jeong (2011) [28]	5.02	$18n^2 + 18n$
Ariba, Gouaisbaut, and Johansson (2010) [42]	5.12	$7n^2 + 4n$
Seuret and Gouaisbaut (2013) [18]	6.059	$3n^2 + 2n$
Hien and Trinh (2015) [43]	6.16	$19.5n^2 + 4.5n$
Liu and Seuret (2017) Theorem 1 [38]	6.1664	$79.5n^2 + 4.5n$
Theorem 1 (m=0)	4.472	$1.5n^2 + 1.5n$
Theorem 1 (m=1)	6.059	$3.5n^2 + 2.5n$
Theorem 1 (m=2)	6.167	$6n^2 + 3n$
Theorem 1 (m=3)	6.1719	$9.5n^2 + 3.5n$
Theorem 1 (m=4)	6.1725	$14n^2 + 4n$

Example 2: We consider the dynamics of machining chatter with following coefficient matrices as firstly studied in [36]:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10-K & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, \quad A_h = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where K denotes a parameter.

It's obviously that the system is stable with K less than some upper bound. Here we try to the upper bound in various delays. It's shown that **Lemma 1** and **Lamme 2** yield more stability region than those derived from Jensen and Wirtinger-based Lemma, as illustrated in Figure 1. When the parameter $K \leq 0.295$, the system is still stable even the delay is very large, such as $h = 500$.

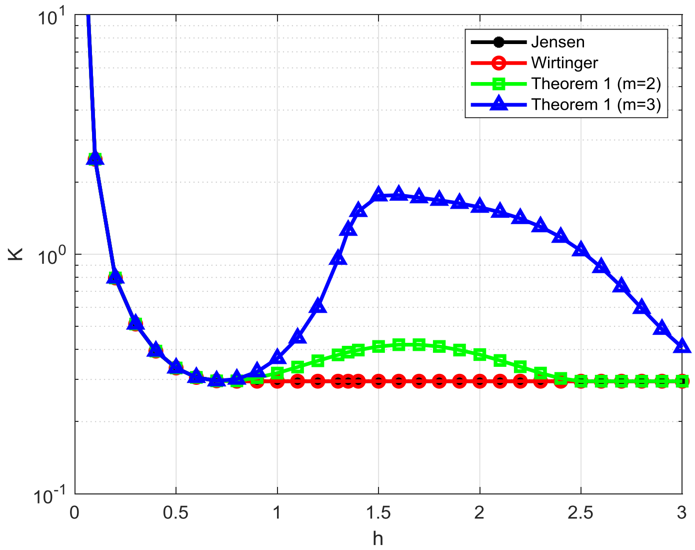

Figure 1. Allowable upper K with variable delay h.

4. Applications to systems with time-varying delays

4.1. Systems with time-varying delays

Let us consider the following system with interval time-varying delay:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_h x(t - h(t)) \\ x(t) &= \varphi(t), \quad t \in [-h_2, 0] \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbf{R}^n$ denotes the state vector of the system with n dimensions, A and A_h are real known constant matrices with appropriate dimensions, the continuously differentiable functions $h(t)$ and $\varphi(t)$ denote the system's time-varying delay and the initial condition, respectively.

Assumption 1: The delay function $h(t)$ and its differential $\dot{h}(t)$ both have finite bounds, i.e., there exist scales $h_2 \geq h_1 > 0$ and $\mu_1 \leq \mu_2 \leq 1$ such that

$$\begin{cases} 0 < h_1 \leq h(t) \leq h_2 \\ \mu_1 \leq \dot{h}(t) \leq \mu_2 \leq 1 \end{cases} \tag{4.2}$$

Theorem 2: The system (4.1) is asymptotically stable if there exist matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $R_1 > 0$, $R_2 > 0$, $R_3 > 0$, and $S_1 > 0$, $S_2 > 0$, $S_3 > 0$ such that the following conditions hold [41]:

$$\Phi = \begin{bmatrix} B_2^{\mathrm{T}} P C_2 + C_2^{\mathrm{T}} P B_2 + e_1^{\mathrm{T}}(Q_1 + Q_2 + Q_3)e_1 - e_3^{\mathrm{T}} Q_1 e_3 - e_4^{\mathrm{T}} Q_2 e_4 - (1-\mu_2) e_2^{\mathrm{T}} Q_3 e_2 \\ + h_1 A_e^{\mathrm{T}} R_1 A_e - \frac{1}{h_1} \Psi_1^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_1(R_1) \widehat{W}_m \Psi_1 + h_2 A_e^{\mathrm{T}} R_2 A_e - \frac{1}{h_2} \Psi_2^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_2(R_2) \widehat{W}_m \Psi_2 \\ + h_2 A_e^{\mathrm{T}} R_3 A_e - \frac{(1-\mu_2)}{h_2} \Psi_3^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_3(R_3) \widehat{W}_m \Psi_3 + \frac{h_1^2}{2} A_e^{\mathrm{T}} S_1 A_e - \bar{\Psi}_1^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_1(S_1) \widehat{\bar{W}}_m \bar{\Psi}_1 \\ + \frac{h_2^2}{2} A_e^{\mathrm{T}} S_2 A_e - \bar{\Psi}_2^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_2(S_2) \widehat{\bar{W}}_m \bar{\Psi}_2 + \frac{h_2^2}{2} A_e^{\mathrm{T}} S_3 A_e - (1-\mu_2) \bar{\Psi}_3^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_3(S_3) \widehat{\bar{W}}_m \bar{\Psi}_3 \end{bmatrix} < 0 \tag{4.3}$$

where the notations in (4.2) are intermediate variables that defined properly in previous and in the process of proof, which can be found as B_2 in (4.8), C_2 in (4.11), e_1, e_2, e_3, e_4 in (3.10), h_1, h_2 in (4.6), μ_2 in (4.16), A_e in (3.11), Ψ_1, Ψ_2, Ψ_3 in (4.14), $\bar{\Psi}_1, \bar{\Psi}_2, \bar{\Psi}_3$ in (4.14), W_m in (2.7), $\widehat{W}_m$ in (2.29), $\widehat{\bar{W}}_m$ in (3.7), $\Omega_1, \Omega_2, \Omega_3$ in (4.13), $\bar{\Omega}_m$ in (3.18), $\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3$ in (4.13).

Proof: If the delay h is varying with time t, then we can develop from (3.3)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} y_k(t) &= \frac{\partial y_k(t)}{\partial t} + \frac{\partial y_k(t)}{\partial h}\frac{\partial h}{\partial t} \\ &= \dot{x}(t) - \frac{k}{h} y_{k-1}(t) - \frac{k\dot{h}}{h}\left(y_k(t) - y_{k-1}(t)\right) \\ &= \dot{x}(t) - \frac{(1-\dot{h})k}{h} y_{k-1}(t) - \frac{\dot{h}k}{h} y_k(t) \end{aligned} \tag{4.4}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} y_1(t) &= \dot{x}(t) - \frac{(1-\dot{h})}{h} y_0(t) - \frac{\dot{h}k}{h} y_1(t) \\ &= \left[A - \frac{(1-\dot{h})}{h} I\right] x(t) + \left[A_h + \frac{(1-\dot{h})}{h} I\right] x(t-h) - \frac{\dot{h}k}{h} y_1(t) \end{aligned} \tag{4.5}$$

If $h = h_1$ or $h = h_2$ is a constant variable, (3.3) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \hat{y}_k(h_i, t) &= \dot{x}(t) - \frac{k}{h_i} \hat{y}_{k-1}(h_i, t) \\ &= Ax(t) + A_h(t-h) - \frac{k}{h_i} \hat{y}_{k-1}(h_i, t) \\ \frac{\mathrm{d}}{\mathrm{d}t} \hat{y}_1(h_i, t) &= \dot{x}(t) - \frac{1}{h_i} \hat{y}_0(h_i, t) \\ &= \left(A - \frac{1}{h_i} I\right) x(t) + A_h x(t-h) + \frac{1}{h_i} x(t-h_i) \\ (i &= 1,2) \end{aligned} \tag{4.6}$$

We introduce the following extra-states $\hat{\chi}_m(t)$ and $\hat{\xi}_m(t)$ as follows

$$\hat{\chi}_m(t) = \begin{bmatrix} x(t) \\ \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} \\ \begin{bmatrix} \hat{y}_1(h_1, t) \\ \vdots \\ \hat{y}_m(h_1, t) \end{bmatrix} \\ \begin{bmatrix} \hat{y}_1(h_2, t) \\ \vdots \\ \hat{y}_m(h_2, t) \end{bmatrix} \end{bmatrix}, \quad \hat{\xi}_m(t) = \begin{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \\ x(t-h_1) \\ x(t-h_2) \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} \\ \begin{bmatrix} \hat{y}_1(h_1, t) \\ \vdots \\ \hat{y}_m(h_1, t) \end{bmatrix} \\ \begin{bmatrix} \hat{y}_1(h_2, t) \\ \vdots \\ \hat{y}_m(h_2, t) \end{bmatrix} \end{bmatrix} \tag{4.7}$$

The extra-states can be expressed by $\hat{\xi}_m(t)$

$$\hat{\chi}_m(t) = B_2\hat{\xi}_m(t) \tag{4.8}$$

where

$$B_2(h) = \begin{bmatrix} \begin{bmatrix} I_n & 0_n & 0_n & 0_n \end{bmatrix} & & & \\ & I_{nm} & & \\ & & I_{nm} & \\ & & & I_{nm} \end{bmatrix} \tag{4.9}$$

And the system (3.1) can be rewritten as

$$\dot{x} = A_e\hat{\xi}_m(t) \tag{4.10}$$

where $A_e = \begin{bmatrix} A & A_h & 0_{n\times(nm+2n)} \end{bmatrix}$

The time derivative of $\hat{\chi}_m(t)$ can be obtained as follows

$$\dot{\hat{\chi}}_m(t) = C_2(h,\dot{h})\hat{\xi}_m(t) \tag{4.11}$$

where

$$C_2(h,\dot{h}) = \begin{bmatrix} \begin{bmatrix} A & A_d & 0_n & 0_n \end{bmatrix} & & & \\ M_0 & -\frac{(1-\dot{h})}{h}\Lambda - \frac{\dot{h}}{h}\Gamma & & \\ M_1 & & -\frac{1}{h_1}\Lambda & \\ M_2 & & & -\frac{1}{h_2}\Lambda \end{bmatrix}$$

where

$$\Lambda = \begin{bmatrix} 0 & & & & \\ 2I & 0 & & & \\ & 3I & 0 & & \\ & & \ddots & \ddots & \\ & & & mI & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} I & & & & \\ & 2I & & & \\ & & 3I & & \\ & & & \ddots & \\ & & & & mI \end{bmatrix}$$

$$M_0 = \begin{bmatrix} A - \frac{(1-\dot{h})}{h}I & A_h + \frac{(1-\dot{h})}{h}I & 0_n & 0_n \\ A & A_h & 0_n & 0_n \\ \vdots & \vdots & \vdots & \vdots \\ A & A_h & 0_n & 0_n \end{bmatrix}$$

$$M_1 = \begin{bmatrix} A - \frac{1}{h_1}I & A_h & \frac{1}{h_1}I_n & 0_n \\ A & A_h & 0_n & 0_n \\ \vdots & \vdots & \vdots & \vdots \\ A & A_h & 0_n & 0_n \end{bmatrix}, \quad M_2 = \begin{bmatrix} A - \frac{1}{h_2}I & A_h & 0_n & \frac{1}{h_2}I_n \\ A & A_h & 0_n & 0_n \\ \vdots & \vdots & \vdots & \vdots \\ A & A_h & 0_n & 0_n \end{bmatrix}$$

In order to analysis the stability of the system (4.1), we consider the following Lyapunov-Krasovskii functional (LKF) candidates

$$V(t) = \sum_{k=1}^{10} V_k(t) \tag{4.12}$$

where

$$\begin{aligned}
V_1(t) &= \hat{\chi}(t)^{\mathrm{T}} P \hat{\chi}(t) \\
V_2(t) &= \int_{t-h_1}^{t} x^{\mathrm{T}}(s) Q_1 x(s) \mathrm{d}s \\
V_3(t) &= \int_{t-h_2}^{t} x^{\mathrm{T}}(s) Q_2 x(s) \mathrm{d}s \\
V_4(t) &= \int_{t-h(t)}^{t} x^{\mathrm{T}}(s) Q_3 x(s) \mathrm{d}s \\
V_5(t) &= \int_{t-h_1}^{t} \int_{s}^{t} \dot{x}^{\mathrm{T}}(u) R_1 \dot{x}(u) \mathrm{d}u \mathrm{d}s \\
V_6(t) &= \int_{t-h_2}^{t} \int_{s}^{t} \dot{x}^{\mathrm{T}}(u) R_2 \dot{x}(u) \mathrm{d}u \mathrm{d}s \\
V_7(t) &= \int_{t-h(t)}^{t} \int_{s}^{t} \dot{x}^{\mathrm{T}}(u) R_3 \dot{x}(u) \mathrm{d}u \mathrm{d}s \\
V_8(t) &= \int_{t-h_1}^{t} \int_{\theta}^{t} \int_{s}^{t} \dot{x}^{\mathrm{T}}(u) S_1 \dot{x}(u) \mathrm{d}u \mathrm{d}s \mathrm{d}\theta \\
V_9(t) &= \int_{t-h_2}^{t} \int_{\theta}^{t} \int_{s}^{t} \dot{x}^{\mathrm{T}}(u) S_2 \dot{x}(u) \mathrm{d}u \mathrm{d}s \mathrm{d}\theta \\
V_{10}(t) &= \int_{t-h(t)}^{t} \int_{\theta}^{t} \int_{s}^{t} \dot{x}^{\mathrm{T}}(u) S_3 \dot{x}(u) \mathrm{d}u \mathrm{d}s \mathrm{d}\theta
\end{aligned}$$

Taking the time derivative of $V_k(t)$ yields

$$
\begin{aligned}
\dot{V}_1(t) &= \hat{\chi}_m(t)^{\mathrm{T}} P \dot{\hat{\chi}}_m(t) + \dot{\hat{\chi}}_m(t)^{\mathrm{T}} P \hat{\chi}_m(t) \\
&= \hat{\xi}_m^{\mathrm{T}}(t) \left(B^{\mathrm{T}} P C + C^{\mathrm{T}} P B \right) \hat{\xi}_m(t) \\
\dot{V}_2(t) &= x^T(t) Q_1 x(t) - x^{\mathrm{T}}(t-h_1) Q_1 x(t-h_1) \\
&= \hat{\xi}_m^{\mathrm{T}}(t) \left(e_1^{\mathrm{T}} Q_1 e_1 - e_3^{\mathrm{T}} Q_1 e_3 \right) \hat{\xi}_m(t) \\
\dot{V}_3(t) &= x^{\mathrm{T}}(t) Q_2 x(t) - x^{\mathrm{T}}(t-h_2) Q_2 x(t-h_2) \\
&= \hat{\xi}_m^{\mathrm{T}}(t) \left(e_1^{\mathrm{T}} Q_2 e_1 - e_4^{\mathrm{T}} Q_2 e_4 \right) \hat{\xi}_m(t) \\
\dot{V}_4(t) &= x^{\mathrm{T}}(t) Q_3 x(t) - (1-\dot{h}) x^{\mathrm{T}}(t-h) Q_3 x(t-h) \\
&= \hat{\xi}_m^{\mathrm{T}}(t) \left[e_1^{\mathrm{T}} Q_3 e_1 - (1-\dot{h}) e_2^{\mathrm{T}} Q_3 e_2 \right] \hat{\xi}_m(t) \\
\dot{V}_5(t) &= h_1 \dot{x}^{\mathrm{T}}(t) R_1 \dot{x}(t) - \int_{t-h_1}^{t} \dot{x}^{\mathrm{T}}(s) R_1 \dot{x}(s) \mathrm{d}s \\
&\leq \hat{\xi}_m^{\mathrm{T}}(t) \left(h_1 A_e^{\mathrm{T}} R_1 A_e - \frac{1}{h_1} \Psi_1^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_1(R_1) \widehat{W}_m \Psi_1 \right) \hat{\xi}_m(t) \\
\dot{V}_6(t) &= h_2 \dot{x}^{\mathrm{T}}(t) R_2 \dot{x}(t) - \int_{t-h_2}^{t} \dot{x}^{\mathrm{T}}(s) R_2 \dot{x}(s) \mathrm{d}s \\
&\leq \hat{\xi}_m^{\mathrm{T}}(t) \left(h_2 A_e^{\mathrm{T}} R_2 A_e - \frac{1}{h_2} \Psi_2^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_2(R_2) \widehat{W}_m \Psi_2 \right) \hat{\xi}_m(t) \\
\dot{V}_7(t) &= h(t) \dot{x}^{\mathrm{T}}(t) R_3 \dot{x}(t) - (1-\dot{h}) \int_{t-h(t)}^{t} \dot{x}^{\mathrm{T}}(s) R_3 \dot{x}(s) \mathrm{d}s \\
&\leq \hat{\xi}_m^{\mathrm{T}}(t) \left(h A_e^{\mathrm{T}} R_3 A_e - \frac{1-\dot{h}}{h} \Psi_3^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_3(R_3) \widehat{W}_m \Psi_3 \right) \hat{\xi}_m(t) \\
\dot{V}_8(t) &= \frac{h_1^2}{2} \dot{x}^{\mathrm{T}}(t) S_1 \dot{x}(t) - \int_{t-h_1}^{t} \int_{s}^{t} \dot{x}^{\mathrm{T}}(u) S_1 \dot{x}(u) \mathrm{d}u \mathrm{d}s \\
&\leq \hat{\xi}_m^{\mathrm{T}}(t) \left(\frac{h_1^2}{2} A_e^{\mathrm{T}} S_1 A_e - \bar{\Psi}_1^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_1(S_1) \widehat{\bar{W}}_m \bar{\Psi}_1 \right) \hat{\xi}_m(t) \\
\dot{V}_9(t) &= \frac{h_2^2}{2} \dot{x}^{\mathrm{T}}(t) S_2 \dot{x}(t) - \int_{t-h_2}^{t} \int_{s}^{t} \dot{x}^{\mathrm{T}}(u) S_2 \dot{x}(u) \mathrm{d}u \mathrm{d}s \\
&\leq \hat{\xi}_m^{\mathrm{T}}(t) \left(\frac{h_2^2}{2} A_e^{\mathrm{T}} S_2 A_e - \bar{\Psi}_2^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_2(S_2) \widehat{\bar{W}}_m \bar{\Psi}_2 \right) \hat{\xi}_m(t) \\
\dot{V}_{10}(t) &= \frac{h^2}{2} \dot{x}^{\mathrm{T}}(t) S_3 \dot{x}(t) - (1-\dot{h}) \int_{t-h_1}^{t} \int_{s}^{t} \dot{x}^{\mathrm{T}}(u) S_1 \dot{x}(u) \mathrm{d}u \mathrm{d}s \\
&\leq \hat{\xi}_m^{\mathrm{T}}(t) \left(\frac{h^2}{2} A_e^{\mathrm{T}} S_3 A_e - (1-\dot{h}) \bar{\Psi}_3^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_3(S_3) \widehat{\bar{W}}_m \bar{\Psi}_3 \right) \hat{\xi}_m(t)
\end{aligned}
\tag{4.13}
$$

where

$$\Psi_1 = \begin{bmatrix} I_n & 0_n & -I_n & 0_n & 0_{nm} \\ & & \bar{\Psi}_1 & & \end{bmatrix}, \quad \bar{\Psi}_1 = \begin{bmatrix} 0_{nm\times 4n} & 0_{nm} & I_{nm} & 0_{nm} \end{bmatrix}$$
$$\Psi_2 = \begin{bmatrix} I_n & 0_n & 0_n & -I_n & 0_{nm} \\ & & \bar{\Psi}_2 & & \end{bmatrix}, \quad \bar{\Psi}_2 = \begin{bmatrix} 0_{nm\times 4n} & 0_{nm} & 0_{nm} & I_{nm} \end{bmatrix} \tag{4.14}$$
$$\Psi_3 = \begin{bmatrix} I_n & -I_n & 0_n & 0_n & 0_{nm} \\ & & \bar{\Psi}_3 & & \end{bmatrix}, \quad \bar{\Psi}_3 = \begin{bmatrix} 0_{nm\times 4n} & I_{nm} & 0_{nm} & 0_{nm} \end{bmatrix}$$

Thus the sum of $\dot{V}_k(t)$, $k = 1,2,\cdots,10$ yields

$$\dot{V}(t) = \xi^T(t) \underbrace{\begin{bmatrix} B_2^{\mathrm{T}} P C_2 + C_2^{\mathrm{T}} P B_2 + e_1^{\mathrm{T}}(Q_1 + Q_2 + Q_3)e_1 - e_3^{\mathrm{T}} Q_1 e_3 - e_4^{\mathrm{T}} Q_2 e_4 - (1-\dot{h}) e_2^{\mathrm{T}} Q_3 e_2 \\ +h_1 A_e^{\mathrm{T}} R_1 A_e - \frac{1}{h_1}\Psi_1^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_1(R_1) \widehat{W}_m \Psi_1 + h_2 A_e^{\mathrm{T}} R_2 A_e - \frac{1}{h_2}\Psi_2^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_2(R_2) \widehat{W}_m \Psi_2 \\ +h A_e^{\mathrm{T}} R_3 A_e - \frac{(1-\dot{h})}{h}\Psi_3^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_3(R_3) \widehat{W}_m \Psi_3 + \frac{h_1^2}{2} A_e^{\mathrm{T}} S_1 A_e - \bar{\Psi}_1^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_1(S_1) \widehat{\bar{W}}_m \bar{\Psi}_1 \\ +\frac{h_2^2}{2} A_e^{\mathrm{T}} S_2 A_e - \bar{\Psi}_2^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_2(S_2) \widehat{\bar{W}}_m \bar{\Psi}_2 + \frac{h^2}{2} A_e^{\mathrm{T}} S_3 A_e - (1-\dot{h}) \bar{\Psi}_3^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_3(S_3) \widehat{\bar{W}}_m \bar{\Psi}_3 \end{bmatrix}}_{\Xi(h,\dot{h})} \xi(t)$$
$$< 0 \tag{4.15}$$

Notice that $\Xi(h,\dot{h}) \le \Xi(h_2,\mu_2)$ for all $h \in [h_1,h_2]$ and $\dot{h} \in [\mu_1,\mu_2]$, we can develop that $\dot{V}(t) \le \xi^T(t)\Phi\xi(t) < 0$, where

$$\Phi = \Xi(h_2,\mu_2) = \begin{bmatrix} B_2^{\mathrm{T}} P C_2 + C_2^{\mathrm{T}} P B_2 + e_1^{\mathrm{T}}(Q_1 + Q_2 + Q_3)e_1 - e_3^{\mathrm{T}} Q_1 e_3 - e_4^{\mathrm{T}} Q_2 e_4 - (1-\mu_2) e_2^{\mathrm{T}} Q_3 e_2 \\ +h_1 A_e^{\mathrm{T}} R_1 A_e - \frac{1}{h_1}\Psi_1^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_1(R_1) \widehat{W}_m \Psi_1 + h_2 A_e^{\mathrm{T}} R_2 A_e - \frac{1}{h_2}\Psi_2^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_2(R_2) \widehat{W}_m \Psi_2 \\ +h_2 A_e^{\mathrm{T}} R_3 A_e - \frac{(1-\mu_2)}{h_2}\Psi_3^{\mathrm{T}} \widehat{W}_m^{\mathrm{T}} \Omega_3(R_3) \widehat{W}_m \Psi_3 + \frac{h_1^2}{2} A_e^{\mathrm{T}} S_1 A_e - \bar{\Psi}_1^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_1(S_1) \widehat{\bar{W}}_m \bar{\Psi}_1 \\ +\frac{h_2^2}{2} A_e^{\mathrm{T}} S_2 A_e - \bar{\Psi}_2^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_2(S_2) \widehat{\bar{W}}_m \bar{\Psi}_2 + \frac{h_2^2}{2} A_e^{\mathrm{T}} S_3 A_e - (1-\mu_2) \bar{\Psi}_3^{\mathrm{T}} \widehat{\bar{W}}_m^{\mathrm{T}} \bar{\Omega}_3(S_3) \widehat{\bar{W}}_m \bar{\Psi}_3 \end{bmatrix} \tag{4.16}$$

This complete the proof.

4.2. Examples

Example 1: We also consider the well-known delay dependent stable system (4.1) with following coefficient matrices as given in [29]:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_h = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \tag{4.17}$$

The delay rate bounds $\mu_1 = -\mu$, $\mu_2 = \mu$. We herein calculate the allowable upper bound h_2 for various delay rate μ via Theorem 2, as illustrate in Figure 2. It's shown that h_2 deceases continuously with delay rate μ growing.

The allowable upper bounds h_2 varying with given μ are shown in Table 2. We observe that the upper bounds obtained by **Theorem 2** are significantly better than others. **Theorem 1** provides the least conservative results.

For simulation, let the time-varying delay $h(t) = 3 + 2\cos(0.25t)$, which means that $h_1 = 1$, $h_2 = 5$, $\mu_1 = -0.5$, and $\mu_2 = 0.5$. The initial condition of the system is chosen as $x(0) = [1, -1]^{\mathrm{T}}$. The time history of system states is illustrated in Figure 3. As our expectation, both states asymptotically converge to zero despite the previous vibration.

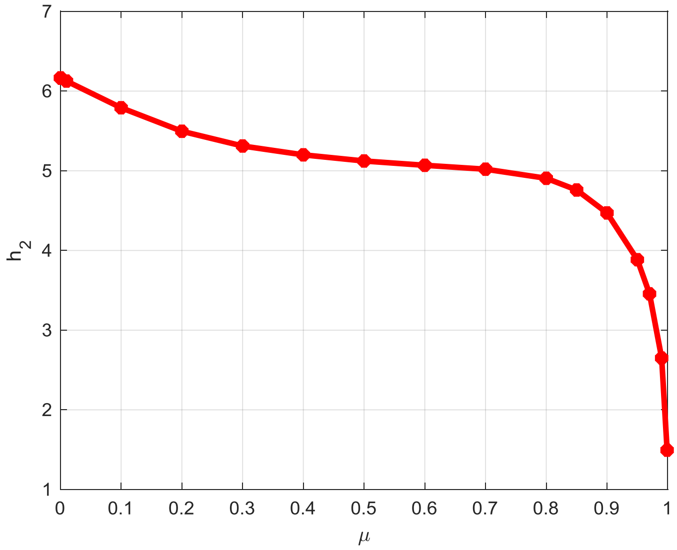

Figure 2. Allowable upper h_2 with variable delay μ.

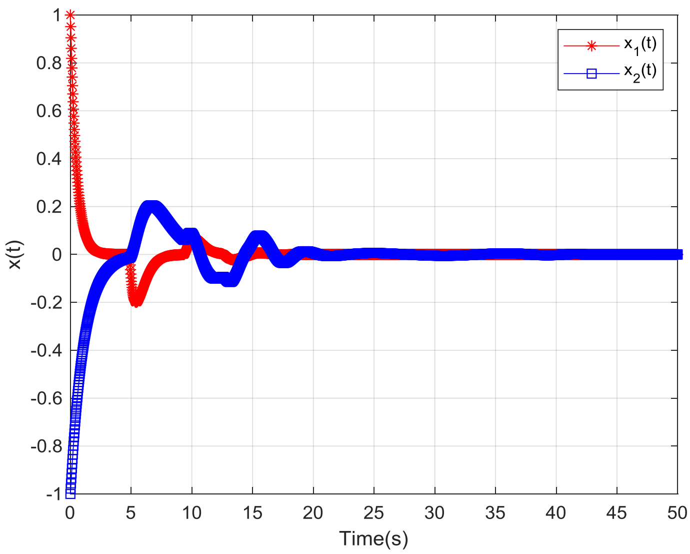

Figure 3. Time history of system states.

Example 2: Consider the time-varying delay system (4.1) with the following parameters [33]:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad A_h = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.18}$$

Table 2. Allowable upper bound h_2 for different μ (example 1).

	μ				
Methods	0.1	0.2	0.5	0.8	Number of variables
Fridman and Uri (2002) [44]	3.604	3.033	2.008	1.364	$5.5n^2 + 1.5n$
He et al. (2007) [16]	3.605	3.039	2.043	1.492	$3n^2 + 3n$
Park and Ko (2007) [45]	3.658	3.163	2.337	1.934	$11.5n^2 + 4.5n$
Ariba and Gouaisbaut (2009) [13]	4.794	3.995	2.682	1.957	$22n^2 + 8n$
Zeng et al. (2013) (N=2) [17]	4.466	3.657	2.375	1.987	$11.5n^2 + 3.5n$
Zeng et al. (2013) (N=3) [17]	4.628	3.766	2.442	2.079	$17n^2 + 5n$
Seuret and Gouaisbaut (2013) [18]	4.703	3.834	2.420	2.137	$10n^2 + 3n$
Zeng et al. (2015) [33]	4.788	4.060	3.055	2.615	$65n^2 + 11n$
Theorem 2 (m=2)	5.791	5.496	5.123	4.906	$14.5n^2 + 4.5n$

When the delay is constant ($\mu = 0$), the analytical upper bound can be obtain $h_{\max} = \pi$. The improvement of our approach is shown in Table 3. It's verified that the advantage of Theorem 2 is over the results in other literatures.

Table 3. Allowable upper bound h_2 for different μ (example 2).

	μ				
Methods	0.1	0.2	0.5	0.8	Number of variables
Park and Ko (2007) [45]	1.99	1.81	1.75	1.61	$11.5n^2 + 4.5n$
Kim (2011) [46]	2.52	2.17	2.02	1.62	$49n^2 + 3n$
Zeng et al. (2015) [33]	3.03	2.57	2.41	1.93	$65n^2 + 11n$
Theorem 2 (m=2)	3.136	3.04	2.95	2.90	$14.5n^2 + 4.5n$

5. Conclusions

New single and double integral inequalities with arbitrary approximation order are developed through the use of shifted Legendre polynomials and Cholesky decomposition. These two inequalities encompass several former well-known integral inequities, such as Jensen inequality, Wirtinger-based inequality, auxiliary function-based integral inequalities, and bring new less-conservative stability criteria by employing proper Lyapunov-Krasovskii functionals. Several numerical examples have been provided which show large improvements compared to existing results in both constant and time-varying delay systems.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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