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Research article

Existence and multiplicity of solutions for a class of damped-like fractional differential system

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Abstract: In this paper, we obtain some results about the existence and multiplicity of weak solutions for a class of damped-like fractional differential system with a parameter λ. When the nonlinear term is subquadratic only near the origin, we obtain that system has a ground state weak solution u_λ if λ is in some given interval, and when the nonlinear term is also even near the origin, then for each $\lambda > 0$, system has infinitely many weak solutions $\{u_n^\lambda\}$ with $\|u_n^\lambda\| \to 0$ as $n \to \infty$. We mainly use Ekeland's variational principle and a variant of Clark's theorem together with a cut-off technique to prove our results.

Keywords: Damped-like fractional differential system; Riemann-Liouville fractional integrals; Ekeland's variational principle; Clark's theorem
Mathematics Subject Classification: 34B15, 34B10

1. Introduction and main results

In this paper, we are concerned with the existence and multiplicity of weak solutions for the damped-like fractional differential system

$$\begin{cases} -\dfrac{d}{dt}\left(p(t)\left(\dfrac{1}{2}\,{}_0D_t^{-\xi}(u'(t)) + \dfrac{1}{2}\,{}_tD_T^{-\xi}(u'(t))\right)\right) \\ \qquad\qquad +r(t)\left(\dfrac{1}{2}\,{}_0D_t^{-\xi}(u'(t)) + \dfrac{1}{2}\,{}_tD_T^{-\xi}(u'(t))\right) + q(t)u(t) = \lambda\nabla F(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where ${}_0D_t^{-\xi}$ and ${}_tD_T^{-\xi}$ are the left and right Riemann-Liouville fractional integrals of order $0 \le \xi < 1$, respectively, $p, r, q \in C([0,T],\mathbb{R})$, $L(t) := \int_0^t (r(s)/p(s))ds$, $0 < m \le e^{-L(t)}p(t) \le M$ and $q(t) - p(t) \ge 0$

for a.e. $t \in [0, T]$, $u(t) = (u_1(t), u_2(t) \cdots, u_n(t))^T$, $(\cdot)^T$ denotes the transpose of a vector, $n \geq 1$ is a given positive integer, $\lambda > 0$ is a parameter, $\nabla F(t, x)$ is the gradient of F with respect to $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, that is, $\nabla F(t, x) = (\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n})^T$, and there exists a constant $\delta \in (0, 1)$ such that $F : [0, T] \times \overline{B_0^\delta} \to \mathbb{R}$ (where $\overline{B_0^\delta}$ is a closed ball in $\mathbb{R}^N$ with center at 0 and radius δ) satisfies the following condition

(F_0) *$F(t, x)$ is continuously differentiable in $\overline{B_0^\delta}$ for a.e. $t \in [0, T]$, measurable in t for every $x \in \overline{B_0^\delta}$, and there are $a \in C(\overline{B_0^\delta}, \mathbb{R}^+)$ and $b \in L^1([0, T]; \mathbb{R}^+)$ such that*

$$|F(t, x)| \leq a(|x|)b(t)$$

and

$$|\nabla F(t, x)| \leq a(|x|)b(t)$$

for all $x \in \overline{B_0^\delta}$ and a.e. $t \in [0, T]$.

In recent years, critical point theory has been extensively applied to investigate the existence and multiplicity of fractional differential equations. An successful application to ordinary fractional differential equations with Riemann-Liouville fractional integrals was first given by [1], in which they considered the system

$$\begin{cases} -\dfrac{d}{dt}\left(\left(\dfrac{1}{2}\, {}_0D_t^{-\xi}(u'(t)) + \dfrac{1}{2}\, {}_tD_T^{-\xi}(u'(t))\right)\right) = \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0. \end{cases} \tag{1.2}$$

They established the variational structure and then obtained some existence results for system (1.2). Subsequently, this topic attracted lots of attention and a series of existence and multiplicity results are established (for example, see [2–12] and reference therein). It is obvious that system (1.1) is more complicated than system (1.2) because of the appearance of damped-like term

$$r(t)\left(\frac{1}{2}\, {}_0D_t^{-\xi}(u'(t)) + \frac{1}{2}\, {}_tD_T^{-\xi}(u'(t))\right).$$

In [13], the variational functional for system (1.1) with $\lambda = 1$ and $N = 1$ has been established, and in [14], they investigated system (1.1) with $\lambda = 1$, $N = 1$ and an additional perturbation term. By mountain pass theorem and symmetric mountain pass theorem in [15] and a local minimum theorem in [16], they obtained some existence and multiplicity results when F satisfies superquadratic growth at infinity and some other reasonable conditions at origin.

In this paper, motivated by the idea in [17, 18], being different from those in [13, 14], we consider the case that F has subquadratic growth only near the origin and no any growth condition at infinity. Our main tools are Ekeland's variational principle in [19], a variant of Clark's theorem in [17] and a cut-off technique in [18]. We obtain that system (1.1) has a ground state weak solution u_λ if λ is in some given interval and then some estimates of u_λ are given, and when $F(t, x)$ is also even about x near the origin for a.e. $t \in [0, T]$, for each given $\lambda > 0$, system (1.1) has infinitely many weak solutions $\{u_n^\lambda\}$ with $\|u_n^\lambda\| \to 0$ as $n \to \infty$. Next, we make some assumptions and state our main results.

(f_0) There exist constants $M_1 > 0$ and $0 < p_1 < 2$ such that

$$F(t, x) \geq M_1 |x|^{p_1} \tag{1.3}$$

for all $x \in \overline{B_0^\delta}$ and a.e. $t \in [0, T]$.
(f_1) There exist constants $M_2 > 0$ and $0 < p_2 < p_1 < 2$ such that

$$F(t, x) \leq M_2 |x|^{p_2} \tag{1.4}$$

for all $x \in \overline{B_0^\delta}$ and a.e. $t \in [0, T]$.
$(f_0)'$ There exist constants $M_1 > 0$ and $0 < p_1 < 1$ such that (1.3) holds.
$(f_1)'$ There exist constants $M_2 > 0$ and $0 < p_2 < p_1 < 1$ such that (1.4) holds.
(f_2) There exists a constant $\eta \in (0, 2)$ such that

$$(\nabla F(t, x), x) \leq \eta F(t, x)$$

for all $x \in \overline{B_0^\delta}$ and a.e. $t \in [0, T]$.
(f_3) $F(t, x) = F(t, -x)$ for all $x \in \overline{B_0^\delta}$ and a.e. $t \in [0, T]$.

Theorem 1.1. *Suppose that* (F_0)*,* (f_0)*,* (f_1) *and* (f_2) *hold. If*

$$0 < \lambda \leq \min\left\{\frac{|\cos(\pi\alpha)|}{2C}, \left(\frac{1}{B}\right)^{2-p_2}\left(\frac{\delta}{2}\right)^{2-p_2}\frac{|\cos(\pi\alpha)|}{2C}\right\},$$

then system (1.1) has a ground state weak solution u_λ *satisfying*

$$\|u_\lambda\|^{2-p_2} \leq \min\left\{1, \left(\frac{1}{B}\right)^{2-p_2}\left(\frac{\delta}{2}\right)^{2-p_2}\right\}, \quad \|u_\lambda\|_\infty^{2-p_2} \leq \left\{B^{2-p_2}, \left(\frac{\delta}{2}\right)^{2-p_2}\right\}.$$

where

$$B = \frac{T^{\frac{2\alpha-1}{2}}}{\sqrt{m}\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}, \quad C = \max\{p, \eta\}\max\{M_1, M_2\}T\max_{t\in[0,T]} e^{-L(t)}\max\{B^{p_1}, B^{p_2}\}.$$

If (f_0) and (f_1) are replaced by the stronger conditions $(f_0)'$ and $(f_1)'$, then (f_2) is not necessary in Theorem 1.1. So we have the following result.

Theorem 1.2. *Suppose that* (F_0)*,* $(f_0)'$ *and* $(f_1)'$ *hold. If*

$$0 < \lambda \leq \min\left\{\frac{|\cos(\pi\alpha)|}{3C^*}, \left(\frac{1}{B}\right)^{2-p_1}\left(\frac{\delta}{2}\right)^{2-p_1}\frac{|\cos(\pi\alpha)|}{3C^*}\right\},$$

then system (1.1) has a ground state weak solution u_λ *satisfying*

$$\|u_\lambda\|^{2-p_1} \leq \min\left\{1, \left(\frac{1}{B}\right)^{2-p_1}\left(\frac{\delta}{2}\right)^{2-p_1}\right\}, \quad \|u_\lambda\|_\infty^{2-p_1} \leq \left\{B^{2-p_1}, \left(\frac{\delta}{2}\right)^{2-p_1}\right\},$$

where $C^* = \max_{t\in[0,T]} e^{-L(t)}\max\left\{(1+\rho_0)a_0 B\int_0^T b(t)dt, M_1 p_1 T B^{p_1}, \rho_0 M_1 T B^{p_1+1}\right\}$*,* $a_0 = \max_{s\in[0,\delta]} a(s)$ *and* $\rho_0 = \max_{s\in[\frac{\delta}{2},\delta]} |\rho'(s)|$ *and* $\rho(s) \in C^1(\mathbb{R}, [0, 1])$ *is any given even cut-off function satisfying*

$$\rho(s) = \begin{cases} 1, & \text{if } |s| \leq \delta/2, \\ 0, & \text{if } |s| > \delta. \end{cases} \tag{1.5}$$

Theorem 1.3. *Suppose that* (F_0), (f_0), (f_1) *and* (f_3) *hold. Then for each* $\lambda > 0$, *system (1.1) has a sequence of weak solutions* $\{u_n^\lambda\}$ *satisfying* $\{u_n^\lambda\} \to 0$, *as* $n \to \infty$.

Remark 1.1. Theorem 1.1-Theorem 1.3 still hold even if $r(t) \equiv 0$ for all $t \in [0, T]$, that is, the damped-like term disappears, which are different from those in [2–12] because all those assumptions with respect to x in our theorems are made only near origin without any assumption near infinity.

The paper is organized as follows. In section 2, we give some preliminary facts. In section 3, we prove Theorem 1.1–Theorem 1.3.

2. Preliminaries

In this section, we introduce some definitions and lemmas in fractional calculus theory. We refer the readers to [1, 9, 20–22]. We also recall Ekeland's variational principle in [19] and the variant of Clark's theorem in [17].

Definition 2.1. (Left and Right Riemann-Liouville Fractional Integrals [22]) Let f be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional integrals of order γ for function f denoted by ${}_aD_t^{-\gamma} f(t)$ and ${}_tD_b^{-\gamma} f(t)$, respectively, are defined by

$$
\begin{aligned}
{}_aD_t^{-\gamma} f(t) &= \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s)ds, \quad t \in [a, b], \quad \gamma > 0,\\
{}_tD_b^{-\gamma} f(t) &= \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} f(s)ds, \quad t \in [a, b], \quad \gamma > 0.
\end{aligned}
$$

provided the right-hand sides are pointwise defined on $[a, b]$, where $\Gamma > 0$ is the Gamma function.
Definition 2.2. ([22]) For $n \in \mathbb{N}$, if $\gamma = n$, Definition 2.1 coincides with nth integrals of the form

$$
\begin{aligned}
{}_aD_t^{-n} f(t) &= \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)ds, \quad t \in [a, b], \quad n \in \mathbb{N},\\
{}_tD_b^{-n} f(t) &= \frac{1}{(n-1)!} \int_t^b (t-s)^{n-1} f(s)ds, \quad t \in [a, b], \quad n \in \mathbb{N}.
\end{aligned}
$$

Definition 2.3. (Left and Right Riemann-Liouville Fractional Derivatives [22]) Let f be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional derivatives of order γ for function f denoted by ${}_aD_t^{\gamma} f(t)$ and ${}_tD_b^{\gamma} f(t)$, respectively, are defined by

$$
\begin{aligned}
{}_aD_t^{\gamma} f(t) &= \frac{d^n}{dt^n} {}_aD_t^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \left(\int_a^t (t-s)^{n-\gamma-1} f(s)ds \right),\\
{}_tD_b^{\gamma} f(t) &= (-1)^n \frac{d^n}{dt^n} {}_tD_b^{\gamma-n} f(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \left(\int_t^b (s-t)^{n-\gamma-1} f(s)ds \right).
\end{aligned}
$$

where $t \in [a, b], n-1 \le \gamma < n$ and $n \in \mathbb{N}$. In particular, if $0 \le \gamma < 1$, then

$$
{}_aD_t^{\gamma} f(t) = \frac{d}{dt} {}_aD_t^{\gamma-1} f(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \left(\int_a^t (t-s)^{-\gamma} f(s)ds \right), \quad t \in [a, b],
$$

$$ {}_tD_b^{\gamma}f(t) = -\frac{d}{dt}{}_tD_b^{\gamma-1}f(t) = -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\left(\int_t^b (s-t)^{-\gamma}f(s)ds\right), \quad t\in[a,b]. $$

Remark 2.1. ([9, 13]) The left and right Caputo fractional derivatives are defined by the above-mentioned Riemann-Liuville fractional derivative. In particular, they are defined for function belonging to the space of absolutely continuous functions, which we denote by $AC([a,b],\mathbb{R}^{\mathbb{N}})$. $AC^k([a,b],\mathbb{R}^{\mathbb{N}})(k=0,1,...)$ are the space of the function f such that $f\in C^k([a,b],\mathbb{R}^{\mathbb{N}})$. In particular, $AC([a,b],\mathbb{R}^{\mathbb{N}}) = AC^1([a,b],\mathbb{R}^{\mathbb{N}})$.

Definition 2.4. (Left and Right Caputo Fractional Derivatives [22]) Let $\gamma \geq 0$ and $n\in\mathbb{N}$.
(i) If $\gamma\in(n-1,n)$ and $f\in AC^n([a,b],\mathbb{R}^{\mathbb{N}})$, then the left and right Caputo fractional derivatives of order γ for function f denoted by ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$ are represented by

$$ {}_a^cD_t^{\gamma}f(t) = {}_aD_t^{\gamma-n}f^n(t) = \frac{1}{\Gamma(n-\gamma)}\left(\int_a^t (t-s)^{n-\gamma-1}f^{(n)}(s)ds\right), $$
$$ {}_t^cD_b^{\gamma}f(t) = (-1)^n\, {}_tD_b^{\gamma-n}f^n(t) = \frac{(-1)^n}{\Gamma(n-\gamma)}\left(\int_t^b (s-t)^{n-\gamma-1}f^{(n)}(s)ds\right), $$

respectively, where $t\in[a,b]$. In particular, if $0<\gamma<1$, then

$$ {}_a^cD_t^{\gamma}f(t) = {}_aD_t^{\gamma-1}f'(t) = \frac{1}{\Gamma(1-\gamma)}\left(\int_a^t (t-s)^{-\gamma}f'(s)ds\right), \quad t\in[a,b], $$
$$ {}_t^cD_b^{\gamma}f(t) = -{}_tD_b^{\gamma-1}f'(t) = -\frac{1}{\Gamma(1-\gamma)}\left(\int_t^b (s-t)^{-\gamma}f'(s)ds\right), \quad t\in[a,b]. $$

(ii) If $\gamma = n-1$ and $f\in AC^n([a,b],\mathbb{R}^N)$, then ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$ are represented by

$$ {}_a^cD_t^{n-1}f(t) = f^{(n-1)}(t), \quad t\in[a,b], $$
$$ {}_t^cD_b^{n-1}f(t) = (-1)^{n-1}f^{(n-1)}(t), \quad t\in[a,b]. $$

In particular, ${}_a^cD_t^0f(t) = {}_t^cD_b^0f(t) = f(t),\ t\in[a,b]$.

Lemma 2.1. ([22]) The left and right Riemann-Liouville fractional integral operators have the property of a semigroup, i.e.

$$ {}_aD_t^{-\gamma_1}({}_aD_t^{-\gamma_2}f(t)) = {}_aD_t^{-\gamma_1-\gamma_2}f(t), $$
$$ {}_tD_b^{-\gamma_1}({}_tD_b^{-\gamma_2}f(t)) = {}_tD_b^{-\gamma_1-\gamma_2}f(t), \quad \forall\ \gamma_1,\gamma_2>0, $$

in any point $t\in[a,b]$ for continuous function f and for almost every point in $[a,b]$ if the function $f\in L^1([a,b],\mathbb{R}^N)$.

For $1\leq r<\infty$, define

$$ \|u\|_{L^r} = \left(\int_0^T |u(t)|^r dt\right)^{\frac{1}{r}} \tag{2.1} $$

and

$$ \|u\|_{\infty} = \max_{t\in[0,T]} |u(t)|. \tag{2.2} $$

Definition 2.5. ([1]) Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by closure of $C_0^\infty\left([0,T],\mathbb{R}^{\mathbb{N}}\right)$ with respect to the norm

$$\|u\|_{\alpha,p} = \left(\int_0^T |u(t)|^p dt + \int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\right)^{\frac{1}{p}}. \tag{2.3}$$

Remark 2.2. ([9]) $E_0^{\alpha,p}$ is the space of functions $u \in L^p([0,T],\mathbb{R}^{\mathbb{N}})$ having an α-order Caputo fractional derivative ${}_0^cD_t^\alpha u(t) \in L^p([0,T],\mathbb{R}^{\mathbb{N}})$ and $u(0) = u(T) = 0$.

Lemma 2.2. ([1]) Let $0 < \alpha \leq 1$ and $1 < p < \infty$. $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

Lemma 2.3. ([1]) Assume that $1 < p < \infty$ and $\alpha > \frac{1}{p}$. Then $E_0^{\alpha,p}$ compactly embedding in $C([0,T],\mathbb{R}^N)$.

Lemma 2.4. ([1]) Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, we have

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left\|{}_0^cD_t^\alpha u\right\|_{L^p}. \tag{2.4}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \left\|{}_0^cD_t^\alpha u\right\|_{L^p}. \tag{2.5}$$

Definition 2.6. ([13]) Assume that X is a Banach space. An operator $A : X \to X^*$ is of type $(S)_+$ if, for any sequence $\{u_n\}$ in X, $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0$ imply $u_n \to u$.

Let $\varphi : X \to \mathbb{R}$. A sequence $\{u_n\} \subset X$ is called (PS) sequence if the sequence $\{u_n\}$ satisfies

$$\varphi(u_n) \text{ is bounded}, \quad \varphi'(u_n) \to 0.$$

Furthermore, if every (PS) sequence $\{u_n\}$ has a convergent subsequence in X, then one call that φ satisfies (PS) condition.

Lemma 2.5. ([19]) Assume that X is a Banach space and $\varphi : X \to \mathbb{R}$ is Gâteaux differentiable, lower semi-continuous and bounded from below. Then there exists a sequence $\{x_n\}$ such that

$$\varphi(x_n) \to \inf_X \varphi, \quad \|\varphi'(x_n)\|_* \to 0.$$

Lemma 2.6. ([17]) Let X be a Banach space, $\varphi \in C^1(X,\mathbb{R})$. Assume φ satisfies the (PS) condition, is even and bounded below, and $\varphi(0) = 0$. If for any $k \in \mathbb{N}$, there exist a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k\cap S_{\rho_k}} \varphi < 0$, where $S_\rho = \{u \in X | \, \|u\| = \rho\}$, then at least one of the following conclusions holds.

(i) There exist a sequence of critical points $\{u_k\}$ satisfying $\varphi(u_k) < 0$ for all k and $\|u_k\| \to 0$ as $k \to \infty$.

(ii) There exists a constant $r > 0$ such that for any $0 < a < r$ there exists a critical point u such that $\|u\| = a$ and $\varphi(u) = 0$.

Remark 2.3. ([17]) Lemma 2.6 implies that there exist a sequence of critical points $u_k \neq 0$ such that $\varphi(u_k) \leq 0$, $\varphi(u_k) \to 0$ and $\|u_k\| \to 0$ as $k \to \infty$.

Now, we establish the variational functional defined on the space $E_0^{\alpha,2}$ with $\frac{1}{2} < \alpha \leq 1$. We follow the same argument as in [13] where the one-dimensional case $N = 1$ and $\lambda = 1$ for system (1.1) was investigated. For reader's convenience, we also present the details here. Note that $L(t) := \int_0^t (r(s)/p(s))ds, 0 < m \leq e^{-L(t)}p(t) \leq M$ and $q(t) - p(t) \geq 0$ for a.e. $t \in [0, T]$. Then system (1.1) is equivalent to the system

$$\begin{cases} -\dfrac{d}{dt}\left(e^{-L(t)}p(t)\left(\dfrac{1}{2}\,{}_0D_t^{-\xi}(u'(t)) + \dfrac{1}{2}\,{}_tD_T^{-\xi}(u'(t))\right)\right) \\ \qquad\qquad +e^{-L(t)}q(t)u(t) = \lambda e^{-L(t)}\nabla F(t,u), \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0. \end{cases} \tag{2.6}$$

By Lemma 2.1, for every $u \in AC([0, T], \mathbb{R}^N)$, it is easy to see that system (2.6) is equivalent to the system

$$\begin{cases} -\dfrac{d}{dt}\left[e^{-L(t)}p(t)\left(\dfrac{1}{2}\,{}_0D_t^{-\frac{\xi}{2}}\left({}_0D_t^{-\frac{\xi}{2}}u'(t)\right) + \dfrac{1}{2}\,{}_tD_T^{-\frac{\xi}{2}}\left({}_tD_T^{-\frac{\xi}{2}}u'(t)\right)\right)\right] \\ \qquad\qquad +e^{-L(t)}q(t)u(t) = \lambda e^{-L(t)}\nabla F(t,u), \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{2.7}$$

where $\xi \in [0, 1)$.

By Definition 2.4, we obtain that $u \in AC([0, T], \mathbb{R}^{\mathbb{N}})$ is a solution of problem (2.7) if and only if u is a solution of the following system

$$\begin{cases} -\dfrac{d}{dt}\left(e^{-L(t)}p(t)\left(\dfrac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - \dfrac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\right)\right) \\ \qquad\qquad +e^{-L(t)}q(t)u(t) = \lambda e^{-L(t)}\nabla F(t,u), \\ u(0) = u(T) = 0, \end{cases} \tag{2.8}$$

for a.e. $t \in [0, T]$, where $\alpha = 1 - \frac{\xi}{2} \in (\frac{1}{2}, 1]$. Hence, the solutions of system (2.8) correspond to the solutions of system (1.1) if $u \in AC([0, T], \mathbb{R}^{\mathbb{N}})$.

In this paper, we investigate system (2.8) in the Hilbert space $E_0^{\alpha,2}$ with the corresponding norm

$$\|u\| = \left(\int_0^T e^{-L(t)}p(t)\left(|{}_0^cD_t^{\alpha}u(t)|^2 + |u(t)|^2\right)dt\right)^{\frac{1}{2}}.$$

It is easy to see that $\|u\|$ is equivalent to $\|u\|_{\alpha,2}$ and

$$m\int_0^T |{}_0^cD_t^{\alpha}u(t)|^2 dt \leq \int_0^T e^{-L(t)}p(t)|{}_0^cD_t^{\alpha}u(t)|^2 dt \leq M\int_0^T |{}_0^cD_t^{\alpha}u(t)|^2 dt.$$

So

$$\|u\|_{L^2} \leq \frac{T^{\alpha}}{\sqrt{m}\Gamma(\alpha+1)}\left(\int_0^T e^{-L(t)}p\,(t)|{}_0^cD_t^{\alpha}u\,(t)|^2\,dt\right)^{\frac{1}{2}},$$

and

$$\|u\|_{\infty} \leq B\|u\|, \tag{2.9}$$

where

$$B = \frac{T^{\frac{2\alpha-1}{2}}}{\sqrt{m}\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}} > 0.$$

(see [13]).

Lemma 2.7. ([13]) If $\frac{1}{2} < \alpha \leq 1$, then for every $u \in E_0^{\alpha,2}$, we have

$$\begin{aligned} |\cos(\pi\alpha)|\|u\|^2 &\leq -\int_0^T e^{-L(t)} p(t) \left({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t)\right) dt + \int_0^T e^{-L(t)} p(t)|u(t)|^2 dt \\ &\leq \max\left\{\frac{M}{m|\cos(\pi\alpha)|}, 1\right\} \|u\|^2. \end{aligned} \tag{2.10}$$

3. Proofs

We follow the idea in [17] and [18]. We first modify and extend F to an appropriate $\widetilde{F}$ defined by

$$\widetilde{F}(t,x) = \rho(|x|)F(t,x) + (1-\rho(|x|))M_1|x|^{p_1}, \quad \text{for all } x \in \mathbb{R}^N,$$

where ρ is defined by (1.5).

Lemma 3.1. *Let* (F_0), (f_0), (f_1) *(or* $(f_0)'$, $(f_1)'$*),* (f_2) *and* (f_3) *be satisfied. Then*
$(\widetilde{F_0})$ $\widetilde{F}(t,x)$ *is continuously differentiable in* $x \in \mathbb{R}^N$ *for a.e.* $t \in [0,T]$*, measurable in* t *for every* $x \in \mathbb{R}^N$*, and there exists* $b \in L^1([0,T];\mathbb{R}^+)$ *such that*

$$\begin{aligned} &|\widetilde{F}(t,x)| \leq a_0 b(t) + M_1|x|^{p_1}, \\ &|\nabla\widetilde{F}(t,x)| \leq (1+\rho_0)a_0 b(t) + M_1 p_1|x|^{p_1-1} + \rho_0 M_1|x|^{p_1} \end{aligned}$$

for all $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$*;*
$(\widetilde{f_0})$ $\widetilde{F}(t,x) \geq M_1|x|^{p_1}$ *for all* $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$*;*
$(\widetilde{f_1})$ $\widetilde{F}(t,x) \leq \max\{M_1, M_2\}(|x|^{p_1} + |x|^{p_2})$ *for all* $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$*;*
$(\widetilde{f_2})$ $(\nabla\widetilde{F}(t,x), x) \leq \theta\widetilde{F}(t,x)$ *for all* $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$*, where* $\theta = \max\{p_1, \eta\}$*;*
$(\widetilde{f_3})$ $\widetilde{F}(t,x) = \widetilde{F}(t,-x)$ *for all* $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$.

Proof. We only prove $(\widetilde{f_0})$, $(\widetilde{f_1})$ and $(\widetilde{f_2})$. $(\widetilde{F_0})$ can be proved by a similar argument by (F_0). By the definition of $\widetilde{F}(t,x)$, (f_0) and (f_1) (or $(f_0)'$ and $(f_1)'$), we have

$$M_1|x|^{p_1} \leq \widetilde{F}(t,x) = F(t,x) \leq M_2|x|^{p_2}, \quad \text{if } |x| \leq \delta/2,$$

$$\widetilde{F}(t,x) = M_1|x|^{p_1}, \quad \text{if } |x| > \delta,$$

$$\widetilde{F}(t,x) \leq F(t,x) + M_1|x|^{p_1} \leq M_1|x|^{p_1} + M_2|x|^{p_2}, \quad \text{if } \delta/2 < |x| \leq \delta$$

and

$$\widetilde{F}(t,x) \geq \rho(|x|)M_1|x|^{p_1} + (1-\rho(|x|))M_1|x|^{p_1} = M_1|x|^{p_1}, \quad \text{if } \delta/2 < |x| \leq \delta.$$

Hence, $(\widetilde{f_1})$ holds. Note that

$$\theta\widetilde{F}(t,x) - (\nabla\widetilde{F}(t,x),x) = \rho(|x|)(\theta F(t,x) - (\nabla F(t,x),x)) + (\theta - p_1)(1-\rho(|x|))M_1|x|^{p_1}$$

$$-|x|\rho'(|x|)(F(t,x)-M_1|x|^{p_1}).$$

It is obvious that the conclusion holds for $0 \le |x| \le \delta/2$ and $|x| > \delta$. If $\delta/2 < |x| \le \delta$, by using $\theta \ge p_1$, (f_2), $(\widetilde{f_1})$ and the fact $s\rho'(s) \le 0$ for all $s \in \mathbb{R}$, we can get the conclusion $(\widetilde{f_2})$. Finally, since $\rho(|x|)$ is even for all $x \in \mathbb{R}^N$, by (f_3) and the definition of $\widetilde{F}(t,x)$, it is easy to get $(\widetilde{f_3})$. □

Remark 3.1. From the proof of Lemma 3.1, it is easy to see that (F_0), (f_0) (or $(f_0)'$) and (f_1) (or $(f_1)'$) independently imply $(\widetilde{F_0})$, $(\widetilde{f_0})$ and $(\widetilde{f_1})$, respectively.

Consider the modified system

$$\begin{cases} -\dfrac{d}{dt}\left(e^{-L(t)}p(t)\left(\dfrac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - \dfrac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\right)\right) \\ \qquad\qquad +e^{-L(t)}q(t)u(t) = \lambda e^{-L(t)}\nabla\widetilde{F}(t,u), \\ u(0)=u(T)=0, \end{cases} \tag{3.1}$$

for a.e. $t \in [0,T]$, where $\alpha = 1-\frac{\xi}{2} \in (\frac{1}{2},1]$.

If the equality

$$\int_0^T e^{-L(t)}\Big[-\frac{1}{2}p(t)(({}_0^cD_t^{\alpha}u(t),\ {}_t^cD_T^{\alpha}v(t)) + ({}_t^cD_T^{\alpha}u(t),\ {}_0^cD_t^{\alpha}v(t))) + p(t)(u(t),v(t)) \\ +(q(t)-p(t))(u(t),v(t)) - \lambda(\nabla\widetilde{F}(t,u(t)),v(t))\Big]dt = 0$$

holds for every $v \in E_0^{\alpha,2}$, then we call $u \in E_0^{\alpha,2}$ is a weak solution of system (3.1).

Define the functional $\widetilde{J}: E_0^{\alpha,2} \to \mathbb{R}$ by

$$\widetilde{J}(u) = \int_0^T e^{-L(t)}\Big[\frac{1}{2}p(t)\left(-({}_0^cD_t^{\alpha}u(t),\ {}_t^cD_T^{\alpha}u(t)) + |u(t)|^2\right) \\ +\frac{1}{2}(q(t)-p(t))|u(t)|^2 - \lambda\widetilde{F}(t,u(t))\Big]dt, \quad \text{for all } u \in E_0^{\alpha,2}.$$

Then $(\widetilde{F_0})$ and Theorem 6.1 in [9] imply that $\widetilde{J} \in C^1(E_0^{\alpha,2},\mathbb{R})$, and for every $u, v \in E_0^{\alpha,2}$, we have

$$\left\langle \widetilde{J}'(u),v\right\rangle = \int_0^T e^{-L(t)}\Big[-\frac{1}{2}p(t)\left(({}_0^cD_t^{\alpha}u(t),\ {}_t^cD_T^{\alpha}v(t)) + ({}_t^cD_T^{\alpha}u(t),\ {}_0^cD_t^{\alpha}v(t))\right) \\ +p(t)(u(t),v(t)) + (q(t)-p(t))(u(t),v(t)) - \lambda(\nabla\widetilde{F}(t,u(t)),v(t))\Big]dt.$$

Hence, a critical point of $\tilde{J}(u)$ corresponds to a weak solution of problem (3.1).

Let

$$\langle Au,v\rangle := \int_0^T e^{-L(t)}\Big[-\frac{1}{2}p(t)\left(({}_0^cD_t^{\alpha}u(t),\ {}_t^cD_T^{\alpha}v(t)) + ({}_t^cD_T^{\alpha}u(t),\ {}_0^cD_t^{\alpha}v(t))\right) \\ +p(t)(u(t),v(t)) + (q(t)-p(t))(u(t),v(t))\Big]dt.$$

Lemma 3.2. ([13])

$$\gamma_1\|u\|^2 \le \langle Au, u\rangle \le \gamma_2\|u\|^2, \quad \text{for all } u \in E_0^{\alpha,2}, \tag{3.2}$$

where $\gamma_1 = |\cos(\pi\alpha)|$ and $\gamma_2 = \left(\max\left\{\frac{M}{m|\cos\pi\alpha|}, 1\right\} + \max_{t\in[0,T]}(q(t) - p(t))\right)$.

Lemma 3.3. *Assume that* (F_0)*,* (f_0) *and* (f_1) *(or* $(f_0)'$ *and* $(f_1)'$*) hold. Then for each* $\lambda > 0$*,* $\widetilde{J}$ *is bounded from below on* $E_0^{\alpha,2}$ *and satisfies (PS) condition.*

Proof. By $(\widetilde{f_1})$, (2.9) and (3.2), we have

$$\begin{aligned}
\widetilde{J}(u) &= \frac{1}{2}\langle Au, u\rangle - \lambda\int_0^T e^{-L(t)}\widetilde{F}(t, u(t))dt \\
&\ge \frac{\gamma_1}{2}\|u\|^2 - \lambda\max\{M_1, M_2\}\int_0^T e^{-L(t)}(|u(t)|^{p_1} + |u(t)|^{p_2})dt \\
&\ge \frac{\gamma_1}{2}\|u\|^2 - \lambda\max\{M_1, M_2\}T\max_{t\in[0,T]} e^{-L(t)}(\|u\|_\infty^{p_1} + \|u\|_\infty^{p_2}) \\
&\ge \frac{\gamma_1}{2}\|u\|^2 - \lambda\max\{M_1, M_2\}T\max_{t\in[0,T]} e^{-L(t)}\left[B^{p_1}\|u\|^{p_1} + B^{p_2}\|u\|^{p_2}\right].
\end{aligned}$$

It follows from $0 < p_2 < p_1 < 2$ that

$$\widetilde{J}(u) \to +\infty, \quad \text{as } \|u\| \to \infty.$$

Hence, $\widetilde{J}$ is coercive and then is bounded from below. Now we prove that $\widetilde{J}$ satisfies the (PS) condition. Assume that $\{u_n\}$ is a (PS) sequence of $\widetilde{J}$, that is,

$$\widetilde{J}(u_n) \text{ is bounded}, \ \ \widetilde{J}'(u_n) \to 0. \tag{3.3}$$

Then by the coercivity of $\widetilde{J}$ and (3.3), there exists $C_0 > 0$ such that $\|u_n\| \le C_0$ and then by Lemma 2.3, there exists a subsequence (denoted again by $\{u_n\}$) such that

$$u_n \rightharpoonup u, \quad \text{weakly in } E_0^{\alpha,2}, \tag{3.4}$$

$$u_n \to u, \quad \text{a.e. in } C([0, T], \mathbb{R}). \tag{3.5}$$

Therefore, the boundness of $\{u_n\}$ and (3.3) imply that

$$\begin{aligned}
\left|\left\langle \widetilde{J}'(u_n), u_n - u\right\rangle\right| &\le \|\widetilde{J}'(u_n)\|_{(E_0^{\alpha,2})^*}\|u_n - u\|, \\
&\le \|\widetilde{J}'(u_n)\|_{(E_0^{\alpha,2})^*}(\|u_n\| + \|u\|) \\
&\to 0,
\end{aligned} \tag{3.6}$$

where $(E_0^{\alpha,2})^*$ is the dual space of $E_0^{\alpha,2}$, and $(\widetilde{F}_0)$, (2.9) together with (3.5) imply that

$$\begin{aligned}
&\left|\lambda\int_0^T (\nabla\widetilde{F}(t, u_n(t)), u_n(t) - u(t))dt\right| \\
\le\ & \lambda\int_0^T |\nabla\widetilde{F}(t, u_n(t))|\,|(u_n(t) - u(t))|dt
\end{aligned}$$

$$
\begin{aligned}
&\le \lambda\|u_n-u\|_\infty \int_0^T \left[(1+\rho_0)\,a_0 b(t) + M_1 p_1 |u_n(t)|^{p_1-1} + \rho_0 M_1 |u_n(t)|^{p_1}\right] dt \\
&\le \lambda\|u_n-u\|_\infty \left[(1+\rho_0)a_0 \int_0^T b(t)dt + M_1 p_1 T B^{p_1-1} C_0^{p_1-1} + M_1 T \rho_0 B^{p_1} C_0^{p_1}\right] \\
&\to 0. \qquad (3.7)
\end{aligned}
$$

Note that

$$
\left\langle \widetilde{J}'(u_n), u_n-u\right\rangle = \langle Au_n, u_n-u\rangle - \lambda \int_0^T \left(\nabla \widetilde{F}(t, u_n(t)), u_n(t)-u(t)\right)dt.
$$

Then (3.6) and (3.7) imply that $\lim_{n\to\infty} \langle Au_n, u_n-u\rangle = 0$. Moreover, by (3.4), we have

$$
\lim_{n\to\infty} \langle Au, u_n-u\rangle = 0.
$$

Therefore

$$
\lim_{n\to\infty} \langle Au_n - Au, u_n-u\rangle = 0.
$$

Since A is of type $(S)_+$ (see [13]), by Definition 2.6, we obtain $u_n \to u$ in $E_0^{\alpha,2}$. □

Define a Nehari manifold by

$$
\mathcal{N}_\lambda = \{u \in E_0^{\alpha,2}/\{0\} | \langle \widetilde{J}_\lambda'(u), u\rangle = 0\}.
$$

Lemma 3.4. *Assume that* (F_0) *and* (f_0) *(or* $(f_0)'$*) hold. For each* $\lambda > 0$, $\widetilde{J}_\lambda$ *has a nontrivial least energy (ground state) weak solution* u_λ, *that is,* $u_\lambda \in \mathcal{N}_\lambda$ *and* $\widetilde{J}_\lambda(u_\lambda) = \inf_{\mathcal{N}_\lambda} \widetilde{J}_\lambda$. *Moreover, the least energy can be estimated as follows*

$$
\widetilde{J}_\lambda(u_\lambda) \le G_\lambda := \frac{(p_1/\gamma_2)^{\frac{p_1}{2-p_1}} [\lambda M_1 \min_{t\in[0,T]} e^{-L(t)} \int_0^T |w_0(t)|^{p_1} dt]^{\frac{2}{2-p_1}} (p_1-2)}{2}.
$$

where $w_0 = \frac{w}{\|w\|}$, *and* $w = \left(\frac{T}{\pi}\sin\frac{\pi t}{T}, 0, \cdots, 0\right) \in E_0^{\alpha,2}$.

Proof. By Lemma 3.3 and $\widetilde{J} \in C^1(E_0^{\alpha,2}, \mathbb{R})$, for each $\lambda > 0$, Lemma 2.5 implies that there exists some $u_\lambda \in E_0^{\alpha,2}$ such that

$$
\widetilde{J}(u_\lambda) = \inf_{v\in E_0^{\alpha,2}} \widetilde{J}(v) \quad \text{and } \tilde{J}'(u_\lambda) = 0. \qquad (3.8)
$$

By (3.2) and $(\widetilde{f_0})$, we have

$$
\begin{aligned}
\widetilde{J}_\lambda(sw_0) &= \frac{1}{2}\langle A(sw_0), sw_0\rangle - \lambda \int_0^T e^{-L(t)} \widetilde{F}(t, sw_0(t))dt \\
&\le \frac{\gamma_2}{2} s^2 \|w_0\|^2 - \lambda \int_0^T e^{-L(t)} M_1 |sw_0(t)|^{p_1} dt \\
&\le \frac{\gamma_2}{2} s^2 - \lambda M_1 \min_{t\in[0,T]} e^{-L(t)} s^{p_1} \int_0^T |w_0(t)|^{p_1} dt. \qquad (3.9)
\end{aligned}
$$

for all $s \in [0, \infty)$. Define $g : [0, +\infty) \to \mathbb{R}$ by

$$g(s) = \frac{\gamma_2}{2} s^2 - \lambda M_1 \min_{t\in[0,T]} e^{-L(t)} s^{p_1} \int_0^T |w_0(t)|^{p_1} dt.$$

Then $g(s)$ achieves its minimum at

$$s_{0,\lambda} = \left(\frac{p_1 \lambda M_1 \min_{t\in[0,T]} e^{-L(t)} \int_0^T |w_0(t)|^{p_1} dt}{\gamma_2} \right)^{\frac{1}{2-p_1}}$$

and

$$g(s_{0,\lambda}) = \frac{(p_1/\gamma_2)^{\frac{p_1}{2-p_1}} [\lambda M_1 \min_{t\in[0,T]} e^{-L(t)} \int_0^T |w_0(t)|^{p_1} dt]^{\frac{2}{2-p_1}} (p_1 - 2)}{2}.$$

Note that $p_1 < 2$. So $g(s_{0,\lambda}) < 0$. Hence, (3.9) implies that

$$\widetilde{J_\lambda}(u_\lambda) = \inf_{v\in E_0^{\alpha,2}} \widetilde{J_\lambda}(v) \le \widetilde{J_\lambda}(s_{0,\lambda} w_0) \le g(s_{0,\lambda}) < 0 = \widetilde{J_\lambda}(0)$$

and then $u_\lambda \neq 0$ which together with (3.8) implies that $u_\lambda \in \mathcal{N}_\lambda$ and $\widetilde{J_\lambda}(u_\lambda) = \inf_{\mathcal{N}_\lambda} \widetilde{J_\lambda}$. □

Lemma 3.5. Assume that (F_0), (f_1) and (f_2) hold. If $0 < \lambda \le \frac{|\cos(\pi\alpha)|}{2C}$, then the following estimates hold

$$\|u_\lambda\|^{2-p_2} \le \frac{2\lambda C}{|\cos(\pi\alpha)|}, \quad \|u_\lambda\|_\infty^{2-p_2} \le \frac{2\lambda C B^{2-p_2}}{|\cos(\pi\alpha)|}.$$

Proof. It follows from Lemma 3.1, (2.9) and $\langle \widetilde{J'}(u_\lambda), u_\lambda \rangle = 0$ that

$$\begin{aligned}
&\int_0^T e^{-L(t)}\Big[- p(t)({}_0^cD_t^\alpha u_\lambda(t),\ {}_t^cD_T^\alpha u_\lambda(t)) + p(t)(u_\lambda(t), u_\lambda(t)) + (q(t) - p(t))(u_\lambda(t), u_\lambda(t))\Big]dt \\
= \ & \lambda \int_0^T e^{-L(t)} (\nabla \widetilde{F}(t, u_\lambda(t)), u_\lambda(t)) dt \\
\le \ & \lambda\theta \int_0^T e^{-L(t)} \widetilde{F}(t, u_\lambda(t)) dt \\
\le \ & \lambda\theta \max\{M_1, M_2\} \max_{t\in[0,T]} e^{-L(t)} \int_0^T (|u_\lambda(t)|^{p_1} + |u_\lambda(t)|^{p_2}) dt \\
\le \ & \lambda\theta \max\{M_1, M_2\} T \max_{t\in[0,T]} e^{-L(t)} (\|u_\lambda\|_\infty^{p_1} + \|u_\lambda\|_\infty^{p_2}) \\
\le \ & \lambda\theta \max\{M_1, M_2\} T \max_{t\in[0,T]} e^{-L(t)} [B^{p_1} \|u_\lambda\|^{p_1} + B^{p_2} \|u_\lambda\|^{p_2}] \\
\le \ & \lambda C (\|u_\lambda\|^{p_1} + \|u_\lambda\|^{p_2}). \qquad (3.10)
\end{aligned}$$

We claim that $\|u_\lambda\| \le 1$ uniformly for all $0 < \lambda \le \frac{|\cos(\pi\alpha)|}{2C}$. Otherwise, we have a sequence of $\{\lambda_n \le \frac{|\cos(\pi\alpha)|}{2C}\}$ such that $\|u_{\lambda_n}\| > 1$. Thus $\|u_{\lambda_n}\|^{p_2} < \|u_{\lambda_n}\|^{p_1}$ since $p_2 < p_1 < 2$. By (2.10) and (3.10), we obtain

$$\int_0^T e^{-L(t)}\Big[- p(t)({}_0^cD_t^\alpha u_{\lambda_n}(t),\ {}_t^cD_T^\alpha u_{\lambda_n}(t)) + p(t)(u_{\lambda_n}(t), u_{\lambda_n}(t)) + (q(t) - p(t))(u_{\lambda_n}(t), u_{\lambda_n}(t))\Big]dt$$

$$\geq |\cos(\pi\alpha)|\|u_{\lambda_n}\|^2 + \int_0^T e^{-L(t)}(q(t)-p(t))|u_{\lambda_n}(t)|^2dt. \tag{3.11}$$

By (3.10) and (3.11), we obtain

$$|\cos(\pi\alpha)|\|u_{\lambda_n}\|^2 + \int_0^T e^{-L(t)}(q(t)-p(t))|u_{\lambda_n}(t)|^2dt \leq \lambda_n C(\|u_{\lambda_n}\|^{p_1} + \|u_{\lambda_n}\|^{p_2}).$$

Since $q(t)-p(t) > 0$,

$$|\cos(\pi\alpha)|\|u_{\lambda_n}\|^2 \leq \lambda_n C(\|u_{\lambda_n}\|^{p_1} + \|u_{\lambda_n}\|^{p_2}) \leq 2\lambda_n C\|u_{\lambda_n}\|^{p_1}.$$

Then

$$\|u_{\lambda_n}\|^{2-p_1} \leq \frac{2\lambda_n C}{|\cos(\pi\alpha)|} \leq 1,$$

which contradicts with the assumption $\|u_{\lambda_n}\| > 1$. Now, from (3.10) we can get

$$\begin{aligned} |\cos(\pi\alpha)|\|u_\lambda\|^2 &\leq \lambda C(\|u_\lambda\|^{p_1} + \|u_\lambda\|^{p_2}) \\ &\leq 2\lambda C\|u_\lambda\|^{p_2}. \end{aligned}$$

So

$$\|u_\lambda\|^{2-p_2} \leq \frac{2\lambda C}{|\cos(\pi\alpha)|}.$$

By (2.9), we can obtain

$$\|u_\lambda\|_\infty \leq B\|u_\lambda\| \leq B\left(\frac{2\lambda C}{|\cos(\pi\alpha)|}\right)^{\frac{1}{2-p_2}}.$$

□

Observe that, in the proof of Lemma 3.5, $(\widetilde{f_2})$ is used only in (3.10). If we directly use $(\widetilde{F_0})$ to rescale $(\nabla\widetilde{F}(t,u_\lambda(t)), u_\lambda(t))$ in (3.10). Then the assumption (f_2) is not necessary but we have to pay the price that $p \in (0,1)$. To be precise, we have the following lemma.

Lemma 3.6. Assume that (F_0) and $(f_0)'$ hold. If $0 < \lambda \leq \frac{|\cos(\pi\alpha)|}{3C^*}$, then the following estimates hold

$$\|u_\lambda\|^{2-p_1} \leq \frac{3\lambda C^*}{|\cos(\pi\alpha)|}, \quad \|u_\lambda\|_\infty^{2-p_1} \leq \frac{3\lambda C^* B^{2-p_1}}{|\cos(\pi\alpha)|}.$$

Proof. It follows from (F_0), Lemma 3.1, Remark 3.1, (2.9) and $\langle\widetilde{J}'(u_\lambda), u_\lambda\rangle = 0$ that

$$\begin{aligned} &\int_0^T e^{-L(t)}\Big[-p(t)({}_0^cD_t^\alpha u_\lambda(t), {}_t^cD_T^\alpha u_\lambda(t)) + p(t)(u_\lambda(t), u_\lambda(t)) + (q(t)-p(t))(u_\lambda(t), u_\lambda(t))\Big]dt \\ =\ & \lambda\int_0^T e^{-L(t)}(\nabla\widetilde{F}(t,u_\lambda(t)), u_\lambda(t))dt \end{aligned}$$

$$
\begin{aligned}
&\le \lambda \max_{t\in[0,T]} e^{-L(t)} \int_0^T |\nabla \widetilde{F}(t,u_\lambda(t))||u_\lambda(t)|dt \\
&\le \lambda \max_{t\in[0,T]} e^{-L(t)} \int_0^T \Big[(1+\rho_0)a_0 b(t)|u_\lambda(t)| + M_1 p_1 |u_\lambda(t)|^{p_1} + \rho_0 M_1 |u_\lambda(t)|^{p_1+1}\Big] dt \\
&\le \lambda \max_{t\in[0,T]} e^{-L(t)} \left[(1+\rho_0)a_0\|u_\lambda\|_\infty \int_0^T b(t)dt + M_1 p_1 \|u_\lambda\|_\infty^{p_1} + \rho_0 M_1 T \|u_\lambda\|_\infty^{p_1+1}\right] \\
&\le \lambda \max_{t\in[0,T]} e^{-L(t)} \left[(1+\rho_0)a_0 B\|u_\lambda\| \int_0^T b(t)dt + M_1 p_1 T B^{p_1}\|u_\lambda\|^{p_1} + \rho_0 M_1 T B^{p_1+1}\|u_\lambda\|^{p_1+1}\right] \\
&\le \lambda C^*(\|u_\lambda\| + \|u_\lambda\|^{p_1} + \|u_\lambda\|^{p_1+1}). \qquad (3.12)
\end{aligned}
$$

We claim that $\|u_\lambda\| \le 1$ uniformly for all $0 < \lambda \le \frac{|\cos(\pi\alpha)|}{3C^*}$. Otherwise, we have a sequence of $\{\lambda_n \le \frac{|\cos(\pi\alpha)|}{3C^*}\}$ such that $\|u_{\lambda_n}\| > 1$. Thus $\|u_{\lambda_n}\|^{p_1} < \|u_{\lambda_n}\| < \|u_{\lambda_n}\|^{p_1+1}$ since $p_1 < 1$. By (2.10) and (3.12), we obtain

$$
\begin{aligned}
&\int_0^T e^{-L(t)}\Big[-p(t)({}_0^cD_t^\alpha u_{\lambda_n}(t),\ {}_t^cD_T^\alpha u_{\lambda_n}(t)) + p(t)(u_{\lambda_n}(t),u_{\lambda_n}(t)) + (q(t)-p(t))(u_{\lambda_n}(t),u_{\lambda_n}(t))\Big]dt \\
&\qquad \ge |\cos(\pi\alpha)|\|u_{\lambda_n}\|^2 + \int_0^T e^{-L(t)}(q(t)-p(t))|u_{\lambda_n}(t)|^2 dt. \qquad (3.13)
\end{aligned}
$$

By (3.12) and (3.13), we obtain

$$
|\cos(\pi\alpha)|\|u_{\lambda_n}\|^2 + \int_0^T e^{-L(t)}(q(t)-p(t))|u_{\lambda_n}|^2 dt \le \lambda_n C^*(\|u_\lambda\| + \|u_\lambda\|^{p_1} + \|u_\lambda\|^{p_1+1}).
$$

Since $q(t) - p(t) > 0$,

$$
|\cos(\pi\alpha)|\|u_{\lambda_n}\|^2 \le \lambda_n C^*(\|u_\lambda\| + \|u_\lambda\|^{p_1} + \|u_\lambda\|^{p_1+1}) \le 3\lambda_n C^*\|u_{\lambda_n}\|^{p_1+1}.
$$

Then

$$
\|u_{\lambda_n}\|^{1-p_1} \le \frac{3\lambda_n C^*}{|\cos(\pi\alpha)|} \le 1,
$$

which contradicts with the assumption $\|u_{\lambda_n}\| > 1$. Now, we can get from (3.12) that

$$
\begin{aligned}
|\cos(\pi\alpha)|\|u_\lambda\|^2 &\le \lambda C^*(\|u_\lambda\| + \|u_\lambda\|^{p_1} + \|u_\lambda\|^{p_1+1}) \\
&\le 3\lambda C^*\|u_\lambda\|^{p_1}.
\end{aligned}
$$

So

$$
\|u_\lambda\|^{2-p_1} \le \frac{3\lambda C^*}{|\cos(\pi\alpha)|}.
$$

By (2.9), we can obtain

$$
\|u_\lambda\|_\infty \le B\|u_\lambda\| \le B\left(\frac{3\lambda C^*}{|\cos(\pi\alpha)|}\right)^{\frac{1}{2-p_1}}.
$$

□

Proof of Theorem 1.1. Since $0 < \lambda \le \min\left\{\frac{|\cos(\pi\alpha)|}{2C}, \left(\frac{1}{B}\right)^{2-p_2}\left(\frac{\delta}{2}\right)^{2-p_2}\frac{|\cos(\pi\alpha)|}{2C}\right\}$, Lemma 3.5 implies that

$$\|u_\lambda\|_\infty \le \frac{\delta}{2}.$$

Therefore, for all $0 < \lambda \le \min\left\{\frac{|\cos(\pi\alpha)|}{2C}, \left(\frac{1}{B}\right)^{2-p_2}\left(\frac{\delta}{2}\right)^{2-p_2}\frac{|\cos(\pi\alpha)|}{2C}\right\}$, we have $\widetilde{F}(t,u(t)) = F(t,u(t))$ and then u_λ is a nontrivial weak solution of the original problem (1.1). Moreover, Lemma 3.5 implies that $\lim_{\lambda\to 0}\|u_\lambda\| = 0$ as $\lambda \to 0$ and

$$\|u_\lambda\|^{2-p_2} \le \min\left\{1, \left(\frac{1}{B}\right)^{2-p_2}\left(\frac{\delta}{2}\right)^{2-p_2}\right\}, \quad \|u_\lambda\|_\infty^{2-p_2} \le B^{2-p_2}\left\{1, \left(\frac{1}{B}\right)^{2-p_2}\left(\frac{\delta}{2}\right)^{2-p_2}\right\}.$$

□

Proof of Theorem 1.2. Note that $0 < \lambda \le \min\left\{\frac{|\cos(\pi\alpha)|}{3C^*}, \left(\frac{1}{B}\right)^{2-p_1}\left(\frac{\delta}{2}\right)^{2-p_1}\frac{|\cos(\pi\alpha)|}{3C^*}\right\}$. Similar to the proof of Theorem 1.1, by Lemma 3.6, it is easy to complete the proof. □

Proof of Theorem 1.3. By Lemma 3.1 and Lemma 3.3, we obtain that $\widetilde{J}$ satisfies (PS) condition and is even and bounded from below, and $\widetilde{J}(0) = 0$. Next, we prove that for any $k \in \mathbb{N}$, there exists a subspace k-dimensional subspace $X_k \subset E_0^{\alpha,2}$ and $\rho_k > 0$ such that

$$\sup_{u\in X^k\cap S_{\rho_k}} \widetilde{J}_\lambda(u) < 0.$$

In fact, for any $k \in \mathbb{N}$, assume that X^k is any subspace with dimension k in $E_0^{\alpha,2}$. Then by (2.10) and Lemma 3.1, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}
\widetilde{J}(u) &\le \max\left\{\frac{M}{m|\cos(\pi\alpha)|}, 1\right\}\|u\|^2 + \frac{1}{2}\int_0^T e^{-L(t)}(q(t)-p(t))|u(t)|^2dt - \lambda\int_0^T e^{-L(t)}\widetilde{F}(t,u(t))dt \\
&\le \max\left\{\frac{M}{m|\cos(\pi\alpha)|}, 1\right\}\|u\|^2 + \frac{C_1}{2}\|u\|_\infty^2 - \lambda C_2\int_0^T \widetilde{F}(t,u(t))dt \\
&\le \max\left\{\frac{M}{m|\cos(\pi\alpha)|}, 1\right\}\|u\|^2 + \frac{C_1B^2}{2}\|u\|^2 - \lambda C_2M_1\int_0^T |u(t)|^{p_1}dt \\
&\le \left[\max\left\{\frac{M}{m|\cos(\pi\alpha)|}, 1\right\} + \frac{C_1B^2}{2}\right]\|u\|^2 - \lambda C_2M_1\|u\|_{L^{p_1}}^{p_1}.
\end{aligned}$$

Since all norms on X^k are equivalent and $p_1 < 2$, for each fixed $\lambda > 0$, we can choose $\rho_k > 0$ small enough such that

$$\sup_{u\in X^k\cap S_{\rho_k}} \widetilde{J}_\lambda(u) < 0.$$

Thus, by Lemma 2.6 and Remark 2.3. $\widetilde{J}_\lambda$ has a sequence of nonzero critical points $\{u_n^\lambda\} \subset E_0^{\alpha,2}$ converging to 0 and $\widetilde{J}_\lambda(u_n^\lambda) \le 0$. Hence, for each fixed $\lambda > 0$, (3.1) has a sequence of weak solutions $\{u_n^\lambda\} \subset E_0^{\alpha,2}$ with $\|u_n^\lambda\| \to 0$, as $n \to \infty$. Furthermore, there exists n_0 large enough such that $\|u_n^\lambda\| \le \frac{\delta}{2B}$ for all $n \ge n_0$ and then (2.9) implies that $\|u_n^\lambda\|_\infty \le \frac{\delta}{2}$ for all $n \ge n_0$. Thus, $\widetilde{F}(t,u(t)) = F(t,u(t))$ and then $\{u_n^\lambda\}_{n_0}^\infty$ is a sequence of weak solutions of the original problem (1.1) for each fixed $\lambda > 0$. □

4. Conclusions

When the nonlinear term $F(t, x)$ is local subquadratic only near the origin with respect to x, system (1.1) with λ in some given interval has a ground state weak solution u_λ. If the nonlinear term $F(t, x)$ is also locally even near the origin with respect to x, system (1.1) with $\lambda > 0$ has infinitely many weak solutions $\{u_n^\lambda\}$.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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