Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(5): 4311–4321.
DOI: 10.3934/math.2020275
Received: 26 December 2019
Accepted: 23 April 2020
Published: 08 May 2020

Research article

L-biconvex sets on some fuzzy algebraic substructures

Hui Yang[1] **and Yi Shi**[1,2,*]

[1] School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, 100081, PR China
[2] Department of Mathematics, Shantou University, Guangdong, 515063, PR China

* **Correspondence:** Email: shiyi7353@126.com; Tel: +8615510087353.

Abstract: With a complete residuated lattice L as the set of truth values, we first introduce the concept of L-biconvex sets. Then we focus on investigating the forms of L-biconvex sets on three fuzzy algebraic substructures which are L-subsemilattices, L-sublattices and L-Boolean sublattices.

Keywords: L-convex structure; L-biconvex set; L-semilattice half-space; L-lattice half-space; L-Boolean lattice half-space
Mathematics Subject Classification: 52A01, 54A40

1. Introduction

Convexity plays a vital role in many mathematical structures, such as vector spaces, posets, lattices, metric spaces, graphs and median algebras. Generalizing the classical theorems in $\mathbb{R}^n$, Helly, Caratheodory et al. abstracted an axiomatic convexity (also called a convex structure). A convex structure [1] on a set X is defined to be a subset $\mathcal{E}$ of 2^X which contains both the empty set $\emptyset$ and X itself and which is closed under arbitrary intersections and directed unions. As a topology-like spatial structure, convex structures have some similar characters with topologies, one kind of the important spatial properties is separation axioms. With the development of fuzzy set theory, the notion of convex structures has been extended to fuzzy case. The notion of fuzzy convex structures [2] was introduced by Rosa in 1994. In recent years, the theory of fuzzy convex spaces become a popular research direction. In what follows, we divide the existing discussions into two groups.

• In 2016, Jin and Li [3] investigated the relationships between convex spaces and stratified L-convex spaces from a categorical viewpoint. Later, Pang and Shi [4] introduced several types of L-convex spaces and investigated their categorical relationships. Afterwards, Shen and Shi [5] provided some new characterizations of L-convex spaces by using the way-below relation in domain theory. Pang et al. [6, 7] proposed fuzzy counterparts of hull operators, interval operators, bases and subbases, which provided some basic tools for the research of L-convex structures.

• From a completely different direction, Shi and Xiu [8] provided a new approach to fuzzification of convex spaces, i.e., M-fuzzifying convex structures. In this situation, each subset of X can be regarded as a convex set to some degree. Furthermore, Shi and Xiu [9] proposed the notion of (L, M)-fuzzy convex spaces, which contains L-convex spaces and M-fuzzifying convex spaces as special cases. Further, Pang [10, 11] introduced (L, M)-fuzzy hull operators, (L, M)-fuzzy bases and (L, M)-fuzzy subbases to characterize (L, M)-fuzzy convex structures.

As we all know, the notion of biconvex sets (half-spaces, hemispaces) was proposed via a general, non-technical way [1]. The existence of sufficiently many of them in a convex structure is required in a series of of high-lever separation axioms (i.e., S_3, S_4). Also, it can be used to handle many examples and constructions, such as semilattices, lattices, cones, and so on. When applied to lattices, the separation and hull properties are equivalent and characterize distributivity. Furthermore, the S_0, S_1, S_2 separation axioms in convex spaces have been already extend to the fuzzy case [12, 13]. However, they did not investigate the biconvex sets. Recently, Xiu et al. [14] introduced the concept of L-convergence structures in L-convex spaces, which will provide a new tool for interpreting separation properties in convex spaces. Inspired by above, we are interested in the fuzzy counterpart of biconvex sets on some concrete algebraic structures in the framework of L-convex spaces. For this reason, we will first generalize the notion of biconvex sets intuitively to L-biconvex sets and give an essential property of L-biconvex sets, which can be used in the subsequent. Then we put emphasis on L-biconvex sets on some algebraic substructures.

As a logical algebra, a residuated lattice is an algebraic structure which is simultaneously a lattice structure. Let us recall the definition of complete residuated lattices.

Definition 1.1. A complete residuated lattice is an algebra $(L; \wedge, \vee, *, \rightarrow)$ such that

(R1) $(L; \wedge, \vee, 0, 1)$ is a complete lattice with the smallest element 0 and the largest element 1,

(R2) $(L, *, 1)$ is a commutative monoid, i.e., $*$ is commutative, associative, and $a * 1 = 1$ holds for each $a \in L$, and

(R3) $*$ and $\rightarrow$ form an adjoint pair, i.e., $a * b \leqslant c \Longleftrightarrow a \leqslant b \rightarrow c$ for all $a, b, c \in L$.

Suppose that L is a complete residuated lattice. For any $a \in L$, we define $\neg a = a \rightarrow 0$, called the *negation* of a. A residuated lattice L is called *regular* [15] if it satisfies the law of double negation, i.e., $\neg\neg a = a$ for all $a \in L$. In this case, we call L a *Girard monoid* .

Throughout this paper, if not otherwise specified, L denotes a complete residuated lattice.

For a nonempty set X, the notation L^X denotes the set of all maps from X to L and every member of L^X is called an *L-subset* of X. An L-subset $A \in L^X$ is called *constant* at $a \in L$ if $A(x) = a$ for each $x \in X$, denoted by a_X. For an operation $* \in \{\vee, \wedge, \otimes, \rightarrow\}$ and for all $A, B \in L^X$, the L-subset $A * B$ is pointwisely defined, that is, $(A * B)(x) = A(x) * B(x)$ $(\forall x \in X)$.

Let $\varphi : X \longrightarrow Y$ be a map between two sets. Define $\varphi_L^{\rightarrow} : L^X \longrightarrow L^Y$ and $\varphi_L^{\leftarrow} : L^Y \longrightarrow L^X$ by $\varphi_L^{\rightarrow}(A)(y) = \bigvee_{\varphi(x)=y} A(x)$ for each $A \in L^X$ and each $y \in Y$ and $\varphi_L^{\leftarrow}(B) = B \circ \varphi$ for each $B \in L^Y$, respectively.

Definition 1.2. [4, 16] Let X be a nonempty set. A *stratified L-convex structure* (or a *stratified L-convexity*) on X is a subset C of L^X such that

(LC1) $a_X \in C$ for every $a \in L$;

(LC2) $\bigwedge_{i\in I} A_i \in C$ for each subset $\{A_i\}_{i\in I}$ of C;

(LC3) $\bigvee_{j\in J} A_j \in C$ for each directed subset $\{A_j\}_{j\in J}$ of C.

A set X equipped with a stratified L-convexity C, denoted by (X, C), is called a *stratified L-convex space.* The members of C are called *L-convex sets.*

Definition 1.3. [9] A map $f : (X, C_X) \longrightarrow (Y, C_Y)$ between L-convex spaces is called *L-convexity preserving* (L-CP, in short) if $f_L^{\leftarrow}(B) \in C_X$ for each $B \in C_Y$.

Definition 1.4. [17,18] Let $(X, \leqslant)$ be a partially ordered set and $A \in L^X$. Then A is called an *L-ordered convexity* if $A(z) \geqslant A(x) \wedge A(y)$ whenever $x \leqslant z \leqslant y$.

Definition 1.5. [17, 18] Let X be a meet (resp. join) semilattice and $A \in L^X$. Then A is called an *L-subsemilattice* if $A(x \wedge y) \geqslant A(x) \wedge A(y)$ (resp. $A(x \vee y) \geqslant A(x) \wedge A(y)$) for all $x, y \in X$.

Definition 1.6. [17, 18] Let X be a meet (resp. join) semilattice and let $A \in L^X$. Then A is called an *L-convex subsemilattice* if A is an L-subsemilattice and L-ordered convexity.

Proposition 1.7. *Let X be a meet (resp. join) semilattice. Define a subset C_S of L^X by*

$$C_S = \left\{C \in L^X \mid C \text{ is an } L\text{-convex subsemiattice}\right\}.$$

Then C_S is a stratified L-convexity on X, called the stratified L-semilattice convexity.

Proof. (LC1) is obvious. The verifications of (LC2) and (LC3) can be found in [18, Theorem 4.3]. □

Definition 1.8. [17, 18] Let X be a lattice and $A \in L^X$. Then A is called an *L-sublattice* if $A(x \vee y) \geqslant A(x) \wedge A(y)$ and $A(x \vee y) \geqslant A(x) \wedge A(y)$ for all $x, y \in X$.

Definition 1.9. [17, 18] Let X be a lattice and let $A \in L^X$. Then A is called an L-convex sublattice if A is an L-sublattice and L-ordered convexity.

Proposition 1.10. *Let X be a lattice. Then (X, C_L) is an L-convex space, where*

$$C_L = \left\{C \in L^X \mid C \text{ is an } L\text{-convex sublattice}\right\}.$$

Then C_L is called the L-lattice convexity and the pair (X, C_L) is called the L-lattice convex space.

Proof. (LC1) is obvious. The verifications of (LC2) and (LC3) can be found in [18, Theorem 5.3]. □

Definition 1.11. [19] Let $(X, \leqslant)$ be a partially ordered set. A map $A : X \longrightarrow L$ is called an upper set on X if for all $x, y \in X$ with $x \leqslant y$, then $A(x) \leqslant A(y)$. Dually, a mapping $A : X \longrightarrow L$ such that for all $x, y \in X$, $x \leqslant y \Rightarrow A(y) \leqslant A(x)$, is called a lower set on X.

Definition 1.12. [17,20] (i) An L-sublattice A is called an *L-ideal* if $A(x) \geqslant A(y)$ whenever $x \leqslant y$. (ii) An L-sublattice A is called an *L-filter* if $A(x) \leqslant A(y)$ whenever $x \leqslant y$.

Definition 1.13. [20] (i) A proper L-ideal $A \in L^X$ is called an *L-prime ideal* if $A(x \wedge y) \leqslant A(x) \vee A(y)$ for all $x, y \in X$. (ii) A proper L-filter $A \in L^X$ is called an *L-prime filter* if $A(x \vee y) \leqslant A(x) \vee A(y)$ for all $x, y \in X$.

It is easy to see that $A \in L^X$ is an L-prime ideal if $\neg A$ is an L-prime filter.

2. L-biconvex sets

The concept of biconvex sets plays an important role in (abstract) convex structures. The existence of sufficiently many of them in convex structures are required in a series of separation axioms. In this section, our aim is to extend biconvex sets to L-fuzzy setting.

Definition 2.1. Let $(X, \mathcal{C})$ be an L-convex space. An L-subset H of X is called an *L-biconvex set* if $H \in \mathcal{C}$ and $\neg H \in \mathcal{C}$. Denote

$$\mathcal{H} = \left\{H \in L^X \mid H \text{ is an } L\text{-biconvex set}\right\},$$

then the pair $(X, \mathcal{H})$ is called an *L-half-space*.

Remark 2.2. It is easy to see that $(X, \{0_X, 1_X\})$ is an L-half-space on X. All other L-half-spaces are said to be non-trivial.

We present an elementary result which will be used in the subsequent.

Proposition 2.3. *If $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is L-CP and $(Y, \mathcal{H}_Y)$ is an L-half-space, then $f_L^{\leftarrow}(H)$ is an L-biconvex set on X for each $H \in \mathcal{H}_Y$.*

Proof. Since f is an L-CP map and $(Y, \mathcal{H}_Y)$ is an L-half-space, it follows that $f_L^{\leftarrow}(H) \in \mathcal{C}_X$ and $f_L^{\leftarrow}(\neg H) \in \mathcal{C}_X$ for each $H \in \mathcal{H}_Y$. Hence $\neg f_L^{\leftarrow}(H) \in \mathcal{C}_X$ by $f_L^{\leftarrow}(\neg H) = \neg f_L^{\leftarrow}(H)$. This shows that $f_L^{\leftarrow}(H)$ is an L-biconvex set on X for each $H \in \mathcal{H}_Y$. □

3. Main results

3.1. L-semilattice half-spaces

Lemma 3.1. *Let $(X, \leqslant)$ be a partially ordered set. The following statements hold.*

(1) An L-subset $A \in L^X$ is an upper set iff $A = A^{\uparrow}$, where $A^{\uparrow}(y) = \bigvee_{x \leqslant y} A(x)$ for all $y \in X$.

(2) An L-subset $A \in L^X$ is a lower set iff $A = A^{\downarrow}$, where $A^{\downarrow}(y) = \bigvee_{y \leqslant x} A(x)$ for all $y \in X$.

Proof. (1) The sufficiency of this proof is obvious. It remains to show its necessity. By Definition 1.11, it is obvious that $A(y) \geqslant A^{\uparrow}(y)$ whenever $x \leqslant y$ for all $y \in X$. And we can see that $\bigvee_{x \leqslant y} A(x) \geqslant A(y)$. Hence $A = A^{\uparrow}$ by the arbitrariness of y.

(2) The proof is dually analogous to that of (1). □

Theorem 3.2. *Let $(X, \mathcal{C}_S)$ be an (meet) L-semilattice convex space. Denote*

$$\mathcal{H}_S = \{C^{\uparrow} \in L^X \mid C \in \mathcal{C}_S\}.$$

Then $(X, \mathcal{H}_S)$ is an L-half-space of a semilattice. In this case, we call it an (meet) L-semilattice half-space.

Proof. We need to show that $C^{\uparrow} \in \mathcal{C}_S$ and $\neg C^{\uparrow} \in \mathcal{C}_S$, and consequently the proof can be solved in two steps.

Step1. $C^{\uparrow}$ is an L-convex subsemilattice.

Let $x \leqslant z \leqslant y$. It holds that

$$C^{\uparrow}(z) = \bigvee_{a \leqslant z} C(a) \geqslant \bigvee_{a \leqslant x} C(a) \geqslant \bigvee_{a \leqslant x} C(a) \wedge \bigvee_{b \leqslant y} C(b) = C^{\uparrow}(x) \wedge C^{\uparrow}(y).$$

Thus $C^{\uparrow}$ is an L-ordered convexity.

Since $C(x \wedge y) \geqslant C(x) \wedge C(y)$, we have

$$\begin{aligned} C^{\uparrow}(x) \wedge C^{\uparrow}(y) &= \bigvee_{z_2 \leqslant x} C(z_2) \wedge \bigvee_{z_3 \leqslant y} C(z_3) = \bigvee_{z_2 \leqslant x} \bigvee_{z_3 \leqslant y} C(z_2) \wedge C(z_3) \\ &\leqslant \bigvee_{\substack{z_2 \leqslant x \\ z_3 \leqslant y}} C(z_2 \wedge z_3) \leqslant \bigvee_{z_2 \wedge z_3 \leqslant x \wedge y} C(z_2 \wedge z_3) = \bigvee_{z_1 \leqslant x \wedge y} C(z_1) \\ &= C^{\uparrow}(x \wedge y). \end{aligned}$$

This shows that $C^{\uparrow}$ is an L-subsemilattice.

Step2. $\neg C^{\uparrow}$ is an L-convex subsemilattice.

Let $y_1 \leqslant y \leqslant y_2$. Then

$$\begin{aligned} \neg C^{\uparrow}(y_1) \wedge \neg C^{\uparrow}(y_2) &= \left(\bigvee_{x \leqslant y_1} C(x) \to 0 \right) \wedge \left(\bigvee_{x \leqslant y_2} C(x) \to 0 \right) \\ &\leqslant \bigvee_{x \leqslant y} C(x) \to 0 = \bigwedge_{x \leqslant y} (C(x) \to 0) = \neg C^{\uparrow}(y). \end{aligned}$$

Hence, $\neg C^{\uparrow}$ is an L-order convexity.

It remains to verify that $\neg C^{\uparrow}$ is an L-subsemilattice. Indeed,

$$\begin{aligned} \neg C^{\uparrow}(x) \wedge \neg C^{\uparrow}(y) &= \left(\bigvee_{z \leqslant x} C(z) \to 0 \right) \wedge \left(\bigvee_{z \leqslant y} C(z) \to 0 \right) \\ &\leqslant \bigvee_{z \leqslant x \wedge y} C(z) \to 0 = \bigwedge_{z \leqslant x \wedge y} (C(z) \to 0) = \neg C^{\uparrow}(x \wedge y). \end{aligned}$$

This completes the proof. □

Remark 3.3. Let $(X, \mathcal{C}_S)$ be an (meet) L-semilattice convex space. Denote

$$\mathcal{H}_S = \{C^{\downarrow} \in L^X \mid C \in \mathcal{C}_S\}.$$

Then $(X, \mathcal{H}_S)$ fails to be an L-half-space in general. This can be seen from the following example.

Example 3.4. Consider the semilattice $X = \{\perp, x, y, z\}$, where $\perp \leqslant x, y, z$ and x, y, z are incomparable. Let $L = [0, 1]$, and let

$$x \oplus y = (x + y) \wedge 1, \quad x' = 1 - x, \quad \text{for all } x, y \in L.$$

Then $(L, \oplus, ', 0)$ is a standard MV-algebra. Define a map $A : X \longrightarrow L$ as follows:

$$A(\perp) = 0.5, A(x) = 0.8, A(y) = 0.3, A(z) = 0.4.$$

It is easy to check that $(X, \{A\})$ is an (meet) L-semilattice convex space. And we can obtain the following results:

$$A^{\downarrow}(\bot) = 0.8, \qquad A^{\downarrow}(x) = 0.8,$$
$$A^{\downarrow}(y) = 0.3, \qquad A^{\downarrow}(z) = 0.4,$$
$$\neg A^{\downarrow}(\bot) = 0.2, \qquad \neg A^{\downarrow}(x) = 0.2,$$
$$\neg A^{\downarrow}(y) = 0.7, \qquad \neg A^{\downarrow}(z) = 0.6.$$

Hence $\neg A(y \wedge z) = 0.2 \ngeqslant \neg A(y) \wedge \neg A(z) = 0.6$. Therefore $A^{\downarrow}$ is not an L-biconvex set.

We can dually obtain the following corollary.

Corollary 3.5. *Let (X, C_S) be an (join) L-semilattice convex space. Denote*

$$\mathcal{H}_S = \{C^{\downarrow} \in L^X \mid C \in C_S\}.$$

Then $(X, \mathcal{H}_S)$ is an L-half-space, and in this case we call it an (join) L-semilattice half-space.

Theorem 3.6. *Let $(X, \mathcal{H}_S)$ be an L-semilattice half-space. Then H is a lower set or an upper set for any $H \in \mathcal{H}_S$.*

Proof. Suppose that H is neither a lower set nor an upper set for any $H \in \mathcal{H}_S$. Then, by Definition 1.11, we can obtain that the following results hold.

(1) There exist $x_1, y_1 \in X$ with $x_1 \leqslant y_1$ such that $H(x_1) \nleqslant H(y_1)$.

(2) There exist $x_2, y_2 \in X$ with $x_2 \leqslant y_2$ such that $H(y_2) \nleqslant H(x_2)$.

To this end, we first claim that $H(x_1 \wedge x_2) \ngeqslant H(x_1) \wedge H(x_2)$. Otherwise, we have either $H(x_1 \wedge x_2) \geqslant H(x_2)$ or $H(x_1 \wedge x_2) \geqslant H(x_1)$. For the first circumstance, since $x_1 \wedge x_2 \leqslant x_2 \leqslant y_2$, it implies from the definition of L-order convex that $H(x_2) \geqslant H(y_2) \wedge H(x_1 \wedge x_2)$. Hence $H(y_2) \leqslant H(x_2)$, that is a contradiction. Dually, we can obtain that $H(x_1) \leqslant H(y_1)$ is also a contradiction.

Since $x_1 \wedge x_2 \leqslant x_1 \wedge y_2 \leqslant x_1 \leqslant y_1$ and $\neg H$ is still n am$L$-ordered convexity, it follows that

$$\neg H(x_1 \wedge y_2) \geqslant \neg H(x_1 \wedge x_2) \wedge \neg H(y_1).$$

That is,

$$H(x_1 \wedge y_2) \to 0 \geqslant (H(x_1 \wedge x_2) \to 0) \wedge (H(y_1) \to 0).$$

This implies

$$H(x_1 \wedge y_2) \leqslant H(x_1 \wedge x_2) \vee H(y_1) \ngeqslant H(x_1) \wedge H(x_2).$$

Hence $H(x_1) \wedge H(y_2) \ngeqslant H(x_1) \wedge H(x_2)$, which contradicts to $H(y_2) \nleqslant H(x_2)$.

Therefore the assumption fails to hold, i.e., H is a lower set or an upper set for any $H \in \mathcal{H}_S$. □

As the following example shows, the converse is not necessarily valid. That is to say, for an L-half-space in a semilattice $(X, \mathcal{H}_S)$, H being a lower or an upper set can not always imply $H \in \mathcal{H}_S$.

We will give an example that satisfies:

- $(X, \mathcal{H}_S)$ is an L-half-space.

- H is a lower set, but $H \notin \mathcal{H}_S$, or
- H is an upper set, but $H \notin \mathcal{H}_S$.

Example 3.7. Consider the semilattice $X = \{\perp, x, y, z\}$, where $\perp \leqslant x, y, z$ and x, y, z are incomparable.. Let $(L, \oplus, \prime, 0)$ be a standard MV-algebra. Denote $\mathcal{H}_S = \{\chi_\emptyset, A, B, C, \chi_X\}$, where A, B, C are defined in Table 1.

Table 1. An L-semilattice half-space $(X, \mathcal{H}_S)$.

	$\chi_\emptyset$	A	B	C	χ_X
x	0	0.3	0.5	0.5	1
y	0	0.7	0.4	0.5	1
z	0	0.2	0.8	0.8	1
$\perp$	0	0.5	0.5	0.3	1

In realities, we have $A \in \mathcal{H}_S$, but A is neither an upper set nor a lower set.

3.2. L-lattice half-spaces

Theorem 3.8. *Let X be a lattice. Denote*

$$\mathcal{H}_L = \{H \in L^X \mid H \text{ is an } L\text{-prime ideal (filter)}\}.$$

Then $(X, \mathcal{H}_L)$ is an L-half-space of a lattice , and in this case we call it an L-lattice half-space.

Proof. Since the negation of an L-prime ideal is an L-prime filter, it suffices to show that $H \in C_L$ and $\neg H \in C_L$ whenever H is an L-prime ideal. And consequently the proof can be seen as follows.

Since H is an L-prime ideal, it follows that H is an L-sublattice and $H(y) \leqslant H(z) \leqslant H(x)$ whenever $x \leqslant z \leqslant y$. Hence $H(z) \geqslant H(x) \wedge H(y)$, and thus H is an L-convex sublattice, i.e., $H \in C_L$.

Let $y_1 \leqslant y \leqslant y_2$. Then

$$\neg H(y_1) \wedge \neg H(y_2) = (H(y_1) \to 0) \wedge (H(y_2) \to 0) \leqslant H(y) \to 0 = \neg H(y).$$

Hence, $\neg H$ is L-ordered convex.

For any $x, y \in X$, the following hold that

$$\begin{aligned} \neg H(x \wedge y) &= H(x \wedge y) \to 0 \\ &\geqslant (H(x) \vee H(y)) \to 0 \\ &= (H(x) \to 0) \wedge (H(y) \to 0) \\ &= \neg H(x) \wedge \neg H(y), \end{aligned}$$

and

$$\neg H(x \vee y) = H(x \vee y) \to 0 \geqslant (H(x) \to 0) \wedge (H(y) \to 0) = \neg H(x) \wedge \neg H(y).$$

Thus $\neg H$ is an L-sublattice. □

Theorem 3.9. *Let $(X, \mathcal{H}_L)$ is an L-lattice half-space. Then H is an L-prime ideal or an L-prime filter for any $H \in \mathcal{H}_L$.*

Proof. From Theorem 3.6, we know that H is a lower set or an upper set for any $H \in \mathcal{H}_L$. Then we just need to show that H is an L-prime ideal (an L-prime filter) while H is a lower set (an upper set) for any $H \in \mathcal{H}_L$. Obviously, H is an L-ideal. Suppose that H is not an L-prime ideal. It means that $H(x \wedge y) \nleqslant H(x) \vee H(y)$ for all $x, y \in X$. Then $\neg H(x \wedge y) \ngeqslant \neg H(x) \wedge \neg H(y)$. But $\neg H$ is an L-convex lattice. Therefore the assumption fails to hold. □

3.3. L-Boolean lattice half-spaces

In this section, considering L as Giraid monoid, we shall present the concept of L-Boolean lattice half-spaces. Basic notions related to Boolean lattice from the classical order theory are listed in the sequel. For more comprehensive presentation, one can refer to [21, 22].

Definition 3.10. [21, 22] Let X be a lattice. X is said to be *distributive* if it satisfies the following one of distributive laws:

(D1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in X$,

(D2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in X$.

Definition 3.11. [21, 22] Let X be a lattice with universal bounds $0, 1$. For $x \in X$, we say $y \in X$ is a *complement* of x if $x \wedge y = 0$ and $x \vee y = 1$. If x has a unique complement, we denote this complement by x'.

Remark 3.12. In a distributive lattice an element can have at most one complement.

Definition 3.13. [21, 22] A lattice X is called a *Boolean lattice* if it satisfies the following conditions:

(B1) X is distributive;

(B2) X has 0_X and 1_X;

(B3) each $x \in X$ has a (necessarily unique) complement $x' \in X$.

Definition 3.14. Let X be a Boolean lattice.

(i) A proper L-ideal I of X is called an *L-maximal ideal* if $I \leqslant J$ implies $I = J$ for all proper L-ideal J of X.

(ii) A proper L-filter F of X is called an *L-maximal filter* if $F \leqslant G$ implies $F = G$ for all proper L-filter G of X.

Theorem 3.15. *Let X be a Boolean lattice and I be a proper L-ideal. Then the following conditions are equivalent.*

(1) I is an L-maximal ideal.

(2) I is an L-prime ideal

(3) For all $x \in X$, $I(x) = \neg I(x')$.

Proof. (1) ⇒ (2). Let I be an L-maximal ideal of X and $x, y \in X$. Assume $I(x \wedge y) \nleqslant I(x)$. It remains to prove $I(x \wedge y) \leqslant I(y)$. Define $I_x(k) = \bigvee_{k \leqslant (x \vee z)} I(z)$. Then we can easily obtain that I_x is an L-ideal and $I \leqslant I_x$. Because I is maximal, we have $I_x = I$. In particular, $I(1) = \bigvee_{x \vee z = 1} I(z)$, so $x \vee w = 1$ for some $w \in X$ with $I(w) \leqslant I(1)$. Then $I((x \wedge y) \vee w) = I((x \vee w) \wedge (y \vee w)) = I(y \vee w)$. Since $y \leqslant y \vee w$, we have $I(y) \geqslant I(y \vee w) \geqslant I(y) \wedge I(w)$. Hence $I(w) \geqslant I(y)$. In addition, $I(y) \geqslant I(x \wedge y) \wedge I(w)$, and thus $I(y) \geqslant I(x \wedge y)$.

(2) ⇒ (3). Note that $x \wedge x' = 0_X$ for all $x \in X$. Because I is an L-prime ideal, we have $I_L = I(0_X) \leqslant I(x) \vee I(x')$, i.e., $I(x) \vee I(x') = 1_L$. By I is an L-ideal, we have $I(x) \leqslant I(x') \to 0_L \Leftrightarrow I(x) \wedge I(x') \leqslant 0_L$. Hence,$I(x) \wedge I(x') = 0_L$. Thus we have $I(x) = \neg I(x')$

(3) ⇒ (1). Let J be a proper L-ideal of X with $I \leqslant J$. It suffices to show that $I \geqslant J$. Fix $x \in X$ with $I(x) \ngeqslant J(x)$. Then $\neg\neg I(x) \ngeqslant \neg\neg J(x)$. By (3), we have $\neg I(x') \ngeqslant \neg J(x')$, so $I(x') \nleqslant J(x')$, which contradicts to $I \leqslant J$. Therefore I is maximal. □

We can dually obtain the following corollary.

Corollary 3.16. *Let X be a Boolean lattice and I be a proper L-filter. Then the following conditions are equivalent.*

(1) *I is an L-maximal filter;*

(2) *I is an L-prime filter;*

(3) *for all $x \in X$, it is the case that $F(x) = \neg F(x')$.*

By Theorems 3.8, 3.9, 3.15 and 3.16, we easily obtain the following corollaries.

Corollary 3.17. *Let X be a Boolean lattice. Denote*

$$\mathcal{H}_B = \{H \in L^X \mid H \text{ is an } L\text{-maximal ideal (or filter)}\}.$$

Then $(X, \mathcal{H}_B)$ is an L-half-space of a Boolean lattice, and in this case we call it an L-Boolean lattice half-space.

Corollary 3.18. *Let $(X, \mathcal{H}_L)$ be an L-Boolean lattice half-space. Then H is an L-maximal ideal or an L-maximal filter for any $H \in \mathcal{H}_L$.*

4. Conclusions

In this paper, we introduced the notion of L-biconvex sets. Further we gave the corresponding formulates of L-biconvex sets on three fuzzy algebraic substructures which are L-subsemilattices, L-sublattices and L-Boolean sublattices. We hope that the results obtained here can be applied in the separation axioms in the framework of L-convex spaces and also on other important algebraic structures, such G-algebras, BCK/BCI algebras [23–26].

Acknowledgments

The authors would like to express their sincere thanks to the anonymous reviewers and the area editor for their careful reading and constructive comments. The project is supported by the National Natural Science Foundation of China (11871097).

Conflict of interest

The authors declare no conflicts of interest.

References

1. M. van de Vel, *Theory of Convex Structures*, North-Holland, Amsterdam, 1993.
2. M. V. Rosa, *On fuzzy topology fuzzy convexity spaces and fuzzy local convexity*, Fuzzy Sets Syst., **62** (1994), 97–100.
3. Q. Jin, L. Q. Li, *On the embedding of L-convex spaces in stratified L-convex spaces*, SpringerPlus, **5** (2016), 1610.
4. B. Pang, F. G. Shi, *Subcategories of the category of L-convex spaces*, Fuzzy Sets Syst., **313** (2017), 61–74.
5. C. Shen, F. G. Shi, *Characterizations of L-convex spaces via domain theory*, Fuzzy Sets Syst., **380** (2020), 44–63.
6. B. Pang, F. G. Shi, *Fuzzy counterparts of hull spaces and interval spaces in the framework of L-convex spaces*, Fuzzy Sets Syst., **369** (2019), 20–39.
7. B. Pang, Z. Y. Xiu, *An axiomatic approach to bases and subbases in L-convex spaces and their applications*, Fuzzy Sets Syst., **369** (2019), 40–56.
8. F. G. Shi, Z. Y. Xiu, *A new approach to the fuzzification of convex structures*, J. Appl. Anal., **2014** (2014), 1–12.
9. F. G. Shi, Z. Y. Xiu, (L, M)*-fuzzy convex structures*, J. Nonlinear Sci. Appl., **10** (2017), 3655–3669.
10. B. Pang, *Hull operators and interval operators in the* (L, M)*-fuzzy convex spaces*, Fuzzy Sets Syst., 2019, DOI: 10.1016/j.fss.2019.11.010.
11. B. Pang, *Bases and subbases in* (L, M)*-fuzzy convex spaces*, Comput. Appl. Math., **39** (2020), 41.
12. C. Y. Liang, F. H. Li, *A degree approach to separation axioms in M-fuzzifying convex spaces*, J. Intell. Fuzzy Syst., **36** (2019), 2885–2893.
13. C. Y. Liang, F. H. Li, J. Zhang, *Separation axioms in* (L, M)*-fuzzy convex spaces*, J. Intell. Fuzzy Syst., **36** (2019), 3649–3660.
14. Z. Y. Xiu, Q. H. Li, B. Pang, *Fuzzy convergence structures in the framework of L-convex spaces*, Iran. J. Fuzzy Syst., 2020, DOI:10.22111/IJFS.2020.5232.
15. D. W. Pei, *The characterization of residuated lattices and regular residuated lattices*, Acta Mathematica Sinica, **45** (2002), 271–278. (In Chinese)
16. Y. Maruyama, *Lattice-valued fuzzy convex geometry*, RIMS Kokyuroku, **1641** (2009), 22–37.
17. N. Ajmal, K. V. Thomas, *Fuzzy lattice*, Inform. Sci., **79** (1994), 271–291.
18. Y. Zhong, F. G. Shi, *Formulations of L-convex hulls on some algebraic structures*, J. Intell. Fuzzy Syst., **33** (2017), 1–11.
19. J. Jiménez, S. Montes, B. Šešeljia, et al. *On lattice valued upsets and down-sets*, Fuzzy Sets Syst., **161** (2010), 1699–1710.

20. Y. Bo, W. W. Ming, *Fuzzy ideals on a distributive lattice*, Fuzzy Sets Syst., **35** (1990), 231–240.
21. G. Birkhoff, *Lattice Theory*, American Mathematical Society, Providence RI, 1967.
22. B. A. Davey, H. A. Priestley, *Introduction to Lattice and Order (2 Eds.)*, Cambridge: Cambridge University Press, 2002.
23. M. Bhowmik, T. Senapati, M. Pal, *Intuitionistic L-fuzzy ideals of BG-algebras*, Afr. Mat., **25** (2014), 577–590.
24. C. Janab, M. Pal, T. Senapatia, et al. *Atanassov's Intuitionistic L-fuzzy G-subalgebras of G-algebras*, J. Fuzzy Math., **23** (2015), 325–340.
25. J. Meng, Y. B. Jun, *BCK-Algebras*, Kyung Moon Sa Co., Seoul, Korea, 1994.
26. T. Senapatia, C. Janab, M. Bhowmikc, et al. *L-fuzzy G-subalgebras of G-algebras*, J. Egypt. Math. Soc., **23** (2015), 219–223.