



Research article

Post-quantum trapezoid type inequalities

Muhammad Amer Latif¹, Mehmet Kunt², Sever Silvestru Dragomir^{3,4} and İmdat İşcan^{5,*}

¹ Department of Basic Sciences, Deanship of Preparatory Year, King Faisal University, Hofuf 31982, Al-Hasa, Saudi Arabia

² Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, 61080, Trabzon, Turkey

³ Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia

⁴ School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa

⁵ Department of Mathematics, Faculty of Sciences and Arts, Giresun University, 28200, Giresun, Turkey

* **Correspondence:** Email: m_amer_latif@hotmail.com; Tel: +966565329016.

Abstract: In this study, the assumption of being differentiable for the convex function f in the (p, q) -Hermite-Hadamard inequality is removed. A new identity for the right-hand part of (p, q) -Hermite-Hadamard inequality is proved. By using established identity, some (p, q) -trapezoid integral inequalities for convex and quasi-convex functions are obtained. The presented results in this work extend some results from the earlier research.

Keywords: Hermite-Hadamard inequality; q -integral inequalities; (p, q) -integral inequalities; (p, q) -derivative; (p, q) -integration; convexity; quasi-convexity

Mathematics Subject Classification: 26A51, 26D15, 34A08

1. Introduction

Quantum calculus or briefly q -calculus is a study of calculus without limits. Post-quantum or (p, q) -calculus is a generalization of q -calculus and it is the next step ahead of the q -calculus. Quantum Calculus is considered an incorporative subject between mathematics and physics, and many researchers have a particular interest in this subject. Quantum calculus has many applications in various mathematical fields such as orthogonal polynomials, combinatorics, hypergeometric functions, number theory and theory of differential equations etc. Many scholars researching in the

field of inequalities have started to take interest in quantum calculus during the recent years and the active readers are referred to the articles [2, 3, 7–9, 11, 12, 16, 17, 21, 24, 25, 27–29] and the references cited in them for more information on this topic. The authors explore various integral inequalities in all of the papers mentioned above by using q -calculus and (p, q) -calculus for certain classes of convex functions.

In this paper, the main motivation is to study trapezoid type (p, q) -integral inequalities for convex and quasi-convex functions. In fact, we prove that the assumption of the differentiability of the mapping in the (p, q) -Hermite-Hadamard type integral inequalities given in [12] can be eliminated. The relaxation of the differentiability of the mapping in the (p, q) -Hermite-Hadamard type integral inequalities proved in [12] also indicates the originality of results established in our research and these findings have some relationships with those results proved in earlier works.

2. Preliminaries

The basic concepts and findings which will be used in order to prove our results are addressed in this section.

Let $I \subset \mathbb{R}$ be an interval of the set of real numbers. A function $f : I \rightarrow \mathbb{R}$ is called as a convex on I , if the inequality

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

holds for every $x, y \in I$ and $t \in [0, 1]$.

A $f : I \rightarrow \mathbb{R}$ known to be a quasi-convex function, if the inequality

$$f(tx + (1 - t)y) \leq \sup\{f(x), f(y)\}$$

holds for every $x, y \in I$ and $t \in [0, 1]$.

The following properties of convex functions are very useful to obtain our results.

Definition 2.1. [19] A function f defined on I has a support at $x_0 \in I$ if there exists an affine function $A(x) = f(x_0) + m(x - x_0)$ such that $A(x) \leq f(x)$ for all $x \in I$. The graph of the support function A is called a line of support for f at x_0 .

Theorem 2.1. [19] A function $f : (a, b) \rightarrow \mathbb{R}$ is a convex function if and only if there is at least one line of support for f at each $x_0 \in (a, b)$.

Theorem 2.2. [4] If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, then f is continuous on (a, b) .

Perhaps the most famous integral inequalities for convex functions are known as Hermite-Hadamard inequalities and are expressed as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}, \quad (2.1)$$

where the function $f : I \rightarrow \mathbb{R}$ is convex and $a, b \in I$ with $a < b$.

By using the following identity, Pearce and Pečarić proved trapezoid type inequalities related to the convex functions in [18] and [6]. Some trapezoid type inequalities related to quasi-convex functions are proved in [1] and [9].

Lemma 2.3. [6] Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° (I° is the interior of I), $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt = \frac{b-a}{2} \int_0^1 (1-2t) f'(ta + (1-t)b) dt. \quad (2.2)$$

Some definitions and results for (p, q) -differentiation and (p, q) -integration of the function $f : [a, b] \rightarrow \mathbb{R}$ in the papers [12, 22, 23].

Definition 2.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function and $0 < q < p \leq 1$, then (p, q) -derivative of f at $t \in [a, b]$ is characterized by the expression

$${}_a D_{p,q} f(t) = \frac{f(pt + (1-p)a) - f(qt + (1-q)a)}{(p-q)(t-a)}, \quad t \neq a. \quad (2.3)$$

The function f is said to be (p, q) -differentiable on $[a, b]$, if ${}_a D_{p,q} f(t)$ exists for all $t \in [a, b]$. It should be noted that

$${}_a D_{p,q} f(a) = \lim_{t \rightarrow a} {}_a D_{p,q} f(t).$$

It is clear that if $p = 1$ in (2.3), then

$$\begin{aligned} {}_a D_q f(t) &= \frac{f(t) - f(qt + (1-q)a)}{(1-q)(t-a)}, \quad t \neq a. \\ {}_a D_q f(a) &= \lim_{t \rightarrow a} {}_a D_q f(t) \end{aligned} \quad (2.4)$$

the q -derivative of the function f defined on $[a, b]$ (see [16, 21, 25, 26]).

Remark 2.1. If one takes $a = 0$ in (2.3), then ${}_0 D_{p,q} f(t) = D_{p,q} f(t)$, where $D_{p,q} f(t)$ is the (p, q) -derivative of f at $t \in [0, b]$ (see [5, 10, 20]) defined by the expression

$$D_{p,q} f(t) = \frac{f(pt) - f(qt)}{(p-q)t}, \quad t \neq 0. \quad (2.5)$$

Remark 2.2. If for $a = 0$ and $p = 1$ in (2.3), then ${}_0 D_q f(x) = D_q f(t)$, where $D_q f(t)$ is the q -derivative of f at $t \in [0, b]$ (see [15]) given by the expression

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0. \quad (2.6)$$

Definition 2.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function and $0 < q < p \leq 1$. The definite (p, q) -integral of the function f on $[a, b]$ is defined as

$$\int_a^b f(t) {}_a d_{p,q} t = (p-q)(b-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} b + \left(1 - \frac{q^n}{p^{n+1}}\right) a\right) \quad (2.7)$$

If $c \in (a, b)$, then the definite (p, q) -integral of the function f on $[c, b]$ is defined as

$$\int_c^b f(t) {}_a d_{p,q} t = \int_a^b f(t) {}_a d_{p,q} t - \int_a^c f(t) {}_a d_{p,q} t. \quad (2.8)$$

Remark 2.3. Let $p = 1$ be in (2.7), then

$$\int_a^b f(t) {}_a d_q t = (1 - q)(b - a) \sum_{n=0}^{\infty} q^n f(q^n b + (1 - q^n)a) \quad (2.9)$$

the definite q -integral of the function f defined on $[a, b]$ (see [16, 21, 25, 26]).

Remark 2.4. Suppose that $a = 0$ in (2.7), then

$$\int_0^b f(t) {}_0 d_{p,q} t = \int_0^b f(t) d_{p,q} t = (p - q)b \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}b\right) \quad (2.10)$$

the definite (p, q) -integral of f on $[0, b]$ (see [20, 22, 23]). We notice that for $a = 0$ and $p = 1$ in (2.7), then

$$\int_0^b f(t) {}_0 d_q t = \int_0^b f(t) d_q t = (1 - q)b \sum_{n=0}^{\infty} q^n f(q^n b) \quad (2.11)$$

is the definite q -integral of f over the interval $[0, b]$ (see [15]).

Remark 2.5. When we take $a = 0$ and $p = 1$, then the existing definitions in the literature are obtained, hence the Definition 2.2 and Definition 2.3 are well defined.

Quantum trapezoid type inequalities are obtained by Noor et al. [16] and Sudsutad [21] by applying the definition convex and quasi-convex functions on the absolute values of the q -derivative over the finite interval of the set of real numbers.

Lemma 2.4. Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and $0 < q < 1$. If ${}_a D_q f$ is a q -integrable function on (a, b) , then the equality holds:

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(t) {}_a d_q t - \frac{qf(a) + f(b)}{1+q} \\ &= \frac{q(b-a)}{1+q} \int_0^1 (1 - (1+q)t) {}_a D_q f(tb + (1-t)b) {}_0 d_q t. \end{aligned} \quad (2.12)$$

The (p, q) -Hermite-Hadamard type inequalities were proved in [12].

Theorem 2.5. Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex differentiable function on $[a, b]$ and $0 < q < p \leq 1$. Then we have

$$f\left(\frac{qa + pb}{p+q}\right) \leq \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q} x \leq \frac{qf(a) + pf(b)}{p+q}. \quad (2.13)$$

In this paper, we remove the (p, q) -differentiability assumption of the function f in Theorem 2.5 and establish (p, q) -analog of the Lemma 2.4 and Lemma 2.3. We obtain (p, q) -analog of the trapezoid type integral inequalities by applying the established identity, which generalize the inequalities given in [1, 6, 9, 16, 18, 21].

3. Main results

Throughout this section let $I \subset \mathbb{R}$ be an interval, $a, b \in I^\circ$ (I° is the interior of I) with $a < b$ (in other words $[a, b] \subset I^\circ$) and $0 < q < p \leq 1$ are constants. Let us start proving the inequalities (2.13), with the lighter conditions for the function f .

Theorem 3.1. *Let $f : I \rightarrow \mathbb{R}$ be a convex function on I and $a, b \in I^\circ$ with $a < b$, then the following inequalities hold:*

$$f\left(\frac{qa + pb}{p + q}\right) \leq \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q}x \leq \frac{qf(a) + pf(b)}{p+q}. \quad (3.1)$$

Proof. Since f is convex function on the interval I , by Theorem 2.2 f is continuous on I° and $[a, b] \subset I^\circ$, the function f is continuous on $[a, b]$. By using Theorem 2.1, there is at least one line of support for f at each $x_0 \in (a, b)$. Since $x_0 = \frac{qa+pb}{p+q} \in (a, b)$, using Definition 2.1

$$A(x) = f\left(\frac{qa + pb}{p + q}\right) + m\left(x - \frac{qa + pb}{p + q}\right) \leq f(x) \quad (3.2)$$

For all $x \in [a, b]$ and some $m \in \left[f'_-\left(\frac{qa+pb}{p+q}\right), f'_+\left(\frac{qa+pb}{p+q}\right)\right]$. In the proof of the Theorem 2.5 the authors used the tangent line at the point of $x_0 = \frac{qa+pb}{p+q}$. Similarly, using the inequality (3.2) and a similar method with the proof of the Theorem 2.5 we have (3.1) but we omit the details. Thus the proof is accomplished. \square

We will use the following identity to prove trapezoid type (p, q) -integral inequalities for convex and quasi-convex functions.

Lemma 3.2. *Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function on I° and $a, b \in I^\circ$ with $a < b$. If ${}_a D_{p,q}f$ is continuous on $[a, b]$, then the equality:*

$$\begin{aligned} & \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q}x - \frac{pf(b) + qf(a)}{p+q} \\ &= \frac{q(b-a)}{p+q} \int_0^1 (1 - (p+q)t) {}_a D_{p,q}f(tb + (1-t)a) {}_0 d_{p,q}t \end{aligned} \quad (3.3)$$

holds.

Proof. Since f is continuous on I° and $a, b \in I^\circ$, the function f is continuous on $[a, b]$. Hence, clearly $a < pb + (1-p)a \leq b$ for $0 < p \leq 1$ and $[a, pb + (1-p)a] \subset [a, b]$. Hence f is continuous on $[a, pb + (1-p)a]$ and hence according to the condition of the Definition 2.3, the function f is (p, q) -integrable on $[a, pb + (1-p)a]$. This means that the (p, q) -integral

$$\int_a^{pb+(1-p)a} f(x) {}_a d_{p,q}x$$

is well defined and exists.

Since f is continuous on $[a, b]$. Hence from Definition 2.2, f is (p, q) -differentiable on $[a, b]$. Thus the (p, q) -derivative of f given by the expression

$$\begin{aligned} {}_aD_{p,q}f(tb + (1-t)a) &= \frac{[f(p[tb + (1-t)a] + (1-p)a) - f(q[tb + (1-t)a] + (1-q)a)]}{(p-q)[tb + (1-t)a - a]} \quad (3.4) \\ &= \frac{f(ptb + (1-pt)a) - f(qtb + (1-qt)a)}{t(p-q)(b-a)}, t \neq 0 \end{aligned}$$

is well defined and exists.

Since $(1 - (p + q)t)$ is continuous on $[0, 1]$ and ${}_aD_{p,q}f$ is continuous on $[a, b]$, then

$$(1 - (p + q)t) {}_aD_{p,q}f(tb + (1-t)a)$$

is continuous on $[0, 1]$ and from Definition 2.3. Thus $(1 - (p + q)t) {}_aD_{p,q}f(tb + (1-t)a)$ is (p, q) -integrable on $[0, 1]$ and the (p, q) -integral

$$\int_0^1 (1 - (p + q)t) {}_aD_{p,q}f(tb + (1-t)a) {}_0d_{p,q}t$$

is well defined and exists.

By using (2.7) and (3.4), we get

$$\begin{aligned} & \frac{q(b-a)}{p+q} \int_0^1 (1 - (p + q)t) {}_aD_{p,q}f(tb + (1-t)a) {}_0d_{p,q}t \\ &= \frac{q(b-a)}{p+q} \int_0^1 (1 - (p + q)t) \frac{f(ptb + (1-pt)a) - f(qtb + (1-qt)a)}{t(p-q)(b-a)} {}_0d_{p,q}t \\ &= \frac{q}{p+q} \left[\frac{1}{(p-q)} \int_0^1 \frac{f(ptb + (1-pt)a) - f(qtb + (1-qt)a)}{t} {}_0d_{p,q}t \right. \\ & \quad \left. - \frac{(p+q)}{(p-q)} \int_0^1 f(ptb + (1-pt)a) - f(qtb + (1-qt)a) {}_0d_{p,q}t \right] \\ &= \frac{q}{(p+q)(p-q)} \left[\int_0^1 \frac{f(ptb + (1-pt)a)}{t} {}_0d_{p,q}t - \int_0^1 \frac{f(qtb + (1-qt)a)}{t} {}_0d_{p,q}t \right. \\ & \quad \left. - (p+q) \int_0^1 f(ptb + (1-pt)a) {}_0d_{p,q}t + (p+q) \int_0^1 f(qtb + (1-qt)a) {}_0d_{p,q}t \right] \\ &= \frac{q}{(p+q)} \left[\sum_{n=0}^{\infty} f\left(\frac{q^n}{p^n}b + \left(1 - \frac{q^n}{p^n}\right)a\right) - \sum_{n=0}^{\infty} f\left(\frac{q^{n+1}}{p^{n+1}}b + \left(1 - \frac{q^{n+1}}{p^{n+1}}\right)a\right) \right. \\ & \quad \left. - (p+q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^n}b + \left(1 - \frac{q^n}{p^n}\right)a\right) \right. \\ & \quad \left. + (p+q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^{n+1}}{p^{n+1}}b + \left(1 - \frac{q^{n+1}}{p^{n+1}}\right)a\right) \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{q}{(p+q)} \left[f(b) - f(a) - \frac{(p+q)}{p} \sum_{n=0}^{\infty} \frac{q^n}{p^n} f\left(\frac{q^n}{p^n}b + \left(1 - \frac{q^n}{p^n}\right)a\right) \right. \\
&\quad \left. + \frac{(p+q)}{q} \sum_{n=0}^{\infty} \frac{q^{n+1}}{p^{n+1}} f\left(\frac{q^{n+1}}{p^{n+1}}b + \left(1 - \frac{q^{n+1}}{p^{n+1}}\right)a\right) \right] \\
&= \frac{q}{(p+q)} \left[f(b) - f(a) - \frac{(p+q)}{q} f(b) - \frac{(p+q)}{p} \sum_{n=0}^{\infty} \frac{q^n}{p^n} f\left(\frac{q^n}{p^n}b + \left(1 - \frac{q^n}{p^n}\right)a\right) \right. \\
&\quad \left. + \frac{(p+q)}{q} \sum_{n=-1}^{\infty} \frac{q^{n+1}}{p^{n+1}} f\left(\frac{q^{n+1}}{p^{n+1}}b + \left(1 - \frac{q^{n+1}}{p^{n+1}}\right)a\right) \right] \\
&= \frac{q}{(p+q)} \left[-f(a) - \frac{p}{q} f(b) - \frac{(p+q)}{p} \sum_{n=0}^{\infty} \frac{q^n}{p^n} f\left(\frac{q^n}{p^n}b + \left(1 - \frac{q^n}{p^n}\right)a\right) \right. \\
&\quad \left. + \frac{(p+q)}{q} \sum_{n=0}^{\infty} \frac{q^n}{p^n} f\left(\frac{q^n}{p^n}b + \left(1 - \frac{q^n}{p^n}\right)a\right) \right] \\
&= \frac{q}{(p+q)} \left[-f(a) - \frac{p}{q} f(b) - \left(\frac{(p+q)}{p} - \frac{(p+q)}{q}\right) \sum_{n=0}^{\infty} \frac{q^n}{p^n} f\left(\frac{q^n}{p^n}b + \left(1 - \frac{q^n}{p^n}\right)a\right) \right] \\
&= \frac{q}{(p+q)} \left[-f(a) - \frac{p}{q} f(b) - \left(\frac{(p+q)}{p} - \frac{(p+q)}{q}\right) \frac{1}{(p-q)(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q}x \right] \\
&= \frac{1}{p(b-q)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q}x - \frac{pf(b) + qf(a)}{p+q}.
\end{aligned}$$

This completes the proof. \square

Remark 3.1. The subsequent observations are important to note from the result of Lemma 3.2:

1. If $p = 1$, we recapture Lemma 2.4,
2. If $p = 1$ and $q \rightarrow 1^-$, we recapture Lemma 2.3.

We can now prove some quantum estimates of (p, q) -trapezoidal integral inequalities by using convexity and quasi-convexity of the absolute values of the (p, q) -derivatives.

Theorem 3.3. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function on I° and $a, b \in I^\circ$ with $a < b$ such that ${}_a D_{p,q}f$ is continuous on $[a, b]$ and $0 < q < p \leq 1$. If $|{}_a D_{p,q}f|^r$ is a convex function on $[a, b]$ for $r \geq 1$, then

$$\begin{aligned}
&\left| \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q}x - \frac{pf(b) + qf(a)}{p+q} \right| \tag{3.5} \\
&\leq \frac{q(b-a)}{p+q} \left[\frac{2(p+q-1)}{(p+q)^2} \right]^{1-\frac{1}{r}} \left[\gamma_1(p, q) |{}_a D_{p,q}f(b)|^r + \gamma_2(p, q) |{}_a D_{p,q}f(a)|^r \right]^{\frac{1}{r}}
\end{aligned}$$

holds, where

$$\gamma_1(p, q) = \frac{q[(p^3 - 2 + 2p) + (2p^2 + 2)q + pq^2] + 2p^2 - 2p}{(p+q)^3(p^2 + pq + q^2)}$$

and

$$\gamma_2(p, q) = \frac{q[(5p^3 - 4p^2 - 2p + 2) + (6p^2 - 4p - 2)q + (5p - 2)q^2 + 2q^3] + (2p^4 - 2p^3 - 2p^2 + 2p)}{(p+q)^3(p^2 + pq + q^2)}.$$

Proof. Taking absolute value on both sides of (3.3), applying the power-mean inequality and by using the convexity of $|{}_aD_{p,q}f|^r$ for $r \geq 1$, we obtain

$$\begin{aligned} & \left| \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q}x - \frac{pf(b) + qf(a)}{p+q} \right| \quad (3.6) \\ & \leq \frac{q(b-a)}{p+q} \int_0^1 |1 - (p+q)t| |{}_a D_{p,q}f(tb + (1-t)a)| {}_0 d_{p,q}t \\ & \leq \frac{q(b-a)}{p+q} \left(\int_0^1 |1 - (p+q)t| {}_0 d_{p,q}t \right)^{1-\frac{1}{r}} \left(\int_0^1 |1 - (p+q)t| |{}_a D_{p,q}f(tb + (1-t)a)|^r {}_0 d_{p,q}t \right)^{\frac{1}{r}} \\ & \leq \frac{q(b-a)}{p+q} \left(\int_0^1 |1 - (p+q)t| {}_0 d_{p,q}t \right)^{1-\frac{1}{r}} \\ & \times \left[|{}_a D_{p,q}f(b)|^r \int_0^1 t|1 - (p+q)t| {}_0 d_{p,q}t + |{}_a D_{p,q}f(a)|^r \int_0^1 (1-t)|1 - (p+q)t| {}_0 d_{p,q}t \right]^{\frac{1}{r}}. \end{aligned}$$

We evaluate the definite (p, q) -integrals as follows

$$\int_0^1 |1 - (p+q)t| {}_0 d_{p,q}t = \int_0^{\frac{1}{p+q}} (1 - (p+q)t) {}_0 d_{p,q}t - \int_{\frac{1}{p+q}}^1 (1 - (p+q)t) {}_0 d_{p,q}t \quad (3.7)$$

$$= 2 \int_0^{\frac{1}{p+q}} (1 - (p+q)t) {}_0 d_{p,q}t - \int_0^1 (1 - (p+q)t) {}_0 d_{p,q}t = \frac{2(p+q-1)}{(p+q)^2},$$

$$\int_0^1 t|1 - (p+q)t| {}_0 d_{p,q}t = 2 \int_0^{\frac{1}{p+q}} t(1 - (p+q)t) {}_0 d_{p,q}t - \int_0^1 t(1 - (p+q)t) {}_0 d_{p,q}t \quad (3.8)$$

$$= \frac{2p^2 + 2pq + 2q^2 - 2p - 2q}{(p+q)^3(p^2 + pq + q^2)} - \frac{-pq}{(p+q)(p^2 + pq + q^2)}$$

$$= \frac{p^3q + 2p^2q^2 + pq^3 + 2p^2 + 2pq + 2q^2 - 2p - 2q}{(p+q)^3(p^2 + pq + q^2)}$$

$$= \frac{q[(p^3 - 2 + 2p) + (2p^2 + 2)q + pq^2] + 2p^2 - 2p}{(p+q)^3(p^2 + pq + q^2)} = \gamma_1(p, q)$$

and

$$\begin{aligned}
 \int_0^1 (1-t)|1-(p+q)t| {}_0d_{p,q}t &= \int_0^1 |1-(p+q)t| {}_0d_{p,q}t - \int_0^1 t|1-(p+q)t| {}_0d_{p,q}t \quad (3.9) \\
 &= \frac{2(p+q-1)}{(p+q)^2} - \frac{[p^3q + 2p^2q^2 + pq^3 + 2p^2 + 2pq + 2q^2 - 2p - 2q]}{(p+q)^3(p^2 + pq + q^2)} \\
 &= \frac{q[(5p^3 - 4p^2 - 2p + 2) + (6p^2 - 4p - 2)q + (5p - 2)q^2 + 2q^3] + (2p^4 - 2p^3 - 2p^2 + 2p)}{(p+q)^3(p^2 + pq + q^2)} \\
 &= \gamma_2(p, q).
 \end{aligned}$$

Making use of (3.7), (3.8) and (3.9) in (3.6), gives us the desired result (3.5). The proof is thus accomplished. \square

Corollary 3.1. *We can get the following subsequent results from (3.5) proved in Theorem 3.3:*

- (1). Suppose $p = 1$ and $r = 1$, then we acquire the inequality proved in [21, Theorem 4.1] (see also [13, inequality (5)]):

$$\begin{aligned}
 &\left| \frac{1}{(b-a)} \int_a^b f(x) {}_a d_q x - \frac{qf(a) + f(b)}{1+q} \right| \quad (3.10) \\
 &\leq \frac{q^2(b-a)}{(1+q)^4(1+q+q^2)} \left[[1+4q+q^2] |{}_a D_q f(b)| + [1+3q^2+2q^3] |{}_a D_q f(a)| \right].
 \end{aligned}$$

- (2). Letting $p = 1$, provides the inequality established in [16, Theorem 3.2] (see also [14], [21, Theorem 4.2] and [13]):

$$\begin{aligned}
 &\left| \frac{1}{(b-a)} \int_a^b f(x) {}_a d_q x - \frac{qf(a) + f(b)}{1+q} \right| \leq \frac{q(b-a)}{1+q} \left[\frac{2q}{(1+q)^2} \right]^{1-\frac{1}{r}} \quad (3.11) \\
 &\times \left[\frac{q[1+4q+q^2]}{(1+q)^3(1+q+q^2)} |{}_a D_q f(b)|^r + \frac{q[1+3q^2+2q^3]}{(1+q)^3(1+q+q^2)} |{}_a D_q f(a)|^r \right]^{\frac{1}{r}}.
 \end{aligned}$$

- (3). Taking $p = 1$ and letting $q \rightarrow 1^-$, gives the inequality proved in [18, Theorem 1]:

$$\left| \frac{1}{(b-a)} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)}{4} \left[\frac{|f'(a)|^r + |f'(b)|^r}{2} \right]^{\frac{1}{r}}. \quad (3.12)$$

- (4). Suppose $r = 1$, $p = 1$ and letting $q \rightarrow 1^-$, we obtain the inequality proved in [6, Theorem 2.2]:

$$\left| \frac{1}{(b-a)} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)[|f'(a)| + |f'(b)|]}{8}. \quad (3.13)$$

Theorem 3.4. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function on I° and $a, b \in I^\circ$ with $a < b$. If ${}_a D_{p,q} f$ is continuous on $[a, b]$, $0 < q < p \leq 1$ and $|{}_a D_{p,q} f|^r$ is a convex function on $[a, b]$ for $r > 1$, then

$$\left| \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q} x - \frac{pf(b) + qf(a)}{p+q} \right| \quad (3.14)$$

$$\leq \frac{q(b-a)}{p+q} [\gamma_3(p, q; s)]^{\frac{1}{s}} \left(\frac{|{}_a D_{p,q} f(b)|^r + (p+q-1) |{}_a D_{p,q} f(a)|^r}{p+q} \right)^{\frac{1}{r}},$$

where

$$\gamma_3(p, q; s) = \int_0^1 |1 - (p+q)t|^s {}_0 d_{p,q} t$$

and $\frac{1}{r} + \frac{1}{s} = 1$.

Proof. Taking absolute value on both sides of (3.3), applying the Hölder inequality and using the convexity of $|{}_a D_{p,q} f|^r$ for $r > 1$, we get

$$\left| \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q} x - \frac{pf(b) + qf(a)}{p+q} \right| \quad (3.15)$$

$$\leq \frac{q(b-a)}{p+q} \left(\int_0^1 |1 - (p+q)t|^s {}_0 d_{p,q} t \right)^{\frac{1}{s}}$$

$$\times \left[|{}_a D_{p,q} f(b)|^r \int_0^1 t {}_0 d_{p,q} t + |{}_a D_{p,q} f(a)|^r \int_0^1 (1-t) {}_0 d_{p,q} t \right]^{\frac{1}{r}}$$

$$= \frac{q(b-a)}{p+q} (\gamma_3(p, q; s))^{\frac{1}{s}} \left[|{}_a D_{p,q} f(b)|^r \int_0^1 t {}_0 d_{p,q} t + |{}_a D_{p,q} f(a)|^r \int_0^1 (1-t) {}_0 d_{p,q} t \right]^{\frac{1}{r}}.$$

We evaluate the definite (p, q) -integrals as follows

$$\int_0^1 t {}_0 d_{p,q} t = \frac{1}{p+q}$$

and

$$\int_0^1 (1-t) {}_0 d_{p,q} t = \frac{p+q-1}{p+q}.$$

By using the values of the above definite (p, q) -integrals in (3.15), we get what is required. \square

Corollary 3.2. In Theorem 3.4;

(1). If we take $p = 1$, then

$$\left| \frac{1}{(b-a)} \int_a^b f(x) {}_a d_q x - \frac{qf(a) + f(b)}{1+q} \right| \quad (3.16)$$

$$\leq \frac{q(b-a)}{1+q} [\gamma_3(1, q; s)]^{\frac{1}{s}} \left(\frac{|{}_a D_q f(b)|^r + q |{}_a D_q f(a)|^r}{1+q} \right)^{\frac{1}{r}}.$$

(2). If we take $p = 1$ and letting $q \rightarrow 1^-$, then

$$\left| \frac{1}{(b-a)} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)}{2(s+1)^{\frac{1}{s}}} \left[\frac{|f'(a)|^r + |f'(b)|^r}{2} \right]^{\frac{1}{r}}. \quad (3.17)$$

Remark 3.2. The inequality (3.17) has been established in [6, Theorem 2.3].

Theorem 3.5. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function on I° and $a, b \in I^\circ$ with $a < b$. Suppose that ${}_a D_{p,q} f$ is continuous on $[a, b]$, $0 < q < p \leq 1$ and $|{}_a D_{p,q} f|^r$ is a convex function on $[a, b]$ for $r > 1$, then

$$\left| \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q} x - \frac{pf(b) + qf(a)}{p+q} \right| \quad (3.18)$$

$$\leq \frac{q(b-a)}{p+q} \left[\frac{2(p+q-1)}{(p+q)^2} \right]^{\frac{1}{s}} \left[\gamma_1(p, q) |{}_a D_{p,q} f(b)|^r + \gamma_2(p, q) |{}_a D_{p,q} f(a)|^r \right]^{\frac{1}{r}},$$

where $\gamma_1(p, q)$, $\gamma_2(p, q)$ are defined as in Theorem 3.3 and $\frac{1}{r} + \frac{1}{s} = 1$.

Proof. Taking absolute value on both sides of (3.3), applying the Hölder inequality and using the convexity of $|{}_a D_{p,q} f|^r$ for $r > 1$, we have that

$$\left| \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q} x - \frac{pf(b) + qf(a)}{p+q} \right| \quad (3.19)$$

$$= \frac{q(b-a)}{p+q} \left| \int_0^1 (1-(p+q)t) {}_a D_{p,q} f(tb + (1-t)a) {}_0 d_{p,q} t \right|$$

$$= \frac{q(b-a)}{p+q} \left| \int_0^1 (1-(p+q)t)^{\frac{1}{s}} (1-(p+q)t)^{\frac{1}{r}} {}_a D_{p,q} f(tb + (1-t)a) {}_0 d_{p,q} t \right|$$

$$\leq \frac{q(b-a)}{p+q} \left[\int_0^1 |1-(p+q)t| {}_0 d_{p,q} t \right]^{\frac{1}{s}} \left[\int_0^1 |1-(p+q)t| |{}_a D_{p,q} f(tb + (1-t)a)|^r {}_0 d_{p,q} t \right]^{\frac{1}{r}}$$

$$\leq \frac{q(b-a)}{p+q} \left[\int_0^1 |1-(p+q)t| {}_0d_{p,q}t \right]^{\frac{1}{s}}$$

$$\times \left[\left| {}_aD_{p,q}f(b) \right|^r \int_0^1 t|1-(p+q)t| {}_0d_{p,q}t + \left| {}_aD_{p,q}f(a) \right|^r \int_0^1 (1-t)|1-(p+q)t| {}_0d_{p,q}t \right]^{\frac{1}{r}}.$$

Making use of (3.7), (3.8) and (3.9) in (3.19), gives us the desired result (3.18). The proof is thus accomplished. \square

Corollary 3.3. *The following results are the consequences of Theorem 3.5:*

(1). Taking $p = 1$, we obtain the inequality proved in [16, Theorem 3.3] (see also [14, inequality (8)]):

$$\left| \frac{1}{(b-a)} \int_a^b f(x) {}_ad_qx - \frac{qf(a) + f(b)}{1+q} \right| \leq \frac{q(b-a)}{1+q} \left[\frac{2q}{(1+q)^2} \right]^{\frac{1}{s}} \quad (3.20)$$

$$\times \left[\frac{q[1+4q+q^2]}{(1+q)^3(1+q+q^2)} \left| {}_aD_qf(b) \right|^r + \frac{q[1+3q^2+2q^3]}{(1+q)^3(1+q+q^2)} \left| {}_aD_qf(a) \right|^r \right]^{\frac{1}{r}},$$

(2). Taking $p = 1$ and letting $q \rightarrow 1^-$, we obtain the inequality proved in [18, Theorem 1]:

$$\left| \frac{1}{(b-a)} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)}{2^{s+1}} \left[\frac{|f'(b)|^r + |f'(a)|^r}{4} \right]^{\frac{1}{r}}. \quad (3.21)$$

Some results related for quasi-convexity are presented in the following theorems.

Theorem 3.6. *Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function on I° and $a, b \in I^\circ$ with $a < b$. If ${}_aD_{p,q}f$ is continuous on $[a, b]$, where $0 < q < p \leq 1$ and $\left| {}_aD_{p,q}f \right|^r$ is a quasi-convex function on $[a, b]$ $r \geq 1$, then*

$$\left| \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_ad_{p,q}x - \frac{pf(b) + qf(a)}{p+q} \right| \quad (3.22)$$

$$\leq \frac{q(b-a)}{p+q} \left[\frac{2(p+q-1)}{(p+q)^2} \right] \sup \left\{ \left| {}_aD_{p,q}f(a) \right|, \left| {}_aD_{p,q}f(b) \right| \right\}.$$

Proof. Taking absolute value on both sides of (3.3), applying the power mean inequality and using the quasi-convexity of $\left| {}_aD_{p,q}f \right|^r$ on $[a, b]$ for $r \geq 1$, we have that

$$\left| \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_ad_{p,q}x - \frac{pf(b) + qf(a)}{p+q} \right|$$

$$\begin{aligned}
&\leq \frac{q(b-a)}{p+q} \int_0^1 |1-(p+q)t| \left| {}_a D_{p,q} f(tb+(1-t)a) \right| {}_0 d_{p,q} t \\
&\leq \frac{q(b-a)}{p+q} \left(\int_0^1 |1-(p+q)t| {}_0 d_{p,q} t \right)^{1-\frac{1}{r}} \\
&\quad \times \left(\int_0^1 |1-(p+q)t| {}_0 d_{p,q} t \sup \left\{ \left| {}_a D_{p,q} f(a) \right|^r, \left| {}_a D_{p,q} f(b) \right|^r \right\} \right)^{\frac{1}{r}} \\
&= \frac{q(b-a)}{p+q} \left(\int_0^1 |1-(p+q)t| {}_0 d_{p,q} t \right) \sup \left\{ \left| {}_a D_{p,q} f(a) \right|, \left| {}_a D_{p,q} f(b) \right| \right\}.
\end{aligned}$$

From (3.7), we have

$$\int_0^1 |1-(p+q)t| {}_0 d_{p,q} t = \frac{2(p+q-1)}{(p+q)^2}.$$

Hence the inequality (3.22) is established. \square

Corollary 3.4. *In Theorem 3.6*

(1). If we let $p = 1$, then:

$$\begin{aligned}
&\left| \frac{1}{(b-a)} \int_a^b f(x) {}_a d_q x - \frac{qf(a)+f(b)}{1+q} \right| \\
&\leq \frac{q(b-a)}{1+q} \left[\frac{2q}{(1+q)^2} \right] \sup \left\{ \left| {}_a D_q f(a) \right|, \left| {}_a D_q f(b) \right| \right\}.
\end{aligned} \tag{3.23}$$

(2). If we take $p = 1$ and letting $q \rightarrow 1^-$, then:

$$\left| \frac{1}{(b-a)} \int_a^b f(x) dx - \frac{f(a)+f(b)}{2} \right| \leq \frac{(b-a)}{4} \sup \{|f'(a)|, |f'(b)|\}. \tag{3.24}$$

Remark 3.3. From the results of Corollary 3.4, we can observe the following consequences

- (1). The result of the inequality (3.23) has also been obtained in [16, Theorem 3.4] (see also [14, inequality (9)]),
- (2). The result of the inequality (3.24) was established in [1, Theorem 6] and [9, Theorem 1].

Theorem 3.7. *Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function on I° and $a, b \in I^\circ$ with $a < b$. If ${}_a D_{p,q} f$ is continuous on $[a, b]$, where $0 < q < p \leq 1$ and $\left| {}_a D_{p,q} f \right|^r$ is a quasi-convex function on $[a, b]$ for $r > 1$, then:*

$$\left| \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q} x - \frac{pf(b)+qf(a)}{p+q} \right| \tag{3.25}$$

$$\leq \frac{q(b-a)}{p+q} [\gamma_3(p, q; s)]^{\frac{1}{s}} \left(\sup \left\{ \left| {}_a D_{p,q} f(a) \right|, \left| {}_a D_{p,q} f(b) \right| \right\} \right),$$

where $\gamma_3(p, q; s)$ is as defined in Theorem 3.4 and $\frac{1}{r} + \frac{1}{s} = 1$.

Proof. Taking absolute value on both sides of (3.3), applying the Hölder inequality and using the quasi-convexity of $\left| {}_a D_{p,q} f \right|^r$ on $[a, b]$ for $r > 1$, we have that

$$\begin{aligned} & \left| \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q} x - \frac{pf(b) + qf(a)}{p+q} \right| \\ & \leq \frac{q(b-a)}{p+q} \int_0^1 |1 - (p+q)t| {}_a D_{p,q} f(tb + (1-t)a) | {}_0 d_{p,q} t \\ & \leq \frac{q(b-a)}{p+q} \left[\int_0^1 |1 - (p+q)t|^s {}_0 d_{p,q} t \right]^{\frac{1}{s}} \\ & \quad \times \left[\int_0^1 \left| {}_a D_{p,q} f(tb + (1-t)a) \right|^r {}_0 d_{p,q} t \right]^{\frac{1}{r}} \\ & \leq \frac{q(b-a)}{p+q} \left[\int_0^1 |1 - (p+q)t|^s {}_0 d_{p,q} t \right]^{\frac{1}{s}} \sup \left\{ \left| {}_a D_{p,q} f(b) \right|, \left| {}_a D_{p,q} f(a) \right| \right\} \\ & = \frac{q(b-a)}{p+q} [\gamma_3(p, q; s)]^{\frac{1}{s}} \left(\sup \left\{ \left| {}_a D_{p,q} f(a) \right|, \left| {}_a D_{p,q} f(b) \right| \right\} \right) \end{aligned}$$

The inequality (3.25) is proved. \square

Corollary 3.5. In Theorem 3.7;

(1). If $p = 1$, then we obtain the inequality proved in [9, Theorem 2]:

$$\begin{aligned} & \left| \frac{1}{(b-a)} \int_a^b f(x) {}_a d_q x - \frac{qf(a) + f(b)}{1+q} \right| \\ & \leq \frac{q(b-a)}{1+q} [\gamma_3(1, q; s)]^{\frac{1}{s}} \left(\sup \left\{ \left| {}_a D_q f(a) \right|, \left| {}_a D_q f(b) \right| \right\} \right), \end{aligned} \quad (3.26)$$

(2). If $p = 1$ and letting $q \rightarrow 1^-$, then:

$$\left| \frac{1}{(b-a)} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)}{2(s+1)^{\frac{1}{s}}} \sup \{ |f'(a)|, |f'(b)| \}. \quad (3.27)$$

acknowledgement

The authors would like to thank the referee for his/her careful reading of the manuscript and for making valuable suggestions.

Conflict of interest

The authors declare to have no conflict of interest.

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