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Research article

Invertible weighted composition operators preserve frames on Dirichlet type spaces

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Abstract: Some characterizations for weighted composition operators to be invertible on Dirichlet type spaces $\mathcal{D}_\rho$ are given in this paper when ρ is finite lower type greater than 0 and upper type less than 1. In particular, the equivalence between invertible and preserve frames is established. Moreover, weighted composition operators that preserve tight frames and normalized tight frames on the Dirichlet type space $\mathcal{D}_\alpha$ $(0 < \alpha < 1)$ are also investigated.

Keywords: Dirichlet type space; weighted composition operator; frame; bounded below; multiplier
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1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ denote the set of functions analytic in $\mathbb{D}$. As usual, let H^∞ be the space of bounded analytic functions in $\mathbb{D}$ and $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$. Let φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. The weighted composition operator $W_{\psi,\varphi}$, induced by φ and ψ, is defined on $H(\mathbb{D})$ by

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z)), \qquad f \in H(\mathbb{D}).$$

When $\psi \equiv 1$, we get the composition operator, denoted by C_φ. We refer the readers to [1, 2] for the theory of composition operators and weighted composition operators.

Let $\rho : [0, \infty) \to [0, \infty)$ be a right continuous and nondecreasing function with $\rho(0) = 0$. We say that an $f \in H(\mathbb{D})$ belongs to the Dirichlet type space $\mathcal{D}_\rho$, if

$$\|f\|^2_{\mathcal{D}_\rho} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \rho\left(1 - |z|^2\right) dA(z) < \infty,$$

where dA is the area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$. When $\rho(t) = t$, $\mathcal{D}_\rho = H^2$, the Hardy space. If $\rho(t) = t^\alpha$ and $\alpha > 1$, $\mathcal{D}_\rho$ is the Bergman space $A^2_{\alpha-2}$. When $\rho(t) = t^\alpha$ and $\alpha < 1$, it gives the weighted Dirichlet space $\mathcal{D}_\alpha$. Many properties of $\mathcal{D}_\rho$ spaces were studied by Aleman in [3] and Kerman, Sawyer in [4]. Carleson measure for $\mathcal{D}_\rho$ spaces was studied by Arcozzi, Rochberg and Sawyer in [5]. For more information about $\mathcal{D}_\rho$ and $\mathcal{D}_\alpha$, we refer to [6–12].

Recall that a weight ρ is of upper (resp.lower) type γ ($0 \le \gamma < \infty$) ([13]), if

$$\rho(st) \le Cs^\gamma \rho(t), \quad s \ge 1 \ \ (\text{resp.} s \le 1) \ \text{ and } \ 0 < t < \infty.$$

We say that ρ is of upper type less than γ if it is of upper type δ for some $\delta < \gamma$ and ρ is of lower type greater than β if it is of lower type δ for some $\delta > \beta$. From [13], we see that an increasing function ρ is of finite upper type if and only if $\rho(2t) \le C\rho(t)$.

Let $\mathcal{H}$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|_{\mathcal{H}}$. A family $\{f_k\}_{k=0}^\infty$ in $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2_{\mathcal{H}} \le \sum_{k=0}^{\infty} |\langle f, f_k\rangle|^2 \le B\|f\|^2_{\mathcal{H}}, \quad \text{for all } f \in \mathcal{H}.$$

A (resp. B) is called the lower (resp. upper) frame bounded. When $A = B$, the family $\{f_k\}_{k=0}^\infty$ is called a tight frame. If $A = B = 1$, we call it a normalized tight frame. The notion of frame was first introduced by Duffin and Schaeffer in [14]. Tight frame is especially popular and now widely used in compressive sensing, image and signal processing, since it provide stable decompositions similar to orthonormal bases (see [15]). We say that a linear operator T on a Hilbert space $\mathcal{H}$ preserves frames if $\{Tf_k\}_{k=0}^\infty$ is a frame in $\mathcal{H}$ for any frame $\{f_k\}_{k=0}^\infty \subseteq \mathcal{H}$. Similarly, we call T on $\mathcal{H}$ preserves (normalized) tight frames if $\{Tf_k\}_{k=0}^\infty$ is a (normalized) tight frame in $\mathcal{H}$ for any (normalized) tight frame $\{f_k\}_{k=0}^\infty \subseteq \mathcal{H}$. See [16] for more information.

Recently, Manhas, Prajitura and Zhao studied weighted composition operators that preserve frames in [16]. Especially, they build the equivalence between preserve frames and invertible of weighted composition operators on weighted Bergman spaces A^2_α in the unit ball.

In this paper, we give some characterizations for invertible weighted composition operators on Dirichlet type spaces $\mathcal{D}_\rho$ when ρ is finite lower type greater than 0 and upper type less than 1. In particular, our result shows that weighted composition operators are invertible if and only if they preserve frames on $\mathcal{D}_\rho$. Moreover, we also investigate weighted composition operators that preserve normalized tight frames and tight frames in the weighted Dirichlet space $\mathcal{D}_\alpha$ ($0 < \alpha < 1$).

Throughout this paper, we say that $A \lesssim B$ if there exists a constant C (independent of A and B) such that $A \le CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Auxiliary results

Let us firstly recall the following construction in [17]. Suppose $c_n > 0$ for $n = 0, 1, \cdots$. Define an inner product on $H(\mathbb{D})$ by

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \frac{1}{c_n} a_n \bar{b}_n,$$

where $f(z) = \sum a_n z^n \in H(\mathbb{D})$ and $g(z) = \sum b_n z^n \in H(\mathbb{D})$. Let $R(z) = \sum_{n=0}^{\infty} c_n z^n$. Denote by $\mathcal{H}_R$ for the Hilbert space of analytic functions with

$$\langle f, f \rangle = \|f\|^2 < \infty.$$

Let $R_\zeta(z) = R(\bar{\zeta} z)$, $\zeta \in \mathbb{D}$. Then $R_\zeta(z)$ is the reproducing kernel of $\mathcal{H}_R$ at $\zeta \in \mathbb{D}$, that is, $f(\zeta) = \langle f, R_\zeta \rangle$ for any $f \in \mathcal{H}_R$.

Lemma 1. Let ρ be of finite upper type less than 1. Set

$$R^\rho(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{n\rho(\frac{1}{n})} z^n, \ z \in \mathbb{D}.$$

Then $\mathcal{H}_{R^\rho} = \mathcal{D}_\rho$ and $R_\zeta^\rho(z) = R^\rho(\bar{\zeta} z)$ is the reproducing kernel for $\mathcal{H}_{R^\rho}$ space at $\zeta \in \mathbb{D}$. Moreover, when $f(z) = \sum_{n=0}^{\infty} a_n z^n$,

$$\|f\|_{\mathcal{D}_\rho}^2 \approx |a_0|^2 + \sum_{n=1}^{\infty} n\rho\left(\frac{1}{n}\right)|a_n|^2.$$

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$\begin{aligned} \|f\|_{\mathcal{D}_\rho}^2 &= |a_0|^2 + \int_{\mathbb{D}} |f'(z)|^2 \rho\left(1 - |z|^2\right) dA(z) \\ &\approx |a_0|^2 + \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_0^1 r^{2n-1} \rho(1 - r^2) dr. \end{aligned}$$

By [9, Lemma 2], for $n > 0$, we have

$$\int_0^1 r^{2n-1} \rho(1 - r^2) dr \approx \int_0^1 r^{2n-1} \rho\left(\log \frac{1}{r}\right) dr \approx \frac{1}{n} \rho\left(\frac{1}{n}\right),$$

the constants occur here depend only on ρ. Therefore,

$$\|f\|_{\mathcal{D}_\rho}^2 \approx |a_0|^2 + \sum_{n=1}^{\infty} n\rho\left(\frac{1}{n}\right)|a_n|^2.$$

By the definition, $f \in \mathcal{H}_{R^\rho}$ if and only if

$$\|f\|_{\mathcal{H}_{R^\rho}}^2 = \langle f, f \rangle = |a_0|^2 + \sum_{n=1}^{\infty} n\rho\left(\frac{1}{n}\right)|a_n|^2 < \infty.$$

Thus, $\mathcal{H}_{R^\rho} = \mathcal{D}_\rho$. The proof is complete. □

Lemma 2. Let ρ be of finite lower type greater than 0 and upper type less than 1. Then there exist constants C_1 and C_2 which depending only on ρ such that

$$C_1\left(1 + \sum_{n=1}^{\infty} \frac{1}{n\rho(\frac{1}{n})} t^n\right) \le \frac{1}{\rho(1-t)} \le C_2\left(1 + \sum_{n=1}^{\infty} \frac{1}{n\rho(\frac{1}{n})} t^n\right)$$

for all $0 \le t < 1$.

Proof. Without loss of generality, we assume that $1/2 < t < 1$. Then,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n\rho(\frac{1}{n})} t^n &\approx \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{t^{\frac{1}{x}}}{x\rho(x)} dx \approx \int_0^1 \frac{t^{\frac{1}{x}}}{x\rho(x)} dx \\
&\approx \int_1^{\infty} \frac{t^y}{y\rho(\frac{1}{y})} dy \quad \left(y = \frac{x}{-\ln t}\right) \\
&\approx \int_{-\ln t}^{\infty} \frac{e^{-x}}{x\rho(\frac{1}{x}\ln\frac{1}{t})} dx \\
&\approx \frac{1}{\rho(\ln\frac{1}{t})} \int_{-\ln t}^{\infty} \frac{e^{-x}\rho(\ln\frac{1}{t})}{x\rho(\frac{1}{x}\ln\frac{1}{t})} dx.
\end{aligned}$$

Since ρ is of finite lower type greater than 0 and upper type less than 1, there exist γ and δ, satisfied $0 < \gamma < \delta < 1$, such that

$$\rho(st) \lesssim s^{\gamma}\rho(t), \quad s \le 1, \tag{1}$$

and

$$\rho(st) \lesssim s^{\delta}\rho(t), \quad s \ge 1, \tag{2}$$

where $0 < t < \infty$. Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n\rho(\frac{1}{n})} t^n &\lesssim \frac{1}{\rho(\ln\frac{1}{t})} \left(\int_0^{\infty} e^{-x} x^{\gamma-1} dx + \int_0^{\infty} e^{-x} x^{\delta-1} dx \right) \\
&\approx \frac{1}{\rho(1-t)} \left(\Gamma(\gamma) + \Gamma(\delta) \right),
\end{aligned}$$

where $\Gamma(.)$ is the Gamma function. Hence

$$1 + \sum_{n=1}^{\infty} \frac{1}{n\rho(\frac{1}{n})} t^n \lesssim \frac{1}{\rho(1-t)}.$$

On the other hand, since ρ is nondecreasing, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n\rho(\frac{1}{n})} t^n &\approx \frac{1}{\rho(\ln\frac{1}{t})} \int_{-\ln t}^{\infty} \frac{e^{-x}\rho(\ln\frac{1}{t})}{x\rho(\frac{1}{x}\ln\frac{1}{t})} dx \\
&\gtrsim \frac{1}{\rho(\ln\frac{1}{t})} \int_{\ln 2}^{\infty} \frac{e^{-x}\rho(\ln\frac{1}{t})}{x\rho(\frac{1}{x}\ln\frac{1}{t})} dx \\
&\gtrsim \frac{1}{\rho(1-t)} \int_{\ln 2}^{\infty} e^{-x} x^{-1} dx \approx \frac{1}{\rho(1-t)}.
\end{aligned}$$

The proof is complete. □

Lemma 3. Let ρ be of finite lower type greater than 0 and upper type less than 1. Let $r_z^{\rho} = \frac{R_{\xi}^{\rho}(z)}{\|R_{\xi}^{\rho}(z)\|_{D_\rho}}$ denote the normalized reproducing kernel for $\mathcal{H}_{R^\rho}$. Then $r_z^{\rho} \to 0$ weakly in $\mathcal{D}_\rho$ as $|z| \to 1$.

Proof. By Lemmas 1 and 2, we have

$$\|R_z^\rho\|_{\mathcal{D}_\rho} \approx \frac{1}{\sqrt{\rho(1-|z|^2)}}.$$

Take any complex polynomial p, we deduce that

$$\langle p, r_z^\rho\rangle = \langle p, \frac{R_z^\rho}{\|R_z^\rho\|_{\mathcal{D}_\rho}}\rangle = \frac{p(z)}{\|R_z^\rho\|_{\mathcal{D}_\rho}} \approx p(z)\sqrt{\rho(1-|z|^2)}.$$

Since a polynomial is bounded on $\mathbb{D}$, we obtain

$$\lim_{|z|\to 1}\langle p, r_z^\rho\rangle = 0.$$

It is well known that polynomials are dense in $\mathcal{D}_\rho$. So $r_z^\rho \to 0$ weakly in $\mathcal{D}_\rho$ as $|z| \to 1$. The proof is complete. □

Let μ be a finite positive Borel measure on $\mathbb{D}$. Recall that μ is a $\mathcal{D}_\rho$-Carleson measure if the inclusion map $i : \mathcal{D}_\rho \to L^2(\mu)$ is bounded, that is

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C\|f\|^2_{\mathcal{D}_\rho}$$

for all $f \in \mathcal{D}_\rho$. The best constant C, denoted by $\|\mu\|_\rho$, is said to be the norm of μ. The following lemma can be found in [9, Lemma 7].

Lemma 4. Let ρ be of finite lower type greater than 0 and upper type less than 1. Then $g \in H(\mathbb{D})$ is a multiplier of $\mathcal{D}_\rho$ if and only if $g \in H^\infty$ and $|g(z)|^2\rho(1-|z|^2)dA(z)$ is $\mathcal{D}_\rho$-Carleson measure.

Lemma 5. ([1]) Let φ be an analytic self-map of $\mathbb{D}$. Then, for any $z \in \mathbb{D}$,

$$\frac{1-|\varphi(0)|}{1+|\varphi(0)|} \leq \frac{1-|\varphi(z)|}{1-|z|}.$$

Lemma 6. Let ρ be of finite lower type greater than 0 and upper type less than 1. Suppose that φ is an automorphism on $\mathbb{D}$. Then $f \circ \varphi \in \mathcal{D}_\rho$ for all $f \in \mathcal{D}_\rho$.

Proof. Suppose that $\varphi(z) = \eta\frac{a-z}{1-\bar{a}z}$, where $a, z \in \mathbb{D}$ and $|\eta| = 1$. Then

$$\begin{aligned}
\|C_\varphi f\|^2_{\mathcal{D}_\rho} =& |f\circ\varphi(0)|^2 + \int_{\mathbb{D}} |(f\circ\varphi)'(z)|^2\rho(1-|z|^2)dA(z)\\
=& |\langle f, R^\rho_{\varphi(0)}\rangle|^2 + \int_{\varphi(\mathbb{D})} |f'(w)|^2\rho(1-|\varphi^{-1}(w)|^2)dA(w)\\
=& |\langle f, R^\rho_{\varphi(0)}\rangle|^2 + \int_{\varphi(\mathbb{D})} |f'(w)|^2\rho(1-|w|^2)\frac{\rho(1-|\varphi^{-1}(w)|^2)}{\rho(1-|w|^2)}dA(w),
\end{aligned}$$

where $z = \varphi^{-1}(w)$. Noting that

$$\|R^\rho_{\varphi(0)}\|_{\mathcal{D}_\rho} \approx \frac{1}{\sqrt{\rho(1-|a|^2)}},$$

we get

$$|\langle f, R^{\rho}_{\varphi(0)}\rangle|^2 \lesssim \|f\|^2_{\mathcal{D}_\rho}\|R^{\rho}_{\varphi(0)}\|^2_{\mathcal{D}_\rho} \approx \frac{\|f\|^2_{\mathcal{D}_\rho}}{\rho(1-|a|^2)} < \infty.$$

Since ρ is of finite lower type greater than 0 and upper type less than 1, similar to Lemma 2, we can deduce that

$$\begin{aligned}\frac{\rho(1-|\varphi^{-1}(w)|^2)}{\rho(1-|w|^2)} &\lesssim \left(\frac{1-|\varphi^{-1}(w)|^2}{1-|w|^2}\right)^{\gamma} + \left(\frac{1-|\varphi^{-1}(w)|^2}{1-|w|^2}\right)^{\delta} \\ &\lesssim \left(\frac{1-|\varphi^{-1}(w)|}{1-|w|}\right)^{\gamma} + \left(\frac{1-|\varphi^{-1}(w)|}{1-|w|}\right)^{\delta}.\end{aligned}$$

Combined with Lemma 5 again, we obtain

$$\frac{1-|\varphi^{-1}(w)|}{1-|w|} = \frac{1-|z|}{1-|\varphi(z)|} \leq \frac{1+|a|}{1-|a|}.$$

That is,

$$\frac{\rho(1-|\varphi^{-1}(w)|^2)}{\rho(1-|w|^2)} \lesssim \left(\frac{1+|a|}{1-|a|}\right)^{\gamma} + \left(\frac{1+|a|}{1-|a|}\right)^{\delta}.$$

Noting the fact that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, we get the desired result. The proof is complete. □

Lemma 7. [16] *Suppose that T is a bounded linear operator on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent.*

(i) T preserves frames on $\mathcal{H}$.

(ii) T is surjective on $\mathcal{H}$.

(iii) T is bounded below on $\mathcal{H}$.

Lemma 8. [16] *Suppose that T is a bounded linear operator on a Hilbert space $\mathcal{H}$. Then T preserves tight frames on $\mathcal{H}$ if and only if there is constant $\lambda > 0$ such that $\|T^*f\|_{\mathcal{H}} = \lambda\|f\|_{\mathcal{H}}$ for any $f \in \mathcal{H}$.*

Lemma 9. [16] *Suppose that T is a bounded linear operator on a Hilbert space $\mathcal{H}$. Then T preserves normalized tight frames on $\mathcal{H}$ if and only if $\|T^*f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}$ for any $f \in \mathcal{H}$.*

The following results can be deduced by [18, Corollary 3.6].

Lemma 10. *Suppose that $\psi \in H(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$ such that $W_{\psi,\varphi}$ is bounded on a Hilbert space $\mathcal{H}_\gamma$ ($0 < \gamma < \infty$) with reproducing kernel functions $\frac{1}{(1-\overline{w}z)^\gamma}$, $w, z \in \mathbb{D}$. Then the following statements are equivalent.*

(i) $W_{\psi,\varphi}$ is co-isometry on $\mathcal{H}_\gamma$, that is, $W_{\psi,\varphi}W^*_{\psi,\varphi} = I$.

(ii) $W_{\psi,\varphi}$ is an unitary operator on $\mathcal{H}_\gamma$.

(iii) φ is an automorphism on $\mathbb{D}$ and $\psi = \xi r^{\alpha}_{\varphi^{-1}(0)}$, where $|\xi| = 1$.

3. Main results and proofs

In this section, we state and prove the main result in this paper.

Theorem 1. *Let ρ be of finite lower type greater than* 0 *and upper type less than* 1. *Suppose that $\psi \in H(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$ such that $W_{\psi,\varphi}$ is bounded on $\mathcal{D}_\rho$. Then the following statements are equivalent.*

(i) $W_{\psi,\varphi}$ preserves frames on $\mathcal{D}_\rho$.

(ii) $W_{\psi,\varphi}$ is surjective on $\mathcal{D}_\rho$.

(iii) $W^*_{\psi,\varphi}$ is bounded below on $\mathcal{D}_\rho$.

(iv) ψ and $\frac{1}{\psi}$ are multipliers of $\mathcal{D}_\rho$ and φ is an automorphism of $\mathbb{D}$.

(v) $W_{\psi,\varphi}$ is invertible on $\mathcal{D}_\rho$.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$. These implications can be deduced by Lemma 7.

$(iii) \Rightarrow (iv)$. Let $w \in \mathbb{D}$ and R^ρ_z be the reproducing kernel function in $\mathcal{D}_\rho$. After a calculation,

$$\begin{aligned} W^*_{\psi,\varphi}(R^\rho_z)(w) =& \langle W^*_{\psi,\varphi}(R^\rho_z), R^\rho_w\rangle = \langle R^\rho_z, W_{\psi,\varphi}(R^\rho_w)\rangle \\ =& \langle R^\rho_z, \psi \cdot R^\rho_w \circ \varphi\rangle = \overline{\psi(z)R^\rho_w(\varphi(z))} \\ =& \overline{\psi(z)}R^\rho_{\varphi(z)}(w). \end{aligned}$$

By the assumption that $W_{\psi,\varphi}$ is bounded and $W^*_{\psi,\varphi}$ is bounded below on $\mathcal{D}_\rho$, we see that there is a constant $C > 0$ such that

$$\|W^*_{\psi,\varphi}f\|_{\mathcal{D}_\rho} \geq C\|f\|_{\mathcal{D}_\rho}, \quad f \in \mathcal{D}_\rho.$$

Thus,

$$\|W^*_{\psi,\varphi}(R^\rho_z)\|_{\mathcal{D}_\rho} \geq C\|R^\rho_z\|_{\mathcal{D}_\rho},$$

that is,

$$|\psi(z)|\|R^\rho_{\varphi(z)}\|_{\mathcal{D}_\rho} \geq C\|R^\rho_z\|_{\mathcal{D}_\rho}.$$

Therefore,

$$|\psi(z)| \geq C\frac{\|R^\rho_z\|_{\mathcal{D}_\rho}}{\|R^\rho_{\varphi(z)}\|_{\mathcal{D}_\rho}} \gtrsim \frac{\sqrt{\rho(1-|\varphi(z)|)}}{\sqrt{\rho(1-|z|)}}.$$

Since

$$\frac{\rho(1-|z|)}{\rho(1-|\varphi(z)|)} \lesssim \left(\frac{1-|z|}{1-|\varphi(z)|}\right)^\gamma + \left(\frac{1-|z|}{1-|\varphi(z)|}\right)^\delta,$$

we have

$$\frac{\sqrt{\rho(1-|\varphi(z)|)}}{\sqrt{\rho(1-|z|)}} \gtrsim \frac{1}{\sqrt{\left(\frac{1-|z|}{1-|\varphi(z)|}\right)^\gamma + \left(\frac{1-|z|}{1-|\varphi(z)|}\right)^\delta}}.$$

By Lemma 5, we obtain

$$\frac{1}{\left(\frac{1-|z|}{1-|\varphi(z)|}\right)^\gamma + \left(\frac{1-|z|}{1-|\varphi(z)|}\right)^\delta} \geq \frac{1}{\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^\gamma + \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^\delta}.$$

So,

$$|\psi(z)| \gtrsim \frac{\rho(1-|z|)}{\rho(1-|\varphi(z)|)} \gtrsim \frac{1}{\sqrt{\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\gamma} + \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\delta}}} > 0.$$

Hence, $\frac{1}{\psi}$ is bounded.

It is well known that a univalent inner function is an automorphism ([1, Corollary 3.8]). To prove that φ is an automorphism, we only need to prove that φ is an inner function and φ is univalent.

First, we prove that φ is an inner function. Since $W_{\psi,\varphi}$ is bounded on $\mathcal{D}_\rho$, applying $W_{\psi,\varphi}$ on the constant function 1, we have $\psi \in \mathcal{D}_\rho$. By Lemma 3, $r_z^\rho \to 0$ weakly in $\mathcal{D}_\rho$. Thus,

$$\lim_{|z|\to 1} \frac{\psi(z)}{\|R_z^\rho\|_{\mathcal{D}_\rho}} = \lim_{|z|\to 1} \left\langle \psi, r_z^\rho \right\rangle = 0.$$

Noting that

$$\frac{|\psi(z)|}{\|R_z^\rho\|_{\mathcal{D}_\rho}} = |\psi(z)| \sqrt{\rho(1-|z|)} \gtrsim \sqrt{\rho(1-|\varphi(z)|)},$$

we get

$$\lim_{|z|\to 1} \rho(1-|\varphi(z)|) = 0,$$

which implies that $\lim_{|z|\to 1} |\varphi(z)| = 1$. In other word, φ is an inner function.

Next we prove that φ is univalent. Suppose $\varphi(z) = \varphi(w)$, where $z, w \in \mathbb{D}$. Then clearly $R^\rho_{\varphi(z)} = R^\rho_{\varphi(w)}$. Since $|\psi| > 0$ and $W^*_{\psi,\varphi} R_z^\rho = \overline{\psi(z)} R^\rho_{\varphi(z)}$, we obtain

$$W^*_{\psi,\varphi}\left(\frac{R_z^\rho}{\psi(z)}\right) = W^*_{\psi,\varphi}\left(\frac{R_w^\rho}{\psi(w)}\right).$$

So $\frac{R_z^\rho}{\psi(z)} = \frac{R_w^\rho}{\psi(w)}$. Let $f = 1$. Then

$$\langle f, \frac{R_z^\rho}{\psi(z)} \rangle = \langle f, \frac{R_w^\rho}{\psi(w)} \rangle,$$

which implies that $\psi(z) = \psi(w)$. Hence, $R_z^\rho = R_w^\rho$. From $\langle \xi, R_z^\rho \rangle = \langle \xi, R_w^\rho \rangle$, we deduce that $z = w$, that is, φ is univalent. Hence, φ is an automorphism.

Since φ is an automorphism, φ^{-1} is also an automorphism. By Lemma 6, $C_{\varphi^{-1}}$ is also bounded on $\mathcal{D}_\rho$. Therefore, $W_{\psi,\varphi} \circ C_{\varphi^{-1}}$ is bounded. For any $f \in \mathcal{D}_\rho$, since

$$W_{\psi,\varphi} \circ C_{\varphi^{-1}} f = \psi \cdot (f \circ \varphi^{-1} \circ \varphi) = \psi f,$$

we see that ψ is multipliers of $\mathcal{D}_\rho$. By Lemma 4, we known that $\psi \in H^\infty$. Moreover, noting that $\frac{1}{\psi} \in H^\infty$, by Lemma 4 again we have

$$\begin{aligned}
&\int_{\mathbb{D}} \left| \left(\frac{f(z)}{\psi(z)}\right)' \right|^2 \rho(1-|z|^2) dA(z) \\
&= \int_{\mathbb{D}} \left| \frac{f'(z)\psi(z) - f(z)\psi'(z)}{\psi^2(z)} \right|^2 \rho(1-|z|^2) dA(z) \\
&\lesssim \int_{\mathbb{D}} |f'(z)|^2 \rho(1-|z|^2) dA(z) + \int_{\mathbb{D}} |f(z)|^2 |\psi'(z)|^2 \rho(1-|z|^2) dA(z) \\
&\lesssim \int_{\mathbb{D}} |f'(z)|^2 \rho(1-|z|^2) dA(z),
\end{aligned}$$

which implies that $\frac{1}{\psi}$ is also a multiplier of $\mathcal{D}_\rho$.

$(iv) \Rightarrow (v)$. From [19, Theorem 3.3] and Lemmas 1 and 6, we only need to verified that

$$\liminf_{n\to\infty} \sqrt[n]{n\rho(\frac{1}{n})} = 1.$$

From (1), we have

$$\rho(1-|z|^2) \lesssim (1-|z|^2)^\gamma \rho(1). \tag{3}$$

Since $\gamma < \delta < 1$, there exist α satisfy $\delta < \alpha < 1$. From [13, Lemma 4], we known that ρ is of finite upper type less than $\alpha < 1$ if and only if

$$\int_t^\infty \rho(s)s^{-\alpha-1}ds \lesssim \rho(t)t^{-\alpha}, \quad 0 < t < \infty.$$

Noted that

$$\int_t^\infty \rho(s)s^{-\alpha-1}ds \geq \rho(t)\int_t^\infty s^{-\alpha-1}ds \approx \frac{\rho(t)}{t^\alpha}.$$

That is,

$$\rho_1(t) := \int_t^\infty \rho(s)s^{-\alpha-1}ds \approx \rho(t)t^{-\alpha}, \quad 0 < t < \infty,$$

and $\rho_1(t)$ is nonincreasing. Therefore,

$$\rho_1(1) \leq \rho_1(1-|z|^2) \approx \rho(1-|z|^2)(1-|z|^2)^{-\alpha}.$$

Thus,

$$\rho(1-|z|^2) \gtrsim (1-|z|^2)^\alpha. \tag{4}$$

Combine with (3) and (4), there exist positive constants C_1 and C_2 such that

$$\left(C_1\sqrt{n\left(\frac{1}{n}\right)^\alpha}\right)^{\frac{1}{n}} \leq \left(\sqrt{n\rho\left(\frac{1}{n}\right)}\right)^{\frac{1}{n}} \leq \left(C_2\sqrt{n\left(\frac{1}{n}\right)^\gamma}\right)^{\frac{1}{n}}.$$

Noting that

$$\lim_{n\to\infty}\sqrt[n]{C_1} = \lim_{n\to\infty}\sqrt[n]{C_2} = 1$$

and

$$\lim_{n\to\infty}\left(\sqrt{n\left(\frac{1}{n}\right)^\alpha}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\sqrt{n\left(\frac{1}{n}\right)^\gamma}\right)^{\frac{1}{n}} = 1,$$

we get the desired result.

$(v) \Rightarrow (iii)$. Since $W_{\psi,\varphi}$ is invertible on $\mathcal{D}_\rho$, we see that $W^*_{\psi,\varphi}$ is also invertible on $\mathcal{D}_\rho$, which implies that $W^*_{\psi,\varphi}$ is bounded below on $\mathcal{D}_\rho$. The proof is complete. □

Next we investigate equivalent characterizations of weighted composition operators $W_{\psi,\varphi}$ preserves normalized tight frames and tight frames on $\mathcal{D}_\rho$. However, we have to restrict ourself on the space $\mathcal{D}_\alpha$ when $0 < \alpha < 1$. Then, we also give the similar results like [16, Theorem 3.7 and Corollary 3.8].

Theorem 2. *Let $0 < \alpha < 1$, $\psi \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Suppose that $W_{\psi,\varphi}$ is bounded on $\mathcal{D}_\alpha$. Then the following statements are equivalent.*

(i) $W_{\psi,\varphi}$ *preserves normalized tight frames on* $\mathcal{D}_\alpha$.

(ii) $W^*_{\psi,\varphi}$ *is an isometry on* $\mathcal{D}_\alpha$.

(iii) $W_{\psi,\varphi}$ *is an unitary operator on* $\mathcal{D}_\alpha$.

(iv) φ *is an automorphism on* $\mathbb{D}$ *and* $\psi = \xi r^\alpha_{\varphi^{-1}(0)}$, *where* $|\xi| = 1$.

Proof. $(i) \Leftrightarrow (ii)$. It follows from Lemma 9.

$(ii) \Leftrightarrow (iv) \Leftrightarrow (iii)$. Noting the fact that the $\mathcal{D}_\alpha$ space is a Hilbert space with the following reproducing kernel:

$$R_w(z) = \frac{1}{(1 - \overline{w}z)^\alpha},$$

then the result can be deduced by Lemma 10. □

Theorem 3. *Let* $0 < \alpha < 1$, $\psi \in H(\mathbb{D})$ *and* φ *be an analytic self-map of* $\mathbb{D}$. *Suppose that* $W_{\psi,\varphi}$ *is bounded on* $\mathcal{D}_\alpha$. *Then the following statements are equivalent.*

(i) $W_{\psi,\varphi}$ *preserves tight frames on* $\mathcal{D}_\alpha$.

(ii) There is a constant $c > 0$ *such that* $\|W^*_{\psi,\varphi} f\|_{\mathcal{D}_\alpha} = c\|f\|_{\mathcal{D}_\alpha}$ *for any* $f \in \mathcal{D}_\alpha$.

(iii) φ *is an automorphism on* $\mathbb{D}$ *and there is a complex number* s *such that* $\psi = s r^\alpha_{\varphi^{-1}(0)}$.

Proof. $(i) \Leftrightarrow (ii)$. It follows by Lemma 8.

$(ii) \Rightarrow (iii)$. Since $\frac{1}{c} W_{\psi,\varphi} = W_{\frac{\psi}{c},\varphi}$, from $\|W^*_{\psi,\varphi} f\|_{\mathcal{D}_\alpha} = c\|f\|_{\mathcal{D}_\alpha}$, we deduce that

$$\|W^*_{\frac{\psi}{c},\varphi} f\|_{\mathcal{D}_\alpha} = \|f\|_{\mathcal{D}_\alpha}.$$

In other word, $W^*_{\frac{\psi}{c},\varphi}$ is an isometry on $\mathcal{D}_\alpha$. By Theorem 2, we get that φ is an automorphism on $\mathbb{D}$ and there exists a complex number ξ with $|\xi| = 1$ such that $\frac{\psi}{c} = \xi r^\alpha_{\varphi^{-1}(0)}$. Setting $s = c\xi$, we get the desired result.

$(iii) \Rightarrow (ii)$. Let $\phi = \frac{\psi}{|s|}$. Then by Theorem 2, $W^*_{\phi,\varphi}$ is an isometry on $\mathcal{D}_\alpha$, which implies that

$$\|W^*_{\phi,\varphi} f\|_{\mathcal{D}_\alpha} = \|f\|_{\mathcal{D}_\alpha}, \quad f \in \mathcal{D}_\alpha.$$

This is clearly the same as

$$\|W^*_{\psi,\varphi} f\|_{\mathcal{D}_\alpha} = |s|\|f\|_{\mathcal{D}_\alpha}, \quad f \in \mathcal{D}_\alpha.$$

The proof is complete. □

4. Conclusions

In this paper, we mainly prove that the weighted composition operator $W_{\psi,\varphi}$ is invertible on Dirichlet type spaces $\mathcal{D}_\rho$ if and only if it preserve frames. We also show that $W_{\psi,\varphi}$ is an unitary operator if and only if $W_{\psi,\varphi}$ preserve normalized tight frames on $\mathcal{D}_\alpha$ $(0 < \alpha < 1)$. Weighted composition operators preserve tight frames on $\mathcal{D}_\alpha$ $(0 < \alpha < 1)$ are also investigated.

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Conflict of interest

We declare that we have no conflict of interest.

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