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Research article

h-Almost Ricci solitons with concurrent potential fields

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Abstract: In this paper, we will focus our attention on the structure of h-almost Ricci solitons. A complete classification of h-almost Ricci solitons with concurrent potential vector fields is given. Also, we obtain conditions on a submanifold of a Riemannian h-almost Ricci soliton to be an h-almost Ricci soliton. Finally, we classify h-almost Ricci soliton on Euclidean hypersurface with $\lambda = h$.

Keywords: Riemannian geometry; concurrent vector fields; h-Almost Ricci soliton
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1. Introduction

Ricci solitons as a generalization of Einstein manifolds introduced by Hamilton in mid 1980s [7, 8]. In the last two decades, a lot of researchers have been done on Ricci solitons. Currently, Ricci solitons have became a crucial tool in studding Riemannian manifolds, especially for manifolds with positive curvature. Chen and Deshmukh in [3] classified the Ricci solitons with concurrent vector fields and introduced a condition for a submanifold to be a Ricci soliton in a Riemannian manifold equipped with a concurrent vector field. Pigola and others have introduced a natural deployment of the concept of gradient Ricci soliton, namely, the Ricci almost soliton [10]. The notion of h-almost Ricci soliton which develops naturally the notion of almost Ricci soliton has been introduced in [6]. It is shown that a compact non-trivial h-almost Ricci soliton of dimension no less than three with having defined signal and constant scalar curvature is isometric to a standard sphere with the potential function well assigned. Ghahremani-Gol showed that a compact non-trivial h-almost Ricci soliton is isometric to an Euclidean sphere with some conditions using the Hodge-de Rham decomposition theorem [5].

In this paper, we will generalize results of [3] for h-almost Ricci solitons. A complete classification of h-almost Ricci soliton with concurrent potential field will be given. When $h = 1$ our results coincide with Chen and Deshmukh results in [3].

This paper is organized as follows: In section 2, we remind definitions of objects and some basic notions which we need throughout the paper. In section 3, we study the structure of h-almost Ricci solitons and classify h-almost Ricci solitons with concurrent vector fields. Moreover, we provide conditions on a submanifold of a Riemannian h-almost Ricci soliton to be an h-almost Ricci soliton. Finally, we study h-almost Ricci soliton on Euclidean hypersurfaces. In the last section, we present proofs of some of the identities provied throughout section 3.

2. Preliminaries and notations

In this section, we shall present some preliminaries which will be needed for the establishment of our desired results.

2.1. Ricci solitons

Definition 2.1. *A Riemannian manifold (M^m, g) is said to be a Ricci soliton if there exists a smooth vector field X on M^m such that*

$$\mathcal{L}_X g + 2Ric = 2\lambda g, \tag{2.1}$$

where λ is a real constant, Ric and $\mathcal{L}_X$ stand for the Ricci tensor and Lie derivative operator, respectively.

We denote a Ricci soliton by (M^m, g, X, λ). The smooth vector field X mentioned above, is called a potential field for the Ricci soliton. A Ricci soliton (M^m, g, X, λ) is said to be steady, shrinking or expanding if $\lambda = 0$, $\lambda > 0$ or $\lambda < 0$, respectively. Also, a Ricci soliton (M^m, g, X, λ) is said to be gradient soliton if there exists a smooth function f on M such that $X = \nabla f$. In this case, f is called a potential function for the Ricci soliton and the Eq (2.1) can be rewritten as follows

$$Ric + \nabla^2 f = \lambda g, \tag{2.2}$$

where ∇^2 is the Hessian of f.

Example 2.2. *Einstein metrics are the most obvious examples of Ricci solitons.*

Definition 2.3. *A Riemannian manifold (M^m, g) is said to be an h-almost Ricci soliton if there exists a smooth vector field X on M^m and a smooth function $h : M^m \to \mathbb{R}$, such that*

$$h\mathcal{L}_X g + 2Ric = 2\lambda g, \tag{2.3}$$

where λ is a smooth function on M, Ric and $\mathcal{L}_X$ stand for the Ricci tensor and Lie derivative, respectively.

We will denote the h-almost Ricci soliton by (M^m, g, X, h, λ). All concepts related to Ricci soliton can be defined for h-almost Ricci soliton, accordingly. An h-almost Ricci soliton is said to be shirinking, steady or expanding if λ is positive, zero or negative, respectively.

Also, if $X = \nabla f$ for a smooth function f, then we say $(M^m, g, \nabla f, h, \lambda)$ is a gradient h-almost Ricci soliton with potential function f. In such cases the Eq (2.3) can be rewritten as follows

$$Ric + h\nabla^2 f = \lambda g, \tag{2.4}$$

where $\nabla^2 f$ denotes the Hessian of f. Note that when the potential function f be a real constant then, the underlying Ricci soltion is simply Einstein metric.

Yano has proved that if the holonomy group of a Riemannian m-manifold leaves a point invariant, then there exists a vector field X on M which satisfies

$$\nabla_Y X = Y, \tag{2.5}$$

for any vector Y tangent to M [15] . We have the following definition.

Definition 2.4. *A vector field X on a Riemannian manifold is said to be concircular vector field if it satisfies an equation of the following form [14]*

$$\nabla_Y X = \rho Y, \tag{2.6}$$

for all smooth vector fields Y, where ρ is a scalar function. If ρ is constant, then X is called a concurrent vector field.

Concurrent vector fields in Riemannian geometry and related topics have been studied by many researchers, for example see [12]. Also, concurrent vector fields have been studied in Finsler geometry [13].

2.2. Sub-Riemannian geometry

Let (M, g) be a Rimannian submanifold of $(\bar{M}, \bar{g})$ and $\psi : M \to \bar{M}$ be an isometric immersion from M into $\bar{M}$. The Levi-Civita connection of $\bar{M}$ and the submanifold M will be denote by $\bar{\nabla}$ and ∇, respectively.

Proposition 2.5. *(The Gauss Formula)[9] If $X, Y \in \chi(M)$ are extended arbitrarily to vector fields on $\bar{M}$, the following formula holds*

$$\bar{\nabla}_X Y = \nabla_X Y + I\!I(X, Y), \tag{2.7}$$

where $I\!I$ is called the second fundamental form.

Lemma 2.6. *(The Weingarten Equation)[9] Suppose $X, Y \in \chi(M)$ and N is normal vector on M. If X, Y, N are extended arbitrarily to $\bar{M}$, the following equations hold at points of M.*

$$< \bar{\nabla}_X N, Y >= - < N, I\!I(X, Y) >, \tag{2.8}$$

$$\tilde{\nabla}_X N = -A_N X + D_X N, \tag{2.9}$$

where A and D denote the shape operator and normal connection of M, respectively.

Recall that the equations of Gauss and Codazzi are given by the following equations

$$\begin{aligned} g(R(X, Y)Z, W) = {} & \tilde{g}(\tilde{R}(X, Y)Z, W) + \tilde{g}(I\!I(X, Y), I\!I(Y, Z)) \\ & - \tilde{g}(I\!I(X, W), I\!I(Y, Z)), \end{aligned} \tag{2.10}$$

$$(\tilde{R}(X, Y)Z)^{\perp} = (\bar{\nabla}_X I\!I)(Y, Z) - (\bar{\nabla}_Y I\!I)(X, Z), \tag{2.11}$$

where X, Y, Z and W are tangent to M.
Also, the mean curvature H of M in $\bar{M}$ is give by

$$H = (\frac{1}{n})\text{trace}(I\!I). \tag{2.12}$$

2.2.1. Warped product

We also need to know the warped product structure of Riemannian manifolds. Let B and E be two Riemannian manifolds equipped with Riemannian metrics g_B and g_E, respectively, and let f be a positive smooth function on B. Consider the product manifold $B \times E$ with its natural projection $\pi : B \times E \to B$ and $\eta : B \times E \to E$.

Definition 2.7. *The warped product $M = B \times E$ is the manifold $B \times E$ equipped with the Riemannian metric structure given by*

$$< X, Y >=< \pi_*(X), \pi_*(Y) > +f^2 < \eta_*(X), \eta_*(Y) >, \tag{2.13}$$

for any tangent vector $X, Y \in TM$. Thus we have

$$g = \pi^* g_B + (f \circ \pi)^2 \eta^* g_E. \tag{2.14}$$

The function f is called the warping function of the warped product.[4]

3. Main results

In this section, we announce our main results and theorems which will be proven in the next section.

Theorem 3.1. *Let (M^m, g, X, h, λ) be a complete h-almost Ricci solition on a Riemannian manifold (M^m, g). Then X is a concurrent field if and only if the following conditions hold:*
1) The h-almost Ricci solition would be shrinking, steady or expanding.
2) The m-dimensional complete manifold M^m is a warped manifold $I \times_s E$, where I is an open interval of arclength s and (E, g_E) is an Einstein manifold with dimension $(m-1)$ whose $Ric_E = (m-2)g_E$, where g_E is the metric tensor of E.

From the above theorem, the following results are obtained immediately.

Corollary 3.2. *Any h-almost Ricci solition (M^m, g, X, h, λ) equipped with a concurrent vector field X would be a gradiant soliton.*

Corollary 3.3. *There are three types of shrinking, steady or expanding h-almost Ricci solition (M^m, g, X, h, λ) equipped with a concurrent vector field X.*

In what follows, we present the existence conditions for a submanifold to be an h-almost Ricci soliton.

In the rest of this paper, suppose that $(P^p, \tilde{g})$ is a Riemannian manifold equiped with a concurrent vector field X. Let $\psi : M^m \to P^p$ is an isometric immersion of a Rimannian submanifold (M^m, g) into $(P^p, \tilde{g})$. We use notations X^T and $X^\perp$ to show the tangential and normal components of X. Also, suppose that A, D and $I\!I$ are the usual notation for the shape operator, normal connection and the second fundamental form of the submanifold M in P, respectively.

Theorem 3.4. *Let $\psi : (M^m, g) \to (P^p, \tilde{g})$ be a hypersurface immersed in $(P^p, \tilde{g})$. Then (M^m, g) has an h-almost Ricci solition structure with a concurrent vector field X^T if only if the following equation holds*

$$Ric(Y, Z) = (\lambda - h)g(Y, Z) - h\tilde{g}(I\!I(Y, Z), X^\perp). \tag{3.1}$$

Using theorem 3.4, we obtain the following results.

Theorem 3.5. *Let $\psi : (M^m, g) \to (P^p, \tilde{g})$ be a minimal immersed in $(P^p, \tilde{g})$. If (M, g) admits a structure of h-almost Ricci soliton (M^m, g, X, h, λ), then M^m has the scaler curvature given by $m(\lambda - h)/2$.*

Also, we can conclude the following theorem.

Theorem 3.6. *Let $\psi : M^m \to \mathbb{R}^{m+1}$ be a hypersurface immersed in $\mathbb{R}^{m+1}$. If (M, g) admits an h-almost Ricci soliton structure $(M^m, g, X^T, h, \lambda)$, then two distinct principal curvatures of M^m are given by the following equation*

$$\kappa_1, \kappa_2 = \frac{m\beta + \theta \pm \sqrt{(m\beta + \theta)^2 + 4h - 4\lambda}}{2}, \tag{3.2}$$

where X^T stands for the tangential component of the vector field X, β is the mean curvature and θ is the support function, that is $H = \beta N$ and $\theta =< N, X >$, where N is a unit normal vector.

Next, using the distinct principal curvatures formula of M^m presented in Proposition (3.6), we classify h-almost Ricci solitons on hypersurface of M^m of $\mathbb{R}^{m+1}$ with $\lambda = h$. In fact, we have the following theorem.

Theorem 3.7. *Let $\psi : M^m \to \mathbb{R}^{m+1}$ be a hypersurface immersed in $\mathbb{R}^{m+1}$. If (M, g) admits an h-almost Ricci soliton structure $(M^m, g, X^T, h, \lambda)$ with $\lambda = h$. Then M^m is an open portion of one of the following hypersurface of $\mathbb{R}^{m+1}$:*
1) A totally umbilical hypersurface.
2) A flat hypersurface generated by lines through the origin 0 of $\mathbb{R}^{m+1}$.
3) A spherical hypercylinder $S^k(\sqrt{k-1}) \times \mathbb{R}^{m-k}, 2 \leq k \leq m - 1$.

4. Proofs of main theorems

In this section, we will prove our results which are established in the previous section.

Proof of Theorem 3.1 Suppose that (M^m, g, X, h, λ) is a h-almost Ricci solition on a Riemannian manifold and X is a concurrent vector field on (M^m, g), then we have

$$\nabla_Y X = Y, \quad \forall Y \in TM. \tag{4.1}$$

From Eq (4.1) and definition of the Lie derivative operator, we can write

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 2g(Y, Z), \tag{4.2}$$

for any X, Y tangent to M. Considering h-almost Ricci soliton equation (2.3) and Eq (4.2) we have

$$\mathrm{Ric}(Y, Z) = (\lambda - h)g(Y, Z), \tag{4.3}$$

which shows that M^m is an Einstein manifold. An straightforward calculation shows that the sectional curvature of (M^n, g) satisfies

$$K(X, Y) = 0, \tag{4.4}$$

for each unit vector Y orthogonal to X. Hence, the Ricci tensor of M satisfies the following equation

$$\mathrm{Ric}(X, X) = 0. \tag{4.5}$$

By comparing Eqs (4.3) and (4.5) we can deduce that M^m is a Ricci flat Riemannian manifold. Therefore, we get $\lambda = h$. Since h is an arbitrary real function, so the h-almost Ricci soliton (M^m, g, X, h, λ) would be a shrinking, steady or expanding.
Let v_1 be a unit vector field tangent to M^m and suppose $\{v_1, ...v_m\}$ is a local orthonormal frame of M^m extended by v_1. We denote the spaces produced by span$\{v_1\}$ and span$\{v_2, \cdots, v_m\}$ with Π_1 and Π_2, respectively. i.e, $\Pi_1 = \text{span}\{v_1\}$ and $\Pi_2 = \text{span}\{v_2, \cdots, v_m\}$.
In [3], it is shown that Π_1 is a totally geodesic distribution. Therefore, the leaves of Π_1 are geodesics of M^m.
Furthermore, a logical argument concludes that the second fundamental form H for each leaf K of Π_2 in M^m is given as follows.

$$H(v_i, v_j) = -\frac{\delta^i_j}{\mu} v_1, \quad 2 \leq i, j \leq m. \tag{4.6}$$

Therefore, the mean curvature of each leaf of K is $-\mu^{-1}$. In addition, we deduce from the above equation that each leaf of Π_2 is a totally umbilical hypersurface of M^m. Setting $X = \mu v_1$, then with an easy calculation we have

$$\mu v_2 = ... = \mu v_m = 0. \tag{4.7}$$

Using the above equations we deduce that Π_2 is a spherical distribution. Since, the mean curvature vector of each totally umbilical leaf is parallel to the normal bundle, we conclude from Pong-Reckziegal (see [11]) that (M^m, g) is locally isometric to a warped product $I \times_{f(s)} E$ equipped with the following warp metric.

$$g = ds^2 + f^2 g_E, \tag{4.8}$$

where $v_1 = \partial/\partial s$.
From definition of above warped metric, the sectional curvature M is obtained as follows

$$K(Y, X) = -\frac{f''(s)}{f(s)}, \tag{4.9}$$

for any unit vector Y orthonormal to X. From Eq (4.4) and the above equation, we obtain $f'' = 0$. So we have

$$f(s) = cs + d, \tag{4.10}$$

for some c, d in $\mathbb{R}$.
The quantity c cannot be equal to zero because if $c = 0$, then $I \times_{f(s)} E$ is a Riemannian product and as a result every leaf Π_2 is totally geodesic in M^m. Hence $\mu = 0$. which is the opposite of the Eq (4.6). Due to the reasons provided, we have $c \neq 0$. With a transfer we can assume $f(s) = s$. That is, f is the identity function. Therefore M^m is locally isometric with a warped product $I \times_s E$.
On the other hand, since M^m is a Ricci flat manifold, the Corollary 4.1 of [2], follows that E is an Einstein manifold with $Ric_E = (n-2)g_E$. An straightforward computation shows the converse of the theorem (see Example 1.1 of [4, page 20]). □

Proof of Corollary 3.2. Using Eq (4.2) we get $(\frac{1}{2}\mathcal{L}_X g) = g$. It follows from the Eq (2.3) and $\lambda = h$ that $Ric = 0$. Setting $f := \frac{1}{2}g(X, X)$, we obtain $Hess(f) = g$.

By Eq (2.3), $Hess(f) = g$ and $(\frac{1}{2}\mathcal{L}_X g) = g$, we have $Ric + hHess(f) = \lambda g$. So the h-almost Ricci solition is gradient. □

In the following, we prove Theorem 3.4.

Proof of Theorem 3.4. Let $\psi : M^m \to P^p$ be an isometric immersion from M into P. We write X based on its tangential and normal components as follows,

$$X = X^T + X^\perp. \tag{4.11}$$

Since X is a concurrent vector field on P, applying Eq (4.11) and the Gauss and Wiengarten equations we have

$$Y = \tilde{\nabla}_Y X^T + \tilde{\nabla}_Y X^\perp \nabla_Y X^T + h(Y, X^T) - A_{X^\perp} Y + D_Y X^\perp, \tag{4.12}$$

for any Y tangent to M. The following equations are established by comparing the tangential and normal components of two sides of the Eq (4.12).

$$\nabla_Y X^T = A_{X^\perp} Y + X, \tag{4.13}$$

$$I\!I(Y, X^T) = -D_Y X^\perp, \tag{4.14}$$

hence,

$$\begin{aligned}(\mathcal{L}_{X^T} g)(Y, Z) &= g(\nabla_Y X^T, Z) + g(\nabla_Z X^T, Z) = 2g(Y, Z) + 2g(A_X^\perp Y, Z) \\ &= 2g(Y, Z) + 2\tilde{g}(I\!I(Y, Z), X^\perp),\end{aligned} \tag{4.15}$$

for Y, Z tangent to M. Equation (2.3) and above equation show that M is an h-almost Ricci soliton if and only if we have

$$Ric(Y, Z) + \frac{h}{2}\mathcal{L}_{X^T} g(Y, Z) = \lambda g(Y, Z), \tag{4.16}$$

$$Ric(Y, Z) + g(Y, Z) + \tilde{g}(I\!I(Y, Z), X^\perp) = \lambda g(Y, Z),$$

$$Ric(Y, Z) + h(g(Y, Z) + \tilde{g}(I\!I(Y, Z), X^\perp)) = \lambda g(Y, Z),$$

$$Ric(Y, Z) = (\lambda - h)g(Y, Z) - h\tilde{g}(I\!I(Y, Z), X^\perp),$$

as required. □

Proof of Theorem 3.5. By Theorem 3.4 we can obtain the following equation.

$$Ric(Y, Z) = (\lambda - h)g(Y, Z) - h\tilde{g}(I\!I(Y, Z), X^\perp), \tag{4.17}$$

for Y, Z tangent to M^m. Since M^m is a minimal submanifold of P^p, then $H = 0$. Specially, we have $\tilde{g}(H, X^\perp) = 0$. So, in the ray of Eq (4.17) we can write

$$\sum_{i=1}^{m} Ric(e_i, e_i) = m(\lambda - h). \tag{4.18}$$

Hence, the scalar curvature given by $m(\lambda - h)/2$. □

Proof of Theorem 3.6. Let M^m a hypersurface of $\mathbb{R}^{m+1}$ be an h-almost Ricci soliton. We choose a local orthonormal frame $\{v_i\}_{i=1}^m$ on M^m such that v_i are eigenvectors of the shape operator A_N, then we have

$$A_N v_i = \kappa_i v_i. \tag{4.19}$$

From the Gauss formula (2.10), we obtain

$$\text{Ric(Y,Z)} = n\tilde{g}(II(Y,Z), H) - \sum_{i=1}^{m} \tilde{g}(II(Y,e_i), II(Z,e_i)), \tag{4.20}$$

where $\tilde{g}$ is the Euclidean metric of $\mathbb{R}^{m+1}$. By an straightforward calculation, using Theorem 3.4, Eq (4.19) and formula (4.20) we deduce that $(M^m, g, X^T, h, \lambda)$ is an h-almost Ricci soliton if and only if we have

$$(m\beta - \kappa_j)\kappa_i\delta_{ij} = (\lambda - h)\delta_{ij} - \theta\kappa_i\delta_{ij}. \tag{4.21}$$

By simplifying two sides of the Eq (4.21), we obtain the following quadratic equation with respect to κ_i

$$\kappa_i^2 - (m\beta + \theta)\kappa_i + (\lambda - h) = 0, \qquad i = 1, ...m. \tag{4.22}$$

Solving the above equation, completes the proof. □

Now we are ready to prove Theorem 3.7.

Proof of Theorem 3.7. Suppose the given condition holds and Let $(M^m, g, X^T, h, \lambda)$ be an h-almost Ricci soliton on hypersurface of $M^m \subset \mathbb{R}^{m+1}$ with $\lambda = h$. using relation (3.2) from Theorem (3.6), M^m has two distinct principal curvatures give by

$$\begin{aligned} \kappa_1 &= \frac{m\beta + \theta + \sqrt{(m\beta+\theta)^2 + 4h - 4\lambda}}{2}, \\ \kappa_2 &= \frac{m\beta + \theta - \sqrt{(m\beta+\theta)^2 + 4h - 4\lambda}}{2}. \end{aligned} \tag{4.23}$$

By combining $\lambda = h$ and (4.23), we obtain $\kappa_1 = m\beta + \theta$ and $\kappa_2 = 0$, respectively. The rest of the proof is similar to the proof of Theorem 6.1 of [3], therefore we omit it. □

Conflict of interest

The authors declare no conflict of interest in this paper.

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