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Research article

Gödel semantics of fuzzy argumentation frameworks with consistency degrees

Jiachao Wu[1,*], **Lingqiang Li**[2] **and Weihua Sun**[3]

[1] School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China
[2] Department of Mathematics, Liaocheng University, Liaocheng 252059, China
[3] School of Mathematics and Statistics, Shandong University, Weihai, Shandong 264209, China

* **Correspondence:** Email: wujiachao1981@163.com; Tel: +8613065083496; Fax: +8653186182501.

Abstract: Argumentation frameworks (AF) play important roles in artificial intelligence. This paper is an exploration in establishing semantics of fuzzy AFs by fuzzy sets. There are many ways to characterize the semantics of fuzzy AFs. In this paper, our work is based on the assumption that some inconsistency of the system is permitted. Firstly, we formalize the conflict-freeness with a consistency degree x and the acceptability with a consistency degree y. Various types of extensions are then defined in a way similar to Dung's approach. The conflict-freeness and acceptability can be seen as an interpretation of the corresponding notion in Janssen's work. Formally, we add the conflict-freeness into the admissible extensions and the preferred extensions. We also introduce the complete extensions and the grounded extensions. Moreover, some basic properties are proven, such as the Fundamental Lemma, the algorithm of the grounded extension, etc. At last, it is proven to be consistent with Dung's original semantics in crisp AFs.

Keywords: argumentation frameworks; fuzzy arguments; fundamental lemma; characteristic function; fuzzy argumentation frameworks
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1. Introduction

The in-put-out-put problem commonly exists in artificial intelligence and control theory [21, 30]. Dung's theory of argumentation frameworks [10] gives a method to deal with the information, when the information is in the form of arguments and there are some attack relations between the arguments. An argumentation framework (AF) [3, 10] consists of a set of arguments and a set of attacks between the arguments. Various types of consistent sets of arguments are selected according to the attack relation. In [10], such sets are called "extensions". In [7], the sets are characterized by "labelling" functions.

Dung's theory has been applied in many fields, such as the mining, games [6], the multi-agent system, the law and so on. The structure of the arguments [26–28] has been formalized by the classic logic. The abstract AFs have also been developed in different ways, for example, the preference-based AFs [1], the bipolar AFs [2, 25], the extended AFs [4, 23, 24], the weighted AFs [5, 11], the possibilistic AFs [15, 18] and the fuzzy AFs [8, 9, 16, 17, 29]. This paper is an investigation of the semantics of fuzzy AFs.

Fuzziness is an important kind of uncertainty. When the arguments or the attacks in an AF become fuzzy, the AF comes to a fuzzy AF. The semantics of fuzzy AFs has been represented in many works. For example, [5, 11, 17]* translate a fuzzy AF into a crisp AF by releasing the attacks weaker than a given budget β, and then seize consistent sets of arguments in the crisp AF as the semantics of the fuzzy AF. [9] introduces a method to determine the values of the arguments by developing the algorithm of the probabilities in [18]. [8] introduces an equation to revise the values of the arguments step by step, and the ultimate values are the required fuzzy degrees of the arguments. [29] introduces the conflict-freeness and the acceptability between the fuzzy degrees of the arguments by developing the method in [8]. Then a variety of extensions are established based on the conflict-freeness and acceptability in Dung's approach, where each extension is a fuzzy set that assigns fuzzy degrees to all the arguments. [16] introduces x-conflict-free sets, y-admissible extensions, y-preferred extensions and z-stable extensions by separate equations. These extensions are also fuzzy sets on all the arguments.

In this paper, we develop the methods in [11] and [16] to characterize the semantics of fuzzy AFs. Distinct to weighted AFs, our discussion lies in the consistency of the degrees of the arguments. Firstly, we assume that some inconsistency of the system is permitted and the consistency degrees are given, for example, the conflict-free degree x and the acceptability degree y. These degrees x, y play roles similar to the budget β in [11]. Our main idea is that "If the conflict-free degree between the degrees of two arguments is more than x, then we recognize it to be conflict-free; if a fuzzy set of arguments accepts the degree of an argument in a degree higher than y, then we recognize the fuzzy set accepts the degree of the argument." Based on this idea, we introduce the x-conflict-free sets and the y-acceptability. Then various types of extensions in the form of fuzzy sets are established in Dung's way.

The contents are arranged as follows: The next section shows the motivation of this article. Then Section 3 shows some basic notions of fuzzy sets, Dung's AFs and Janssen's AFs. In Section 4, the conflict-freeness and acceptability are discussed. In Section 5, the characteristic function and kinds of extensions are introduced and some basic properties of them are discussed. In Section 6, comparability to Dung's original theory and to the Gödel fuzzy argumentation frameworks in [29] are shown. The last section is the conclusion of this paper.

2. Motivation

This section consists of two parts. The first part shows that there indeed exists fuzzy arguments and fuzzy attacks in AFs. And the second part shows the reason to apply our method.

2.1. Fuzziness of arguments and attacks

For the fuzziness in AFs, let's consider a scenario where two friends are discussing whether or not to play football.

*The weight of the attacks in weighted AFs can be seen as a special fuzzy degree of the attacks.

A: Let's go to play football.

B: It is raining now.

C: Yes. But the weather forecast says the rain is stopping soon.

In this scenario, there are three arguments A, B and C. Argument C attacks argument B and argument B attacks argument A. It can be simply characterized by a Dung's AF in the following graph, where A, B, C are arguments and $\rightarrow$ stands for attacks.

$$C \rightarrow B \rightarrow A.$$

As we know, the weather forecast is not always accurate. When they make a decision, the forecast can not be fully trusted. For example, generally, it is right, and the argument C can be associated with a truth degree 0.8. At the same time, different rains will have different influences on playing football. We associate a truth degree to the argument B to show the strength of the rain. For example, if it rains heavily, then the degree of B is 0.9; if it rains light, then B's degree is with 0.2.

When we associated B, C with some truth degrees, the AF becomes a fuzzy AF, where the arguments become fuzzy arguments. If the truth degree of B is 0.9, the truth degree of C is 0.8, and we suppose the truth degree of A is 1, then the fuzzy AF in the scenario is the following graph, where the numbers are the truth degrees of the arguments.

$$\begin{array}{ccccc} C & \longrightarrow & B & \longrightarrow & A \\ 0.8 & & 0.9 & & 1 \end{array}$$

Also, the attacks between the arguments can be fuzzified. There are still some persons play football in the rain, although most people don't. Hence, the attack between "the rain" and "playing football" is generally of a high degree, for example, 0.9.

Let the truth degree of $C \rightarrow B$ be 1, and the truth degree of $B \rightarrow A$ be 0.9, then the fuzzy AF in the scenario becomes the following graph, where the numbers are the truth degrees of the corresponding arguments and attacks.

$$\begin{array}{ccccc} C & \longrightarrow & B & \longrightarrow & A \\ 0.8 & 1 & 0.9 & 0.9 & 1 \end{array}$$

For convenience, an argument A with a fuzzy degree a is denoted by a tuple (A, a). And it is called a fuzzy argument (A, a).

2.2. Consistency degrees

When we build the semantics of fuzzy AFs, there are kinds of ways. In [11], they release the attacks whose weights are less than the given budget β. In [16], for a fuzzy set of arguments, its conflict-free degree must be no less than x, the admissible degree must be no less than y, and the stable degree must be no less than z. In other words, there may exist some attacks whose weights are less than β between the arguments they selected in [11]; there may exist some weak conflict (less than $1-x$), weak inadmissibility (less than $1-y$) and weak instability (less than $1-z$) of the selected fuzzy sets in [16]. In some sense, these are inconsistencies in their semantics. But these inconsistencies are weak and can be tolerated by a rational agent. Therefore, in the papers, the authors build semantics by neglecting them.

In this paper, we will follow and develop the methods in [11] and [16]. Similar to them, we permit or tolerate some weak inconsistencies when building the semantics. Different from the two works, a weak conflict between the degrees of arguments and a weak unacceptability are permitted here. Formally, if the conflict-free degree between the fuzzy degrees of two arguments are no less than x, then we say the two arguments with the fuzzy degrees are x-conflict-free; and if a fuzzy set of arguments accepts the fuzzy degree of an argument with a high degree, for example, no less than y, then we recognize that the fuzzy set of arguments y-accepts the fuzzy degree of the argument. In this paper, they are called x-conflict-freeness and y-acceptability in fuzzy AFs. Then a semantics system is established based on the two in Dung's way.

Next, in this part, we will show the motivation that we introduce the two consistency degrees from the football scenario.

In the scenario, $(A,\ 1)$ stands for "playing football", and $(B,\ 0.5)$ stands for "moderate rain". If the match is very important, like a professional match, then a high conflict between "the moderate rain" and "playing football" is permitted. In other words, a low conflict-free degree between the two is permitted. For example, let the conflict-free degree x be 0.1. Because a moderate rain can not prevent a professional match in general, $(A,\ 1)$ and $(B,\ 0.5)$ are 0.1-conflict-free. On the other hand, if the match is just for fitness, moderate rain can cancel it. In this case, the conflict-free degree should be very high, for instance, $x = 0.9$. $(A,\ 1)$ and $(B,\ 0.5)$ are not 0.9-conflict-free. In a word, for different circumstances, different conflict-free degrees are permitted and different degrees x are selected. The formal definition of x-conflict-freeness will appear in Section 4.

Similarly, let's talk about the acceptability. $(C,\ 0.8)$ can not fully accept or defend $(A,\ 1)$. Instead, $(C,\ 0.8)$ accepts $(A,\ 1)$ within some degree. However, commonly, we acknowledge the acceptability and make a decision according to it. Suppose the degree that $(C,\ 0.8)$ accepts $(A,\ 1)$ is y. Mathematically, if the acceptability degree is no less than y, we acknowledge the acceptability, and call it y-acceptability. For example, $(C,\ 0.8)$ 0.6-accepts $(A,\ 1)$. But $(C,\ 0.8)$ doesn't 0.999-accept $(A,\ 1)$. The formal definition of y-acceptability will appear in Section 4.

3. Basic notions

In this part, we recall some basic notions of fuzzy sets and Dung's abstract AFs.

3.1. Fuzzy sets

In this part, we only recall some basic notions of fuzzy set theory, which will appear in this paper. For more fuzzy theory, please read professional works.

Let U be the universe set. Given a set $S \subseteq U$, its characteristic function χ_S is a mapping from U to $\{0, 1\}$, where for any $x \in U$,

$$\chi_S(x) = \begin{cases} 1, \text{ if } x \in S, \\ 0, \text{ if } x \notin S. \end{cases}$$

A fuzzy set on the universe set U is a function $\mathcal{S}\colon U \longrightarrow [0,\ 1]$. For example, let $U = Args$, which is the set of all the arguments. $S : Args \rightarrow [0,\ 1]$ is *a fuzzy set of arguments* that determine the fuzzy degree of each argument $A \in Args$; and $\twoheadrightarrow\colon Args \times Args \longrightarrow [0,\ 1]$ is *a fuzzy set of attacks* that determine fuzzy degree of each attack between arguments.

For any set $S \subseteq U$, we call S a regular set or a crisp set. And χ_S is a fuzzy set on U, which is the fuzzy form of S.

A fuzzy set $\mathcal{S}$ is a subset of another fuzzy set $\mathcal{S}'$, denoted by $\mathcal{S} \subseteq \mathcal{S}'$, if $\forall x \in U$, $\mathcal{S}(x) \leq \mathcal{S}'(x)$.

Given a fuzzy set $\mathcal{S}$, the set $S = \{A \in U : \mathcal{S}(A) \neq 0\}$ is called the support of $\mathcal{S}$. A fuzzy point is a special fuzzy set $\mathcal{S}$, whose support is a single element set $\{A\}$, i.e., $\mathcal{S}(A) \neq 0$ and $\mathcal{S}(B) = 0, \forall B \in U \setminus \{A\}$. Generally, a fuzzy point is denoted by (A, a), where $a = \mathcal{S}(A) \neq 0$. In fuzzy AFs, a fuzzy point (A, a), where $A \in Args = U$, is called *a fuzzy argument*. Moreover, $(A, a) \subseteq \mathcal{S}$ is in general denoted by $(A, a) \in \mathcal{S}$.

Some operators on $[0, 1]$ are also useful. We list the operators appearing in this paper as follows.

Standard negation: $\neg x = 1 - x$, for any $x \in [0, 1]$.

Gödel t-norm: $x \wedge y = \min\{x, y\}, \forall x, y \in [0, 1]$.

S-implication: $x \rightsquigarrow y = \max\{1 - x, y\}, \forall x, y \in [0, 1]$.

3.2. Dung's argumentation frameworks

A Dung's argumentation framework [3, 10] is a pair $(Args, Atts)$, where $Args$ is a set of arguments and $Atts \subseteq Args \times Args$ is a set of attacks.

The extension semantics are built on the following two concepts:

1. (Acceptability:) An argument $A \in Args$ is acceptable to a set $S \subseteq Args$, if for every $B \in Args$ s.t. $(B, A) \in Atts$, there is some $C \in S$ s.t. $(C, B) \in Atts$.
2. (Conflict-freeness:) A set $S \in Args$ is conflict-free if there are no arguments $A, B \in S$ such that $(A, B) \in Atts$.

The acceptability is also called that S accepts or defends A.

The characteristic function of an AF $(Args, Atts)$ is a function $F : 2^{Args} \to 2^{Args}$, where $\forall S \subseteq Args$, $F(S) = \{A : S \text{ defends } A\}$.

Then a conflict-free set is called

- **Admissible** if $S \subseteq F(S)$.
- **Grounded** if it is the least fixed point of F.
- **Complete** if $S = F(S)$.
- **Preferred** if is a maximal admissible set w.r.t. set inclusion.
- **Stable** if it attacks each argument in $Args \setminus S$.

4. Conflict-freeness and acceptability

In different papers, the definition of fuzzy AFs are different. For example, a fuzzy AF in [8] includes fuzzy arguments and crisp attacks; a fuzzy AF in [11, 16] covers crisp arguments and fuzzy attacks; a fuzzy AFs in [29] consists of fuzzy arguments and fuzzy attacks. In this paper, we will build a semantics for fuzzy AFs with both fuzzy arguments and fuzzy attacks.

Definition 1. *Let Args be a finite set of arguments. A fuzzy AF is a tuple* $(\mathcal{AR}, \rho)$*, where* $\mathcal{AR} : Args \to [0, 1]$ *is a fuzzy set of the arguments and* $\rho : Args \times Args \to [0, 1]$ *is a fuzzy set of attacks.*

For convenience, $\rho((A,\ B))$ is represented by $\rho(A,\ B)$. The set $Args$ is always supposed to be finite in this paper.

4.1. Conflict-free sets

Firstly, let's consider the conflict-freeness between two fuzzy arguments. Suppose the conflict-free degree $x \in [0,\ 1]$ is given. Then the conflict degree is $1 - x$. Take the football scenario for example.

If $\rho(B,\ A) \leq 1 - x$, then the conflict between the two fuzzy arguments is weak enough to be tolerated. And we acknowledge the conflict degree between the two arguments with any degrees. In other words, $\forall a, b \in [0,\ 1]$, $(A,\ a)$ and $(B,\ b)$ are x-conflict-free.

Suppose $\rho(B,\ A) > 1 - x$. Then one will accept the fact with a degree $1 - x$ that, either the rain is very light such that $b \leq 1 - x$ or the football match will be held at a very low degree such that $a \leq 1 - x$.

In other words, at least one of the three values a, b or $\rho(B,\ A)$ should be no more than $1 - x$. It can be shown by the following equation

$$\min\{a,\ b,\ \rho(B,\ A)\} \leq 1 - x.$$

Then the conflict-free sets are introduced by this equation.

Definition 2. *Given a fuzzy AF* $(\mathcal{AR},\ \rho)$ *and a conflict-free degree* $x \in [0,\ 1]$, *a fuzzy set* $\mathcal{S} \subseteq \mathcal{AR}$ *is x-conflict-free, if for any* $(A,\ a),\ (B,\ b) \in \mathcal{S}$,

$$\min\{a,\ b,\ \rho(B,\ A)\} \leq 1 - x.$$

Example 1. *Consider the fuzzy AF in the football scenario, the fuzzy set* $\mathcal{S}_1 = \{(A,\ 1),\ (B,\ 0.3),\ (C,\ 0.8)\}$ *is not 0.8-conflict-free, because* $\min\{0.8,\ 1,\ 0.3\} = 0.3 > 1 - 0.8 = 0.2$, *i.e.,* $(C,\ 0.8)$ *is not 0.8-conflict-free with* $(B,\ 0.2)$. *But* $\mathcal{S}_1$ *is 0.7-conflict-free, for* $\min\{0.8,\ 1,\ 0.3\} = 0.3 \leq 1 - 0.7 = 0.3$ *and* $\min\{0.32,\ 0.9,\ 1\} = 0.3 \leq 1 - 0.7 = 0.3$.

On the other hand, $\mathcal{S}_2 = \{(A,\ 1),\ (B,\ 0.2),\ (C,\ 0.8)\}$ *is 0.8-conflict-free, because* $\min\{0.8,\ 1,\ 0.2\} = 0.2 \leq 1 - 0.8 = 0.2$ *and* $\min\{0.2,\ 0.9,\ 1\} = 0.2 \leq 1 - 0.8 = 0.2$.

Obviously, the following proposition is valid.

Proposition 1. *Given two values* $x_1 \leq x_2 \in [0,\ 1]$, *if a fuzzy set* $\mathcal{S}$ *in a fuzzy AF is* x_2*-conflict-free, then it is* x_1*-conflict-free.*

The x-conflict-freeness can be represented in the following form.

Proposition 2. *Let* $\mathcal{S} \subseteq \mathcal{AR}$ *be a fuzzy set of arguments in a fuzzy AF.* $\mathcal{S}$ *is x-conflict-free if and only if*

$$\sup_{A,\ B \in Args} \min\{\mathcal{S}(A),\ \mathcal{S}(B),\ \rho(A,\ B)\} \leq 1 - x.$$

Proof. **(Sufficiency:)** $\sup_{A,\ B \in Args} \min\{\mathcal{S}(A),\ \mathcal{S}(B),\ \rho(A,\ B)\} \leq 1 - x$ means that

$$\min\{\mathcal{S}(A),\ \mathcal{S}(B),\ \rho(A,\ B)\} \leq 1 - x,\ \forall A,\ B \in Args.$$

It equals that for any $a \leq \mathcal{S}(A)$ and $b \leq \mathcal{S}(B)$, $\min\{a,\ b,\ \rho(A,\ B)\} \leq 1 - x$.

(Necessity:) It can be got by reversing the above process. □

For the absolutely conflict-free sets, i.e., 1-conflict-free sets, we have the following result.

Corollary 1. *Suppose $\mathcal{S}$ a 1-conflict-free set. For any $A,\ B \in Args$,* $\min\{\mathcal{S}(A),\ \mathcal{S}(B), \rho(A,\ B)\} = 0$, *i.e., either A doesn't attacks B or at least one of $A,\ B$ is totally out of the set $\mathcal{S}$.*

Proof. It is directly from Proposition 2. □

4.2. Acceptability and the characteristic functions

In the football scenario, the fact that "the forecast says the rain is going to stop soon" defends or accepts "going to play football", i.e., C defends A.

But in the scenario, we know that the forecast is not so trustworthy. In other words, it only can be trusted to some degree, for example, 0.8. Then can it defend $(A,\ 1)$? Or more exactly, what degree it can defend $(A,\ 1)$ in? It equals another problem: Given $y \in [0,\ 1]$, does $(C,\ 0,8)$ defend $(A,\ 1)$ in a degree no less than y? A more general question is:

> Suppose the system permits or tolerates the acceptability that a fuzzy set $\mathcal{S}$ of arguments defends a fuzzy argument $(A,\ a)$ in a degree no less than $y \in [0,\ 1]$, but rejects the acceptability that $\mathcal{S}$ defends $(A,\ a)$ in a degree strictly less than y. How can we define such acceptability?

One case is that $(A,\ a)$ doesn't need defence, i.e., $1 - a \geq y$ or $1 - \rho(B,\ A) \leq y$. Otherwise, $(A,\ a)$ should be defended by $(C,\ c)$. In this case, c and $\rho(C,\ B)$ should be high enough, for example, $c \geq y$ and $\rho(C,\ B) \geq y$. Put these two cases in an equation, we have

$$\max\{1 - a,\ 1 - \rho(B,\ A),\ \min\{c,\ \rho(C,\ B)\}\} \geq y. \tag{4.1}$$

Then we apply Eq (4.1) to define the y-acceptability.

Definition 3. *Given a fuzzy AF $(\mathcal{AR},\ \rho)$, a fuzzy set $\mathcal{S} \subseteq \mathcal{AR}$ y-defends or y-accepts a fuzzy argument $(A,\ a)$, if and only if for any $B \in Args$, there exists some $(C,\ c) \in \mathcal{S}$, such that Eq (4.1) is valid.*

Moreover, a fuzzy set $\mathcal{C} \subseteq \mathcal{AR}$ y-defends another fuzzy set $\mathcal{A} \subseteq \mathcal{AR}$, if and only if it y-defends every fuzzy argument $(A,\ a) \in \mathcal{A}$

Example 2. *Consider the fuzzy AF in the football scenario, the fuzzy set* $(C,\ 0.8)$ *does not 1-defend* $(A,\ 1)$*, because* $\max\{1 - 1,\ 1 - 0.9,\ \min\{0.8,\ 1\}\} = 0.8 < 1$.

On the other hand, we have $(C,\ 0.8)$ *0.8-defends* $(A,\ 1)$*, because* $\max\{1 - 1,\ 1 - 0.9,\ \min\{0.8,\ 1\}\} = 0.8 \geq 0.8$.

Similarly, $(B,\ 0.2)$ *is 0.8-defended by the empty set* $\emptyset = \{(A,\ 0),\ (B,\ 0), (C,\ 0)\}$*, because* $\max\{1 - 0.2,\ 1 - 1,\ \min\{0,\ 0\}\} = 0.8 \geq 0.8$*, where the last 0 can be seen as the fuzzy degree of the imaginary attack from A to C.*

The y-acceptability can be briefly represented by the S-implication.

Proposition 3. *Let $\mathcal{A},\ \mathcal{C} \subseteq \mathcal{AR}$ be two fuzzy sets of arguments in a fuzzy AF $(\mathcal{AR},\ \rho)$. $\mathcal{C}$ y-defends $\mathcal{A}$, if and only if*

$$\inf_{B \in Args} \big(\sup_{A \in Args} \min\{\mathcal{A}(A),\ \rho(B,\ A)\} \rightsquigarrow \sup_{C \in Args} \min\{\mathcal{C}(C),\ \rho(C,\ B)\}\big) \geq y.$$

Proof. (Necessity:) Because $\mathcal{C}$ y-defends $\mathcal{A}$, then $\forall(A, a) \in \mathcal{A}$, $\forall B \in Args$, $\exists(C, c) \in \mathcal{C}$, such that $\max\{1 - a, 1 - \rho(B, A), \min\{c, \rho(C, B)\}\} \geq y$, i.e.,

$$\max\{1 - \min\{a, \rho(B, A)\}, \min\{c, \rho(C, B)\}\} \geq y.$$

Particularly, this equation is valid for $a = \mathcal{A}(A)$ and $c = \mathcal{C}(C)$, i.e., $\forall A \in Args$, $\forall B \in Args$, $\exists C \in Args$, s.t.

$$\max\{1 - \min\{\mathcal{A}(A), \rho(B, A)\}, \min\{\mathcal{C}(C), \rho(C, B)\}\} \geq y.$$

Because for any $C \in Args$, $\min\{\mathcal{C}(C), \rho(C, B)\} \leq \sup_{C \in Args} \min\{\mathcal{C}(C), \rho(C, B)\}$, we have $\forall A \in Args$, $\forall B \in Args$, s.t.

$$\max\{1 - \min\{\mathcal{A}(A), \rho(B, A)\}, \sup_{C \in Args} \min\{\mathcal{C}(C), \rho(C, B)\}\} \geq y.$$

If for any $A \in Args$, $1 - \min\{\mathcal{A}(A), \rho(B, A)\} \geq y$, then

$$y \leq \inf_{A \in Args} \{1 - \min\{\mathcal{A}(A), \rho(B, A)\}\} = 1 - \sup_{A \in Args} \min\{\mathcal{A}(A), \rho(B, A)\}.$$

Transferring the order of $\forall A \in Args$, $\forall B \in Args$, we have $\forall B \in Args$,

$$\max\{1 - \sup_{A \in Args} \min\{\mathcal{A}(A), \rho(B, A)\}, \sup_{C \in Args} \min\{\mathcal{C}(C), \rho(C, B)\}\} \geq y.$$

Representing it by the S-implication, we have $\forall B \in Args$,

$$\sup_{A \in Args} \min\{\mathcal{A}(A), \rho(B, A)\} \rightsquigarrow \sup_{C \in Args} \min\{\mathcal{C}(C), \rho(C, B)\} \geq y.$$

It equals

$$\inf_{B \in Args} (\sup_{A \in Args} \min\{\mathcal{A}(A), \rho(B, A)\} \rightsquigarrow \sup_{C \in Args} \min\{\mathcal{C}(C), \rho(C, B)\}) \geq y.$$

(Sufficiency:) Reverse the proof of necessity. □

The next corollary is obvious from Proposition 3.

Corollary 2. *Let $\mathcal{S}$ be a fuzzy set and (A, a_0) be a fuzzy argument in a fuzzy AF. If for any $a < a_0$, (A, a) is y-defended by $\mathcal{S}$, then (A, a_0) is y-defended by $\mathcal{S}$.*

The following two propositions show the monotonicity of y-acceptability. They can be directly obtained from the definition of the y-acceptability.

Proposition 4. *Given a fuzzy AF and two values $y_1 \leq y_2 \in [0, 1]$, if a fuzzy set $\mathcal{S}$ y_2-defends another fuzzy $\mathcal{S}'$, then $\mathcal{S}$ y_1-defends $\mathcal{S}'$.*

Proof. Suppose $\mathcal{S}$ y_2-defends (A, a). We have for any $B \in Args$, there exists some $(C, c) \in \mathcal{S}$, such that $\max\{1 - a, 1 - \rho(B, A), \min\{c, \rho(C, B)\}\} \geq y_2 \geq y_1$. Hence, $\mathcal{S}$ y_1-defends (A, a). □

Proposition 5. *Given a fuzzy AF, suppose $\mathcal{S} \subseteq \mathcal{S}'$ are two fuzzy sets of arguments and (A, a) is a fuzzy argument. If $\mathcal{S}$ y-defends (A, a), then $\mathcal{S}'$ y-defends (A, a).*

Proof. For $\mathcal{S}$ y-defends $(A,\ a)$, we have for any $B \in Args$, there exists some $(C,\ c) \in \mathcal{S}$, such that Equation (4.1) is valid. Because $\mathcal{S} \subseteq \mathcal{S}'$, we have $(C,\ c) \in \mathcal{S}'$. Hence, $\mathcal{S}'$ y-defends $(A,\ a)$. □

The characteristic function can be introduced similar to Dung's work. A little difference is that the characteristic function here will depend on the acceptability degree $y \in [0,\ 1]$, and is called y-characteristic function.

Definition 4. *Let $(\mathcal{AR},\ \rho)$ be a fuzzy AF. Given a number $y \in [0,\ 1]$, the function $F_y\colon [0,\ 1]^{Args} \longrightarrow [0,\ 1]^{Args}$ is called the y-characteristic function if for any fuzzy set $\mathcal{S} \subseteq \mathcal{AR}$,*

$$F_y(\mathcal{S}) = \{(A,\ a)\colon (A,\ a) \text{ is } y\text{-defended by } \mathcal{S}\}.$$

From Corollary 2, $F_y(\mathcal{S})$ is well defined.
From Proposition 5, we can get the monotonicity of F_y.

Lemma 1. *Given two fuzzy sets $\mathcal{S} \subseteq \mathcal{S}' \subseteq \mathcal{AR}$, $\forall y \in [0,\ 1]$, $F_y(\mathcal{S}) \subseteq F_y(\mathcal{S}')$.*

Proof. From Proposition 5, if $(A,\ a)$ is in $F(\mathcal{S})$, i.e., $\mathcal{S}$ y-defends $(A,\ a)$, then $\mathcal{S}'$ y-defends $(A,\ a)$, i.e. $(A,\ a) \in F(\mathcal{S}')$. □

5. Semantics in the form of extensions

In this section, we will introduce the semantics in Dung's way, including the admissible extension, the preferred extensions, the complete extensions, the grounded extensions, and the stable extensions.

5.1. Admissible extensions and preferred extensions

Similar to Dung's work, the admissibility is defined by the x-conflict-freeness and y-acceptability.

Definition 5. *Suppose $x,\ y \in [0,\ 1]$ and $(\mathcal{AR},\ \rho)$ is a fuzzy AF. An x-conflict-free set $\mathcal{E} \subseteq \mathcal{AR}$ is $(x,\ y)$-admissible if and only if $\mathcal{E}$ y-defends every fuzzy argument $(A,\ a)$ in it, i.e., $\mathcal{E} \subseteq F_y(\mathcal{E})$.*

Example 3. *Consider the fuzzy AF in the football scenario, the fuzzy set $\mathcal{S}_1 = \{(A,\ 1), (B,\ 0.3), (C,\ 0.8)\}$ is not (0.7, 0.8)-admissible, because $(B, 0.3)$ can not 0.8-defended by $\mathcal{S}_1$.*
On the other hand, the fuzzy set $\mathcal{S}_2 = \{(A,\ 1),\ (B,\ 0.2),\ (C,\ 0.8)\}$ is (0.8, 0.8)-admissible.

From Propositions 1 and 4, we can get the following proposition.

Proposition 6. *Suppose $x_1 \le x_2$ and $y_1 \le y_2$. In a fuzzy AF, if a fuzzy set $\mathcal{S}$ is $(x_2,\ y_2)$-admissible, then it is $(x_1,\ y_1)$-admissible.*

Proof. By Proposition 1, $\mathcal{S}$ is x_1-conflict-free. By Proposition 4, $\mathcal{S}$ y_2-defends every element in it. Hence, $\mathcal{S}$ is $(x_1,\ y_1)$-admissible. □

From this proposition, the fuzzy set $\mathcal{S}_2$ in Example 4 is $(0.8,\ y)$-admissible for all $y \le 0.8$.

Lemma 2 (Fundamental Lemma)**.** *Let $(\mathcal{AR},\ \rho)$ be a fuzzy AF and two values $x,\ y \in [0,\ 1]$ satisfy $1 - y < x \le y$. Suppose $\mathcal{S} \subseteq \mathcal{AR}$ is $(x,\ y)$-admissible. If $\mathcal{S}$ y-defends $(A,\ a) \in \mathcal{AR}$, then*

1. *$\mathcal{S}' = \mathcal{S} \cup (A,\ a)$ is x-conflict-free.*

2. *$\mathcal{S}'$ y-defends $(A,\ a)$, i.e., $\mathcal{S}'$ is $(x,\ y)$-admissible.*

Proof. (1) Let's show $\mathcal{S}'$ is x-conflict-free. For $\mathcal{S}$ is x-conflict-free, it is only necessary to show that for any $(B,\ b) \in \mathcal{S}$,

$$1 - \min\{a,\ b,\ \rho(A,\ B)\} \geq x \text{ and } 1 - \min\{a,\ b,\ \rho(B,\ A)\} \geq x.$$

(a) Suppose there is some $(B,\ b) \in \mathcal{S}$, such that

$$1 - \min\{a,\ b,\ \rho(B,\ A)\} < x. \tag{5.1}$$

For $(A,\ a)$ is y-defended by $\mathcal{S}$, we have $\exists C \in Args$ s.t.

$$\max\{1 - a,\ 1 - \rho(B,\ A),\ \min\{\mathcal{S}(C),\ \rho(C,\ B)\}\} \geq y.$$

By Equation (5.1) and $x \leq y$, we have $1 - \min\{a,\ \rho(B,\ A)\} < x \leq y$. Hence, $1 - a < y,\ 1 - \rho(B,\ A) < y$. Then we have, $\min\{\mathcal{S}(C),\ \rho(C,\ B)\} \geq y$, i.e.

$$1 - \min\{\mathcal{S}(C),\ \rho(C,\ B)\} \leq 1 - y < x.$$

On the other hand, we can get $1 - b < x$ from Equation (5.1). Therefore,

$$1 - \min\{b,\ \mathcal{S}(C),\ \rho(C,\ B)\} < x.$$

Contradict to the x-conflict-freeness of $\mathcal{S}$.

(b) Suppose there is some $(B,\ b) \in \mathcal{S}$, such that

$$1 - \min\{a,\ \rho(A,\ B),\ b\} < x. \tag{5.2}$$

By the y-admissibility of $\mathcal{S}$, similar to Case (a), we can get that there is some $D \in Args$, s.t.

$$1 - \min\{\mathcal{S}(D),\ \rho(D,\ A),\ a\} < x.$$

It comes back to Case (a) and contradicts to the x-conflict-freeness of $\mathcal{S}$.

2. It is obvious from Proposition 5. □

Note, for $x > y$, the Fundamental Lemma and this proposition are not valid. In other words, it means that the conflict-freeness in a higher degree x can not be kept by the acceptability in a lower degree y. In other words, weak acceptability can not keep strong conflict-freeness.

When we consider the admissibility, commonly we only care about the sets whose conflict-free degrees and acceptability degrees are not too week, for example, higher than 0.5. Then the condition $1 - y < x \leq y$ can be refined to $0.5 < x \leq y$ (particular $x = y > 0.5$). In other words, the lemma is suitable for $0.5 < x \leq y$ (particular $x = y > 0.5$), which covers most cases in the application.

By the Fundamental Lemma, the following result is obvious.

Proposition 7. *Let $\mathcal{S} \subseteq \mathcal{AR}$ be an $(x,\ y)$-admissible set with $1 - y < x \leq y$. Then $F_y(\mathcal{S})$ is also $(x,\ y)$-admissible.*

Obviously, the empty set is the least $(x,\ y)$-admissible extension. For the maximal $(x,\ y)$-admissible extensions, we have the following definition.

Definition 6. *An* $(x,\ y)$*-preferred extension is a maximal* $(x,\ y)$*-admissible extension w.r.t. set inclusion.*

In the fuzzy AF in the football scenario, the fuzzy set $\mathcal{S}_2 = \{(A,\ 1),\ (B,\ 0.2), (C,\ 0.8)\}$ is (0.8, 0.8)-preferred.

By the Fundamental Lemma, the next proposition is valid.

Proposition 8. *In a fuzzy AF, suppose the fuzzy set* $\mathcal{S} \subseteq \mathcal{AR}$ *is* $(x,\ y)$*-admissible. Then there exists some* $(x,\ y)$*-preferred extension* $\mathcal{S}' \subseteq \mathcal{AR}$ *s.t.* $\mathcal{S} \subseteq \mathcal{S}'$*.*

Because the empty set is always $(x,\ y)$-admissible, we have the following result.

Proposition 9. *Given a fuzzy AF, there always exists some preferred extension.*

5.2. Complete extensions and grounded extensions

The fixed points of the characteristic function are called complete extensions.

Definition 7. *Given a fuzzy AF* $(\mathcal{AR},\ \rho)$ *and two numbers* $x,\ y \in [0,\ 1]$*, an* $(x,\ y)$*-admissible set* $\mathcal{E} \subseteq \mathcal{AR}$ *is* $(x,\ y)$*-complete, if and only if it includes every fuzzy argument that it defends, i.e.,* $\mathcal{E} = F_y(\mathcal{E})$*.*

Definition 8. *Given a fuzzy AF, the* $(x,\ y)$*-grounded extension is the least* $(x,\ y)$*-complete extension.*

Example 4. *In the football scenario, the unique complete* $(0.8,\ 0.8)$*-complete extension is* $\{(A,\ 0.1),\ (B,\ 0.2),\ (C,\ 0.8)\}$*, which is also* $(0.8,\ 0.8)$*-grounded.*

Example 5. *Consider the fuzzy AF* $(\{(A,\ 1),\ (B,\ 1)\},\ \{(A \to B,\ 1), ((B \to A,\ 1))\})$*, i.e., the classical AF:* $A \leftrightarrows B$*. It is not difficult to check that the empty set* $\emptyset$ *is* $(1,\ 1)$*-grounded and* $(1,\ 1)$*-complete. But it is not* $(1,\ 1)$*-preferred.*

$\{(A,\ 1),\ (B,\ 0.2)\}$ *is* $(0.8,\ 0.8)$*-preferred extensions and* $(0.8,\ 0.8)$*-complete, but it is not* $(0.8,\ 0.8)$*-grounded.*

The relation between preferred extensions and complete extensions follows from Lemma 2.

Proposition 10. *Suppose* $1-y < x \le y$*. In a fuzzy AF, each* $(x,\ y)$*-preferred extension is* $(x,\ y)$*-complete. But not vice versa.*

Proof. Suppose $\mathcal{E}$ is an $(x,\ y)$-preferred extension and $\mathcal{E}$ y-defends $(A,\ a) \in \mathcal{AR}$. Let's show $(A,\ a) \in \mathcal{E}$, which shows the $(x,\ y)$-completeness of $\mathcal{E}$.

If $(A,\ a) \notin \mathcal{E}$, then From Lemma 2, we have $\mathcal{E}' = \mathcal{E} \cup (A,\ a)$ is $(x,\ y)$-admissible and $\mathcal{E} \subsetneq \mathcal{E}'$. Contradicts to the maximum of $\mathcal{E}$. □

Given a fuzzy set $\mathcal{S}$, denote $\mathcal{S}$ by $F_y^0(\mathcal{S})$, $F_y(\mathcal{S})$ by $F_y^1(\mathcal{S})$, and $F_y(F_y^n(\mathcal{S}))$ by $F_y^{n+1}(\mathcal{S})$ for $n \in \mathbb{N}$.

Lemma 3. *Given an* $(x,\ y)$*-admissible set* $\mathcal{S}$*,* $\mathcal{S} \subseteq F_y^n(\mathcal{S}) \subseteq F_y^{n+1}(\mathcal{S})$*,* $\forall n \in \mathbb{N}$*.*

Particularly, for the empty set $\emptyset$*, we have*

$$\emptyset \subseteq F_y(\emptyset) \subseteq ... \subseteq F_y^n(\emptyset) \subseteq F_y^{n+1}(\emptyset) \subseteq ...$$

Proof. Because $\mathcal{S}$ is $(x,\ y)$-admissible, $\mathcal{S} \subseteq F_y(\mathcal{S})$. From the monotonicity of F_y, we have $F_y(\mathcal{S}) \subseteq F_y^2(\mathcal{S})$. Similarly, $F_y^2(\mathcal{S}) \subseteq F_y^3(\mathcal{S})$,..., $F_y^n(\mathcal{S}) \subseteq F_y^{n+1}(\mathcal{S})$,... Therefore, $\mathcal{S} \subseteq F_y^n(\mathcal{S}) \subseteq F_y^{n+1}(\mathcal{S})$, $\forall n \in \mathbb{N}$. □

Proposition 11. *Suppose* $1 - y < x \leq y$. *In a fuzzy AF, the fuzzy set* $\mathcal{GE} = \cup_{n \in \mathbb{N}} F_y^n(\emptyset)$ *is the* (x, y)-*grounded extension.*

Proof. Firstly, let's show $\mathcal{GE}$ is a fixed point of F_y. Suppose $\mathcal{GE}$ y-defends (A, a). Then from Lemma 3, there is some $n \in \mathbb{N}$ s.t. (A, a) is y-defended by $F_y^n(\emptyset)$. Therefore, $(A, a) \in F_y^{n+1}(\emptyset) \subseteq \mathcal{GE}$.

Now let's show $\mathcal{GE}$ is the least fixed point, i.e., for any fixed point $\mathcal{S}$ of F_y, $\mathcal{GE} \subseteq F_y(\mathcal{S})$.

For $\emptyset \subseteq \mathcal{S}$, we have $F_y(\emptyset) \subseteq F_y(\mathcal{S}) = \mathcal{S}$. Inductively, we can get

$$F_y^n(\emptyset) \subseteq F_y(\mathcal{S}) = \mathcal{S}, \ \forall n \in \mathbb{N}.$$

Therefore, $\cup_{n \in \mathbb{N}} F_y^n(\emptyset) \subseteq \mathcal{S}$. □

If there is some $n_0 \in \mathbb{N}$ s.t. $F_y^{n_0}(\emptyset) = F_y^{n_0+1}(\emptyset)$, then by Lemma 3, the grounded extension is $\mathcal{GE} = \cup_{n \in \mathbb{N}} F_y^n(\emptyset) = F_y^{n_0}(\emptyset)$.

Proposition 11 provides us a way to calculate the grounded extensions in fuzzy AFs. Now, let's calculate the $(0.6, 0.7)$-grounded extension of the fuzzy AF in the football scenario.

Example 6. *By the definition of y-acceptability, the empty set* $\emptyset$ *0.7-defends* $\mathcal{S}_1 = \{(A, 0.3), (B, 0.3), (C, 0.8)\}$ *and any fuzzy arguments out of* $\mathcal{S}_1$ *are not 0.7-defended by* $\emptyset$. *Therefore,*

$$F_{0.7}(\emptyset) = \{(0.3), (B, 0.3), (C, 0.8)\}.$$

Similarly, we have

$$F_{0.7}^2(\emptyset) = \{(A, 1), (B, 0.3), (C, 0.8)\},$$

$$F_{0.7}^3(\emptyset) = \{(A, 1), (B, 0.3), (C, 0.8)\}.$$

Because $F^2(\emptyset) = F^3(\emptyset)$, *the* $(0.6, 0.7)$-*grounded extension of the fuzzy AF is*

$$F^2(\emptyset) = \{(A, 1), (B, 0.3), (C, 0.8)\}.$$

Similarly, it is not difficult to calculate that $\{(A, 0.2), (B, 0.2)\}$ is the $(0.8, 0.8)$-grounded extension of the fuzzy AF in Example 5.

5.3. Stable extensions

A stable extension in Dung's theory is a set of arguments, which is conflict-free inside and attacks every argument outside. Here, we introduce a similar notion. We first introduce the notion called "x-sufficient attack".

Definition 9. *Suppose* (A, a) *and* (B, b) *are two fuzzy arguments in a fuzzy AF. If* $\min\{a, b, \rho(B, A)\} > 1 - x$, *then we say* (B, b) *x-sufficiently attacks* (A, a),

Note, the y-acceptability should not be in the following form:

> A fuzzy set $\mathcal{S}$ y-defends a fuzzy argument (A, a), if for any $(B, b) \in \mathcal{AR}$ which y-sufficiently attacks (A, a), there is some $(C, c) \in \mathcal{S}$ s.t. (C, c) y-sufficiently attacks (B, b).

Example 7. *Consider the fuzzy AF in the football scenario. For any* (B, b)*, if* (B, b) 0.8*-sufficiently attacks* $(A, 1)$*, then* $b > 0.2$ *because* $\min\{a, b, \rho(B, A)\} = \min\{1, b, 0.9\} > 1 - 0.8$. $(C, 0.3)$ *0.8-sufficiently attacks* (B, b)*, because* $\min\{c, b, \rho(C, B)\} = \min\{0.3, b, 1\} > 1 - 0.8$.

On the other hand, by Definition 3, $(C, 0.3)$ *doesn't 0.8-defend* $(A, 1)$. *Therefore, the y-acceptability in Definition 3 can not be defined by the y-sufficient attacks.*

Here, we use the y-sufficient attacks to define the stable extensions.

Definition 10. *In a fuzzy AF, a fuzzy set* $\mathcal{S}$ *of arguments is called* (x, y)*-stable, if it is x-conflict-free and it is not y-conflict-free with any fuzzy argument not in it.*

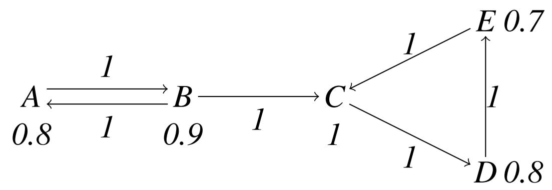
The following example shows an (x, y)-stable extension may not be (x, y)-admissible.

Example 8. *Consider the fuzzy AF in the football scenario. The fuzzy set* $\mathcal{S} = \{(A, 1), (B, 0.3), (C, 0.8)\}$ *is (0.7, 0.8)-stable.*

But it is not (0.7, 0.8)-admissible, because $(B, 0.3) \in \mathcal{S}$ *is not* 0.8*-defended by* $\mathcal{S}$*. As a result, it is not (0.7, 0.8)-complete or (0.7, 0.8)-preferred.*

The AF in the following example is very common in Dung's AFs. Let's check the extensions in it.

Example 9. *Consider the fuzzy AF in the following graph, where the fuzzy degrees of all the attacks are 1 and the fuzzy degrees of the arguments are the number next to the argument.*



The empty set $\emptyset$ *is* $(0.8, 0.8)$*-admissible, but it is not* $(0.8, 0.8)$*-complete.*

From Proposition 11, the $(0.8, 0.8)$*-grounded extension is*

$$\mathcal{S}_1 = F^1_{0.8}(\emptyset) = F^2_{0.8}(\emptyset) = \{(A, 0.2), (B, 0.2), (C, 0.2), (D, 0.2), (E, 0.2)\}.$$

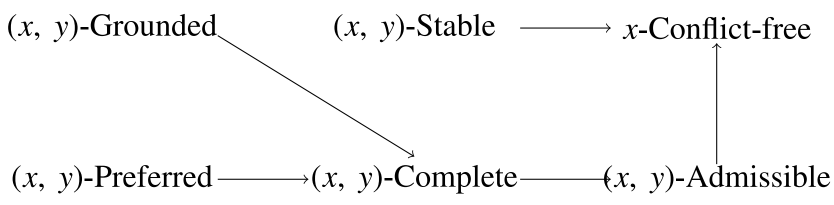
It is also $(0.8, 0.8)$*-complete and* $(0.8, 0.8)$*-admissible, but not* $(0.8, 0.8)$*-preferred.*

It is not difficult to check that $\mathcal{S}_2 = \{(A, 0.8), (B, 0.2), (C, 0.2), (D, 0.2), (E, 0.2)\}$ *is* $(0.8, 0.8)$*-preferred, thus* $(0.8, 0.8)$*-complete and* $(0.8, 0.8)$*-admissible. But it is not* $(0.8, 0.8)$*-stable, because* $(D, 0.3)$ *is not 0.8-sufficiently attacked by any fuzzy argument in* $\mathcal{S}_2$.

On the other hand, the fuzzy set $\mathcal{S}_3 = \{(A, 0.2), (B, 0.9), (C, 0.2), (D, 1), (E, 0.2)\}$ *is both* $(0.8, 0.8)$*-preferred and* $(0.8, 0.8)$*-stable. Note, in* $\mathcal{S}_3$*, the fuzzy argument* $(D, 1)$ *is* 0.8*-defended by* $(B, 0.9)$.

The fuzzy set $\mathcal{S}_2$ in this example shows that an (x, y)-preferred extension may not be (x, y)-stable.

The relation between the extensions can be shown in the following graph. A minor difference to Dung's theory is that the (x, y)-stable extensions may not be (x, y)-preferred.



6. Related works

There are many papers about fuzzy AFs. Some, like [12–14], characterize crisp AFs by fuzzy method. Some, like [11] and its following works, look for crisp semantics for fuzzy AFs. Some, like [8, 9], concentrate on the algorithm to calculate concrete fuzzy degrees for each argument. And some, like [16, 29], establish semantics similar to Dung's work by fuzzy sets. This paper follows the last way.

In this section, we firstly show our semantics is compatible with Dung's semantics. Then we compare our semantics to the semantics in [16, 29]. For convenience, denote our semantics by WSL, Dung's semantics by DUNG, the semantics in [16] by JDV, and the semantics in [29] by WLON.

6.1. Comparability to Dung's argumentation frameworks

A Dung's AF $(Args,\ Atts)$ can be seen as a special fuzzy AF $(\mathcal{AR},\ \rho)$, where $\mathcal{AR} = \chi_{Args}$ and $\rho(A,\ B) = \chi_{Atts}(A,\ B) = \begin{cases} 1, & \text{if } (A,\ B) \in Atts, \\ 0, & \text{otherwise.} \end{cases}$

The next theorem shows the conflict-freeness in WSL is compatible with the conflict-freeness in DUNG.

Theorem 1. *In a Dung's AF, a set of arguments $S \subseteq Args$ is conflict-free in DUNG, if and only if $\chi_S \subseteq \chi_{Args}$ is 1-conflict-free in WSL.*

Proof. (⇒) Suppose S is conflict-free in DUNG. Then $\forall A,\ B \in S$, $(A,\ B) \notin Atts$, i.e., $\rho(A,\ B) = 0$. It equals

$$\min\{\chi_S(A),\ \chi_S(B),\ \rho(A,\ B)\} = \rho(A,\ B) = 0, \forall A,\ B \in S.$$

Together with $\forall A \notin S$, $\chi_S(A) = 0$, we have

$$\min\{\chi_S(A),\ \chi_S(B),\ \rho(A,\ B)\} = 0 \leq 1 - 1, \forall A,\ B \in Args.$$

Hence, χ_S is 1-conflict-free in WSL.

(⇐) Suppose χ_S is 1-conflict-free in WSL, i.e., $\forall A,\ B \in Args$, $\min\{\chi_S(A), \chi_S(B),\ \rho(A,\ B)\} \leq 1 - 1 = 0$. Then for any $A,\ B \in S$, because $\chi_S(A) = \chi_S(B) = 1$, we have $\rho(A,\ B) = 0$, i.e., $(A,\ B) \notin Atts$. □

The next theorem shows the acceptabilities in DUNG and WSL are compatible.

Theorem 2. *In a Dung's AF, $S \subseteq Args$ defends an argument $A \in Args$ in DUNG, if and only if $\chi_S \subseteq \chi_{Args}$ 1-defends the fuzzy argument $(A,\ a)$ with $a > 0$ in WSL.*

Proof. **(Necessity:)** Suppose S defends A in DUNG. Then for any $B \in Args$ with $(B,\ A) \in Atts$, there exists $C \in S$ s.t. $(C,\ B) \in Atts$, i.e., $\chi_S(C) = 1$ $\rho(B,\ A) = \rho(C,\ B) = 1$. Then

$$\max\{1 - a,\ 1 - \rho(B,\ A),\ \min\{\chi_S(C),\ \rho(C,\ B)\}\} \geq 1.$$

Hence, χ_S 1-defends $(A,\ 1)$.

(Sufficiency:) Suppose χ_S 1-defends $(A,\ a)$ with $a > 0$ in WSL. For any $B \in Args$ with $(B,\ A) \in Atts$, $\exists (C,\ c) \in \chi_S$ s.t.

$$\max\{1 - a,\ 1 - \rho(B,\ A),\ \min\{c,\ \rho(C,\ B)\}\} \geq 1.$$

Because $a > 0$ and $\rho(B,\ A) = 1$, we have $1 - a < 1$ and $1 - \rho(B,\ A) = 0 < 1$. Hence, $c = 1$ and $\rho(C,\ B) = 1$, i.e., $C \in S$ and $(C,\ B) \in Atts$. Then we can get that S defends A in DUNG. □

From Theorem 2, the next corollary is directly obtained by the definition of the characteristic function.

Corollary 3. *Given a Dung's AF, suppose F is the characteristic function of DUNG and F_1 is the 1-characteristic function of WSL. Then $\chi_{F(S)} = F_1(\chi_S)$.*

Because the conflict-freeness and the acceptability are equivalent in DUNG and WSL, we can get the next theorem. It shows the our semantics WSL is compatible with Dung's original semantics DUNG in Dung's AFs.

Theorem 3. *In a Dung's AF, $S \subseteq Args$ is admissible, complete, preferred, stable or grounded in DUNG, if and only if χ_S is (1, 1)-admissible, (1, 1)-complete, (1, 1)-preferred, (1, 1)-stable or (1, 1)-grounded in WSL, correspondingly.*

Proof. The stable case is obvious. The admissible, complete and grounded case can be got by Corollary 3 easily. Let's show the preferred case.

(Sufficiency:) It is direct from Theorem 2.

(Necessity:) Suppose S is a preferred extension in DUNG. By Theorem 1, χ_S is 1-conflict-free in WSL.

Assume χ_S is not a maximal (1, 1)-admissible extension in WSL w.r.t. set inclusion. There exists some (1, 1)-admissible set $\mathcal{E} \subseteq \mathcal{AR}$ s.t. $\chi_S \subsetneq \mathcal{E}$. We can get $\exists A \in Args$ s.t. $\mathcal{E}(A) > 0$ and $\chi_S(A) = 0$. On the other hand, from the above part of this theorem, $S' = \{A \in Args\colon \mathcal{E}(A) > 0\}$ is an admissible extension in DUNG and $S \subsetneq S'$. It is contradicting to S is preferred in DUNG.

Therefore, χ_S is (1, 1)-preferred in WSL. □

6.2. Comparability to semantics of Janssen et al.

Firstly, let's remind the JDV. In [16], the fuzzy degrees of arguments and attacks are drawn from a complete lattice $\mathcal{L}$, with a partial order $\leq_{\mathcal{L}}$. And the fuzzy AFs are defined with crisp arguments and fuzzy attacks.

Definition 11. *[Definition 3 in [16]] Given a set of arguments Args, a JAF is a tuple $(Args, \twoheadrightarrow)$, where $\twoheadrightarrow\colon Args \times Args \longrightarrow \mathcal{L}$ is a fuzzy relation over Args.*

Suppose $A,\ B \in Args$ are arguments and $\mathcal{A},\ \mathcal{B} \subseteq Args$ are fuzzy sets. $\twoheadrightarrow$ is extended to represent the degrees to which fuzzy sets of arguments attack each other as follows:

$$\mathcal{A} \twoheadrightarrow B = \sup_{A \in Args} (\mathcal{A}(A) \wedge (A \twoheadrightarrow B)),$$

$$B \twoheadrightarrow \mathcal{A} = \sup_{A \in Args} (\mathcal{A}(A) \wedge (B \twoheadrightarrow A)),$$

$$\mathcal{A} \twoheadrightarrow \mathcal{B} = \sup_{B \in Args} (\mathcal{B}(B) \wedge (\mathcal{A} \twoheadrightarrow B)).$$

Then the extensions in the form of fuzzy sets are defined in the following definition.

Definition 12 (Definition 4 in [16])**.** *Let $(\mathcal{AR}, \twoheadrightarrow)$ be a JAF. Then*

A fuzzy set $\mathcal{E} \subseteq \mathcal{AR}$ is x-conflict-free, $x \in \mathcal{L}$, iff $(\neg(\mathcal{E} \twoheadrightarrow \mathcal{E})) \geq_{\mathcal{L}} x$.

A fuzzy set $\mathcal{E}$ over Args is y-admissible, if it defends itself well enough against all attacks, i.e.,

$$\inf_{B \in Args} ((B \twoheadrightarrow \mathcal{E}) \rightsquigarrow (\mathcal{E} \twoheadrightarrow B)) \geq_{\mathcal{L}} y.$$

A y-preferred extension, $y \in \mathcal{L}$, is a maximal y-admissible extension.

A z-stable extension, $z \in \mathcal{L}$, is a fuzzy set $\mathcal{E}$, that sufficiently attacks all external arguments, i.e.

$$\inf_{B \in Args} (\neg \mathcal{E}(B) \rightsquigarrow (\mathcal{E} \twoheadrightarrow B)) \geq_{\mathcal{L}} z.$$

Comparing the semantics in JDV and WSL, there are three obvious but critical distinctions between them. First, the conflict-free sets, the admissible sets and the stable sets are defined independently in JDV. And the admissible sets and the stable sets in WSL include the conflict-free condition. This makes that all our semantics are conflict-free. Second, each fuzzy extension is an integral fuzzy set and the fuzzy degrees of the arguments can not be discussed separately in JDV. But in WSL, the fuzzy degrees of a single argument can be explored, like in the acceptability. Then the growing up of the admissible sets (the characteristic function) can be introduced in WSL. Last, the WSL is established based on the acceptability and the conflict-free sets. This is similar to Dung's original way in [10]. Then all the semantics in Dung's theory can be introduced in Dung's way, for example, the grounded extensions and the complete extensions are introduced. But the JDV failed to introduce the ground extensions and the complete extensions.

Next, let's compare the admissible extensions in JDV and WSL. Note that the symbol $\twoheadrightarrow$ in Definition 11 is the same as ρ in Definition 1, and $Args$ is a special case of $\mathcal{AR}$ in Definition 1, where $Args = \chi(Args)$. We can get the next proposition, where the y-admissible sets in JDV is defined by the y-defence in WSL.

Proposition 12. *Given a JAF $(\mathcal{AR}, \twoheadrightarrow)$, a fuzzy set $\mathcal{E} \subseteq \mathcal{AR}$ is a y-admissible extensions, if and only if it y-defends each element in it, i.e., for any $(A, a) \in \mathcal{E}$, there exists $(C, c) \in \mathcal{E}$, s.t.*

$$\max\{1 - \min\{B \twoheadrightarrow A, a\}, \min\{c, C \twoheadrightarrow B\}\} \geq y.$$

It is a direct corollary of Proposition 3.

Also, the y-admissible set can be characterized by the y-characteristic function.

Proposition 13. *Let F_y be a characteristic function. $\mathcal{S} \subseteq \mathcal{AR}$ is y-admissible, if and only if $\mathcal{S} \subseteq F_y(\mathcal{S})$.*

Proof. From Corollary 12, $\mathcal{S}$ is y-admissible, iff for any $(A, a) \in \mathcal{S}$, $\mathcal{S}$ y-defends it. From the definition of F_y, $(A, a) \in F_y(\mathcal{S})$, i.e., $\mathcal{S} \subseteq F_y(\mathcal{S})$. □

It is not difficult to find that there is no restriction of the conflict-freeness in the definition of the y-admissible sets. It makes the notion not so natural.

Example 10. *Consider the JAF $(\{A, B\}, \{(A \to B, 1), ((B \to A, 1))\})$, i.e., the classic AF: $A \leftrightarrows B$.*

It is not difficult to check by Corollary 12 that $\mathcal{S} = \{(A, 1), (B, 1)\}$ is y-admissible for any $y \in [0, 1]$.

However, from Proposition 2, $\mathcal{S}$ is not x-conflict-free for any $x \neq 0$.

Then the set $\{(A,\ 1),\ (B,\ 1)\}$ in Example 5 is not $(x,\ y)$-admissible for any $x > 0$.

The $(x,\ y)$-stable extensions have nothing to do with the z-stable extensions in JAFs. As we know, the z-stable extensions $\mathcal{E}$ are defined by the following function:

$$\inf_{B\in Args}(\neg\mathcal{E}(B) \rightsquigarrow (\mathcal{E} \twoheadrightarrow B)) \geq_{\mathcal{L}} z.$$

If the t-norm is the Gödel t-norm, it is the following equation.

$$\inf_{B\in Args} \max\{1-(1-\mathcal{E}(B)),\ \sup_{A\in Args}\{\mathcal{E}(A),\ A \twoheadrightarrow B\}\} \geq z.$$

It is $\forall B \in Args$, $\exists A \in Args$ s.t. $\max\{\mathcal{E}(B),\ \sup\{\mathcal{E}(A),\ A \twoheadrightarrow B\}\} \geq z$, i.e.,

$$\max\{\mathcal{E}(B),\ \mathcal{E}(A),\ A \twoheadrightarrow B\} \geq z.$$

If the AF is in the form of a cycle, for example, $A \leftrightarrows B$, the empty set $\emptyset$ is z-stable in JAF for all $z \in [0,\ 1]$. Therefore, we introduce a totally new definition of stable extensions.

6.3. Comparability to Gödel fuzzy argumentation frameworks

Given a set of arguments $Args$, a GFAF [29] is a pair $(\mathcal{A},\ \rho)$, where $\mathcal{A} \subseteq \chi_{Args}$ is a fuzzy set of arguments and $\rho\colon Args \times Args$ is a fuzzy set of attacks.

A fuzzy set $\mathcal{S} \subseteq \mathcal{A}$ is conflict-free in GFAF, if for any $(A,\ a),\ (B,\ b) \in \mathcal{S}$,

$$\min\{a,\ B \twoheadrightarrow A\} + b \leq 1.$$

A fuzzy set $\mathcal{S}$ weakening defends a fuzzy argument $(A,\ a)$, if for any $(B,\ b) \in \mathcal{A}$ there exists some $(C,\ c) \in \mathcal{S}$ s.t.

$$\min\{1 - min\{c, \rho(C,\ B)\},\ b\} + a \leq 1. \tag{6.1}$$

Then the characteristic function and the admissible, complete, preferred, grounded extensions are introduced in Dung's way.

In [29], given a fuzzy AF, the semantics is uniquely defined. But in our work, the semantics differ when different consistency degrees $x, y \in [0, 1]$ are given. In other words, the semantics here depends on how much inconsistency the system permits.

In the following, we discuss the conflict-free sets and the acceptability.

First, the conflict-freeness is different. Suppose $\mathcal{S}$ is 1-conflict-free in JAF, i.e., $\neg(\mathcal{S} \twoheadrightarrow \mathcal{S}) \geq 1$. It equals that for any $A,\ B \in Args$, $\min\{\mathcal{S}(A),\ \mathcal{S}(B), A \twoheadrightarrow B\} \leq 0$. It concludes $\mathcal{S}(A) = 0$, $\mathcal{S}(B) = 0$ or $A \twoheadrightarrow B = 0$. Then we can get $\mathcal{S}$ is conflict-free in GFAF. But in general, a conflict-free set in GFAF is not a 1-conflict-free set in JAF. For example, suppose $\mathcal{S}(A) = \mathcal{S}(B) = 0.5 = A \twoheadrightarrow B$. Then $(A,\ 0.5)$ does not sufficiently attack $(B,\ 0.5)$ in GFAF; but $(A,\ 0.5)$ does not sufficiently attack $(B,\ 0.5)$ in JAF.

Moreover, the acceptability in the two fuzzy AFs are distinct. From Proposition 3, for two fuzzy sets $\mathcal{A}, C \subseteq \mathcal{AR}$, C 1-defends $\mathcal{A}$, if and only if for any $(A,\ a) \in \mathcal{A}$ and for any $B \in Args$, there exists some $(C,\ c) \in C$ s.t.

$$\max\{1 - \min\{B \twoheadrightarrow A,\ a\},\ \min\{c,\ C \twoheadrightarrow B\}\} \geq 1.$$

It is distinct to Eq 6.1. Thus the 1-defence in JAF is distinct to the weakening defence in GFAF.

As a result, in the two fuzzy AFs, the characteristic function and the extensions (including the admissible extensions, the complete extensions, the grounded extensions, and the preferred extensions) are different. In a word, the two semantics are distinct.

7. Conclusion

Dung's theory of AFs plays an increasingly important role in artificial intelligence. This paper is an exploration of its fuzzy case. In other words, we discuss the semantics of the AFs where arguments and attacks can be associated with fuzzy degrees.

Many works study the semantics of fuzzy AFs, such as [8, 9, 16, 29], etc. In this paper, the semantics is built on the assumption that some inconsistency is permitted in a fuzzy system and the degree of consistency is already known. For example, if the inconsistency of the conflict-freeness is permitted to be no more than the degree $1 - x$,† then we introduce the x-conflict-free sets. Concretely, in the football scenario, if the conflict-free is permitted to be in a degree no less than 0.8, then light rain and playing football can be accepted at the same time in a conflict-free semantics. In our semantics system, it is the 0.8-conflict-freeness of the set $\{(B,\ 0.2), (A,\ 1)\}$. Similarly, when $\mathcal{S}$ defends $(A,\ a)$ in a degree no less than y, we say that $\mathcal{S}$ y-defends or y-accepts $(A,\ a)$. Various types of extensions are then introduced in Dung's way, for instance, the $(x,\ y)$-admissible extensions, the $(x,\ y)$-preferred extensions, the $(x,\ y)$-complete extensions, the $(x,\ y)$-grounded extensions and the $(x,\ y)$-stable extension. The relation between these extensions is the same as the relation of them in Dung's theory, except that the $(x,\ y)$-stable extensions may not be $(x,\ y)$-preferred.

In some sense, the semantics in this paper can be seen as an improvement of Janssen's work, because the conflict-freeness and the acceptability are the same as the corresponding notions in JAFs. A minor difference is that the numbers $x,\ y$ mean the degree of the consistency here and there is no illustration of them in JAFs. In some sense, it is an illustration of the $x,\ y$ in the semantics of JAFs. Formally, the y-admissible sets and the preferred extensions are developed to $(x,\ y)$-admissible sets and $(x,\ y)$-preferred sets, by adding the x-conflict-freeness to the sets. The y-characteristic function, the $(x,\ y)$-complete extensions, the $(x,\ y)$-grounded extensions and the $(x,\ y)$-stable extensions are introduced. Moreover, some basic properties are discussed in this paper. For example, the existence of the $(x,\ y)$-preferred extensions, the Fundamental Lemma and the algorithm of the $(x,\ y)$-grounded extensions.

In [7], the labeling theory of Dung's AF shows the algorithm of the other extensions, such as the complete extensions, etc. In the future, similar works of the fuzzy AFs can be studied. On the other hand, methods in graph theory [22], rough sets theory [32] can be borrowed to study the AFs, and applications in control theory [19, 20, 31] can be explored.

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Conflict of interest

The authors declare no conflict of interest.

†It equals that the degree of the conflict-freeness is no less than x.

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