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Research article

Two different systematic methods for constructing meromorphic exact solutions to the KdV-Sawada-Kotera equation

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Abstract: In this paper, we obtain meromorphic exact solutions of the KdV-Sawada-Kotera equation via two different systematic methods. Applying the $\exp(-\psi(z))$-expansion method, we achieve the trigonometric, exponential, hyperbolic and rational function solutions for the mentioned equation. It is more interesting that we firstly proposed the extended complex method based on the previous work of Yuan *et al.*, and as an example we use it to search exact solutions to the KdV-Sawada-Kotera equation. Dynamic behaviors of solutions obtained by these two different systematic techniques are also shown by some graphs. The results show that these two methods are direct and efficient methods to deal with various differential equations in the applied sciences.

Keywords: differential equation; KdV-Sawada-Kotera equation; symbolic computation; extended complex method; $\exp(-\psi(z))$-expansion method
Mathematics Subject Classification: 30D35, 34A05, 35C07

1. Introduction

Hirota and Ito [1] proposed the following Sawada-Kotera equation to theoretically study the resonances of solitons in one dimension,

$$u_t + b(15u^3 + 15uu_{xx} + u_{xxxx})_x = 0, \tag{1.1}$$

which has a non-vanishing boundary condition

$$u|_{x=\infty} = constant. \tag{1.2}$$

Replace u by $u + \frac{a}{15b}$ and apply the Galilei transformation to remove u_x, then Eq (1.1) changes to the KdV-Sawada-Kotera equation [2]

$$u_t + a(3u^2 + u_{xx})_x + b(15u^3 + 15uu_{xx} + u_{xxxx})_x = 0, \tag{1.3}$$

where a, b are constants. It is a linear combination of the Sawada-Kotera equation and the KdV equation, with considering $a = 0$, Eq (1.3) reduces to the Sawada-Kotera equation, when $b = 0$, Eq (1.3) reduces to the KdV equation. In the past few years, many achievements have been made in the study of KdV-Sawada-Kotera equation. About this equation, conservation laws are investigated by Konno [3], and traveling wave solutions are discovered in [4]. Quasi-periodic wave and exact solitary wave solutions to the KdV-Sawada-Kotera equation are obtained [5].

As we know, nonlinear differential equations (NLDEs) are widely utilized in fluid dynamics, solid state physics, plasma physics, biology, nonlinear optics, chemistry and so on. The study to exact solutions of various NLDEs is extremely important in modern mathematics with ramifications to some areas of physics, mathematics and other sciences. There are many systematic methods to seek exact solutions of NLDEs, for example, Hirota bilinear method [6,7], modified simple equation method [8], generalized (G'/G)-expansion method [9, 10], modified Kudryashov method [11, 12], exp function method [13, 14], modified extended tanh method [15, 16], sine-Gordon expansion method [17, 18], extended sine-Gordon expansion method [19,20], complex method [21–24] and $\exp(-\psi(z))$-expansion method [25–28].

Eremenko showed that all meromorphic solutions of the Kuramoto Sivashinsky equation are elliptic function and its degeneration in [29]. After that, Laurent series were applied by Kudryashov *et al.* [30,31] to obtain meromorphic exact solutions to certain nonlinear differential equations. On the basis of their work, Yuan *et al.* [32,33] established the complex method combining the theories of complex analysis and complex differential equations. It is a powerful approach to obtain exact solutions for NLDEs that admit $\langle p, q\rangle$ condition or are Briot-Bouquet (BB) equations [34]. Following their work, we propose the extended complex method to get meromorphic exact solutions for NLDEs which neither admit $\langle p, q\rangle$ condition nor BB equations. Therefore, the extended complex method is an enhancement of the complex method and should deal with more NLDEs in applied sciences.

The $\exp(-\psi(z))$-expansion approach is an effectual technique to seek analytical solutions for NLDEs. A lot of researchers, for instance, Jafari, Khan, Roshid, etc [25–28], made good use of this method to study NLDEs. In this article, we utilize two different systematic methods mentioned above to seek meromorphic exact solutions of the KdV-Sawada-Kotera equation. Dynamic behaviors of the solutions are shown by some graphs in which the profiles of Weierstrass elliptic function solutions have never been shown in former literatures.

2. Description of the $\exp(-\psi(z))$-expansion method

Consider the following nonlinear PDE:

$$P(u, u_x, u_t, u_{xx}, u_{tt}, \cdots) = 0, \tag{2.1}$$

where P is a polynomial consisted by the unknown function $u(x, t)$ as well as its partial derivatives.

Step 1. Reduce Eq (2.1) to the ODE

$$F(u, u', u'', u''', \cdots) = 0, \tag{2.2}$$

by traveling wave transform

$$u(x,t) = u(z), \quad z = kx + rt.$$

Step 2. Assume that Eq (2.2) has exact solutions as follows:

$$u(z) = \sum_{\tau=0}^{m} B_\tau (\exp(-\psi(z)))^\tau, \tag{2.3}$$

where B_τ $(0 \leq \tau \leq m)$ are constants to be determined latter, such that $B_m \neq 0$ and $\psi = \psi(z)$ admits the following ODE:

$$\psi'(z) = \gamma + \exp(-\psi(z)) + \mu \exp(\psi(z)). \tag{2.4}$$

The solutions of Eq (2.4) are given in the following.
When $\gamma^2 - 4\mu > 0, \mu \neq 0,$

$$\psi(z) = \ln\left(\frac{-\sqrt{(\gamma^2-4\mu)}\tanh(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+c)) - \gamma}{2\mu}\right), \tag{2.5}$$

$$\psi(z) = \ln\left(\frac{-\sqrt{(\gamma^2-4\mu)}\coth(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+c)) - \gamma}{2\mu}\right). \tag{2.6}$$

When $\gamma^2 - 4\mu < 0, \mu \neq 0,$

$$\psi(z) = \ln\left(\frac{\sqrt{(4\mu-\gamma^2)}\tan(\frac{\sqrt{(4\mu-\gamma^2)}}{2}(z+c)) - \gamma}{2\mu}\right), \tag{2.7}$$

$$\psi(z) = \ln\left(\frac{\sqrt{(4\mu-\gamma^2)}\cot(\frac{\sqrt{(4\mu-\gamma^2)}}{2}(z+c)) - \gamma}{2\mu}\right). \tag{2.8}$$

When $\gamma^2 - 4\mu > 0, \gamma \neq 0, \mu = 0,$

$$\psi(z) = -\ln\left(\frac{\gamma}{\exp(\gamma(z+c)) - 1}\right). \tag{2.9}$$

When $\gamma^2 - 4\mu = 0, \gamma \neq 0, \mu \neq 0,$

$$\psi(z) = \ln\left(-\frac{2(\gamma(z+c)+2)}{\gamma^2(z+c)}\right). \tag{2.10}$$

When $\gamma^2 - 4\mu = 0, \gamma = 0, \mu = 0,$

$$\psi(z) = \ln(z+c). \tag{2.11}$$

In Eqs (2.5)–(2.11), $B_m \neq 0, \gamma, \mu, c$ are constants. Taking the homogeneous balance between nonlinear terms and highest order derivatives of Eq (2.2) yields the positive integer m.

Step 3. Insert Eq (2.3) into Eq (2.2) and collect the function $\exp(-\psi(z))$ to yield the polynomial to $\exp(-\psi(z))$. Letting all coefficients with same power of $\exp(-\psi(z))$ be zero to obtain a system of algebraic equations. Solving these equations, we achieve the values of $B_m \neq 0, \gamma, \mu$ and substitute them into Eq (2.3) as well as Eqs (2.5)–(2.11) to accomplish the determination for analytical solutions of the original PDE.

3. Utilization of the exp($-\psi(z)$)-expansion method to the KdV-Sawada-Kotera equation

Substituting

$$u(x,t) = u(z), \quad z = kx + rt,$$

into Eq (1.3) and then integrating it we obtain

$$ru + 3kau^2 + k^3au'' + 15kbu^3 + 15k^3buu'' + k^5bu'''' + \zeta = 0. \tag{3.1}$$

where ζ is the integration constant.

Taking the homogeneous balance between u'''' and uu'' in Eq (3.1) to yields

$$u(z) = B_0 + B_1 \exp(-\psi(z)) + B_2(\exp(-\psi(z)))^2, \tag{3.2}$$

where $B_2 \neq 0$, B_1 and B_0 are constants.

Substituting $u'''', uu'', u'', u^3, u^2, u$ into Eq.(3.1) and equating the coefficients about exp($-\psi(z)$) to zero, we obtain

$$e^{0(-\psi(z))}:$$

$$k^5bB_1\gamma^3\mu + 14\,k^5bB_2\gamma^2\mu^2 + 8\,k^5bB_1\gamma\mu^2 + 16\,k^5bB_2\mu^3 + 15\,k^3bB_0B_1\mu\gamma$$
$$+30\,k^3bB_0B_2\mu^2 + ak^3B_1\mu\gamma + 2\,ak^3B_2^2\mu + 15\,kbB_0{}^3 + 3\,kaB_0^2 + rB_0 + \zeta = 0,$$

$$e^{1(-\psi(z))}:$$

$$B_1b\gamma^4k^5 + 30\,B_2b\gamma^3k^5\mu + 22\,B_1b\gamma^2k^5\mu + 120\,B_2b\gamma k^5\mu^2 + 30\,k^3bB_0B_1\mu$$
$$+15\,B_0B_1b\gamma^2k^3 + 90\,B_0B_2b\gamma k^3\mu + 15\,B_1{}^2b\gamma k^3\mu + 30\,B_1B_2bk^3\mu^2 + 16\,B_1bk^5\mu^2$$
$$+B_1a\gamma^2k^3 + 6\,B_2a\gamma k^3\mu + 2\,ak^3B_1\mu + 45\,B_0^2B_1bk + 6\,B_0B_1ak + B_1r = 0,$$

$$e^{2(-\psi(z))}:$$

$$16\,B_2b\gamma^4k^5 + 15\,B_1b\gamma^3k^5 + 232\,B_2b\gamma^2k^5\mu + 60\,B_1b\gamma k^5\mu + 136\,B_2bk^5\mu^2$$
$$+60\,B_0B_2b\gamma^2k^3 + 15\,B_1{}^2b\gamma^2k^3 + 105\,B_1B_2b\gamma k^3\mu + 30\,B_2{}^2bk^3\mu^2 + 45\,B_0B_1b\gamma k^3$$
$$+120\,B_0B_2bk^3\mu + 30\,B_1^2bk^3\mu + 4\,B_2a\gamma^2k^3 + 3\,B_1a\gamma k^3 + 8\,B_2ak^3\mu + 45\,B_0^2B_2bk$$
$$+45\,B_0B_1^2bk + 6\,B_0B_2ak + 3\,B_1^2ak + B_2r = 0,$$

$$e^{3(-\psi(z))}:$$

$$130\,B_2b\gamma^3k^5 + 50\,B_1b\gamma^2k^5 + 440\,B_2b\gamma k^5\mu + 75\,B_1B_2b\gamma^2k^3 + 40\,B_1bk^5\mu$$
$$+90\,B_2^2b\gamma k^3\mu + 150\,B_0B_2b\gamma k^3 + 45\,B_1^2b\gamma k^3 + 30\,B_0B_1bk^3 + 10\,B_2a\gamma k^3$$
$$+150\,B_1B_2bk^3\mu + 90\,B_0B_1B_2bk + 15\,B_1^3bk + 2\,B_1ak^3 + 6\,B_1B_2ak = 0,$$

$$e^{4(-\psi(z))}:$$

$$330\,B_2b\gamma^2k^5 + 60\,B_1b\gamma k^5 + 60\,B_2^2b\gamma^2k^3 + 240\,B_2bk^5\mu + 195\,B_1B_2b\gamma k^3$$
$$+90\,B_0B_2bk^3 + 30\,B_1^2bk^3 + 45\,B_0B_2^2bk + 45\,B_1^2B_2bk + 6\,B_2ak^3 + 3\,B_2^2ak$$
$$+120\,B_2^2bk^3\mu = 0,$$

$$e^{5(-\psi(z))}:$$

$$336\,B_2\,b\gamma\,k^5+24\,B_1\,bk^5+150\,B_2^2b\gamma\,k^3+120\,B_1\,B_2\,bk^3+45\,B_1\,B_2^2bk=0,$$

$$e^{6(-\psi(z))}:$$

$$120\,B_2\,bk^5+90\,B_2^2bk^3+15\,B_2^3bk=0.$$

We solve the above algebraic equations and derive two different families:

Family 1:

$$B_2=-4k^2,B_1=-4\gamma k^2,B_0=-\frac{5k^2b(\gamma^2+8\mu)+a}{15b},r=-\frac{k(5k^4b^2(\gamma^2-4\mu)^2-a^2)}{5b},\tag{3.3}$$

where γ and μ are arbitrary constants.

Substituting Eq (3.3) into Eq (3.2) yields

$$u(z)=-\frac{5k^2b(\gamma^2+8\mu)+a}{15b}-4k^2\gamma\exp(-\psi(z))-4k^2(\exp(-\psi(z)))^2.\tag{3.4}$$

Applying Eqs (2.5)–(2.11) into Eq (3.4) respectively, we get the following exact solutions of the KdV-Sawada-Kotera equation.

Family 1.1: When $\gamma^2-4\mu>0,\mu\neq0,$

$$u_{11}(z)=-\frac{5k^2b(\gamma^2+8\mu)+a}{15b}+\frac{8k^2\gamma\mu}{\sqrt{(\gamma^2-4\mu)}\tanh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+c)\right)+\gamma}$$
$$-\frac{16k^2\mu^2}{\left(\sqrt{(\gamma^2-4\mu)}\tanh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+c)\right)+\gamma\right)^2},$$

$$u_{12}(z)=-\frac{5k^2b(\gamma^2+8\mu)+a}{15b}+\frac{8k^2\gamma\mu}{\sqrt{(\gamma^2-4\mu)}\coth\left(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+c)\right)+\gamma}$$
$$-\frac{16k^2\mu^2}{\left(\sqrt{(\gamma^2-4\mu)}\coth\left(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+c)\right)+\gamma\right)^2}.$$

Family 1.2: When $\gamma^2-4\mu<0,\mu\neq0,$

$$u_{13}(z)=-\frac{5k^2b(\gamma^2+8\mu)+a}{15b}-\frac{8k^2\gamma\mu}{\sqrt{(4\mu-\gamma^2)}\tan\left(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+c)\right)-\gamma}$$
$$-\frac{16k^2\mu^2}{\left(\sqrt{(4\mu-\gamma^2)}\tan\left(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+c)\right)-\gamma\right)^2},$$

$$u_{14}(z) = -\frac{5k^2b(\gamma^2+8\mu)+a}{15b} - \frac{8k^2\gamma\mu}{\sqrt{(4\mu-\gamma^2)}\cot\left(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+c)\right)-\gamma}$$
$$-\frac{16k^2\mu^2}{\left(\sqrt{(4\mu-\gamma^2)}\cot\left(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+c)\right)-\gamma\right)^2}.$$

Family 1.3: When $\gamma^2-4\mu>0$, $\gamma\neq0$, $\mu=0$,

$$u_{15}(z) = -\frac{5k^2b\gamma^2+a}{15b} - \frac{4k^2\gamma^2}{\exp(\gamma(z+c))-1} - \frac{4k^2\gamma^2}{(\exp(\gamma(z+c))-1)^2}.$$

Family 1.4: When $\gamma^2-4\mu=0$, $\gamma\neq0$, $\mu\neq0$,

$$u_{16}(z) = -\frac{60k^2b\mu+a}{15b} + \frac{2k^2\gamma^3(z+c)}{\gamma(z+c)+2} - \frac{k^2\gamma^4(z+c)^2}{(\gamma(z+c)+2)^2}.$$

Family 1.5: When $\gamma^2-4\mu=0$, $\gamma=0$, $\mu=0$,

$$u_{17}(z) = -\frac{a}{15b} - \frac{4k^2}{(z+c)^2}.$$

Family 2:

$$B_2=-2k^2, B_1=-2\gamma k^2, B_0=-\frac{2a+5k^2b(\gamma^2+8\mu)}{30b}, r=-\frac{k(4a^2+5k^4b^2(\gamma^2-4\mu)^2)}{20b}, \tag{3.5}$$

where γ and μ are arbitrary.

Substituting Eq (3.5) into Eq (3.2) yields

$$u(z) = -\frac{2a+5k^2b(\gamma^2+8\mu)}{30b} - 2k^2\gamma\exp(-\psi(z)) - 2k^2(\exp(-\psi(z)))^2. \tag{3.6}$$

Applying Eqs (2.5)–(2.11) into Eq (3.6) respectively, we get the following exact solutions of the KdV-Sawada-Kotera equation.

Family 2.1: When $\gamma^2-4\mu>0$, $\mu\neq0$,

$$u_{21}(z) = -\frac{2a+5k^2b(\gamma^2+8\mu)}{30b} + \frac{4k^2\gamma\mu}{\sqrt{(\gamma^2-4\mu)}\tanh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+c)\right)+\gamma}$$
$$-\frac{8k^2\mu^2}{\left(\sqrt{(\gamma^2-4\mu)}\tanh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+c)\right)+\gamma\right)^2},$$

$$u_{22}(z) = -\frac{2a+5k^2b(\gamma^2+8\mu)}{30b} + \frac{4k^2\gamma\mu}{\sqrt{(\gamma^2-4\mu)}\coth\left(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+c)\right)+\gamma}$$

$$-\frac{8k^2\mu^2}{\left(\sqrt{(\gamma^2-4\mu)}\coth\left(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+c)\right)+\gamma\right)^2}.$$

Family 2.2: When $\gamma^2-4\mu<0$, $\mu\neq 0$,

$$u_{23}(z)=-\frac{2a+5k^2b(\gamma^2+8\mu)}{30b}-\frac{4k^2\gamma\mu}{\sqrt{(4\mu-\gamma^2)}\tan\left(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+c)\right)-\gamma}$$
$$-\frac{8k^2\mu^2}{\left(\sqrt{(4\mu-\gamma^2)}\tan\left(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+c)\right)-\gamma\right)^2},$$
$$u_{24}(z)=-\frac{2a+5k^2b(\gamma^2+8\mu)}{30b}-\frac{4k^2\gamma\mu}{\sqrt{(4\mu-\gamma^2)}\cot\left(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+c)\right)-\gamma}$$
$$-\frac{8k^2\mu^2}{\left(\sqrt{(4\mu-\gamma^2)}\cot\left(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+c)\right)-\gamma\right)^2}.$$

Family 2.3: When $\gamma^2-4\mu>0$, $\gamma\neq 0$, $\mu=0$,

$$u_{25}(z)=-\frac{2a+5k^2b\gamma^2}{30b}-\frac{2k^2\gamma^2}{\exp(\gamma(z+c))-1}-\frac{2k^2\gamma^2}{(\exp(\gamma(z+c))-1)^2}.$$

Family 2.4: When $\gamma^2-4\mu=0$, $\gamma\neq 0$, $\mu\neq 0$,

$$u_{26}(z)=-\frac{a+30k^2b\mu}{15b}+\frac{k^2\gamma^3(z+c)}{\gamma(z+c)+2}-\frac{k^2\gamma^4(z+c)^2}{2(\gamma(z+c)+2)^2}.$$

Family 2.5: When $\gamma^2-4\mu=0$, $\gamma=0$, $\mu=0$,

$$u_{27}(z)=-\frac{a}{15b}-\frac{2k^2}{(z+c)^2}.$$

Figures 1–6 show the properties of the solutions.

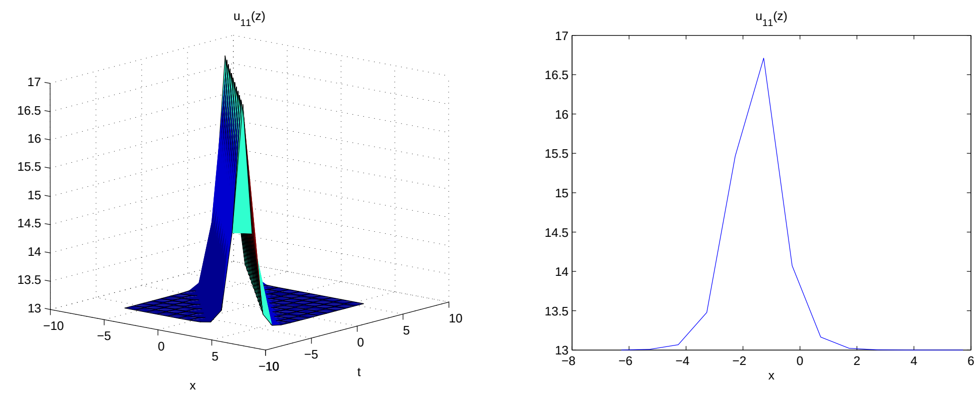

Figure 1. The 3D and 2D surfaces of $u_{11}(z)$ by considering the values $\gamma=4$, $\mu=3$, $k=1$, $r=1$, $c=1$, $b=1$, $a=-215$ and $t=0$ for the 2D graphic.

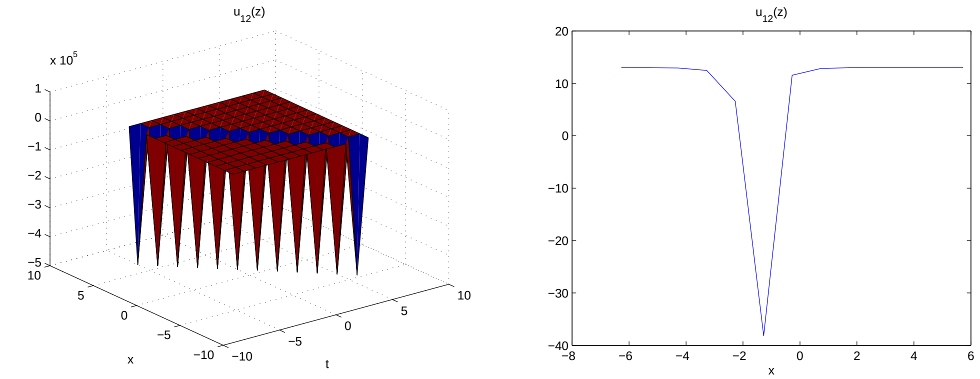

Figure 2. The 3D and 2D surfaces of $u_{12}(z)$ by considering the values $\gamma = 4$, $\mu = 3$, $k = 1$, $r = 1$, $c = 1$, $b = 1$, $a = -215$ and $t = 0$ for the 2D graphic.

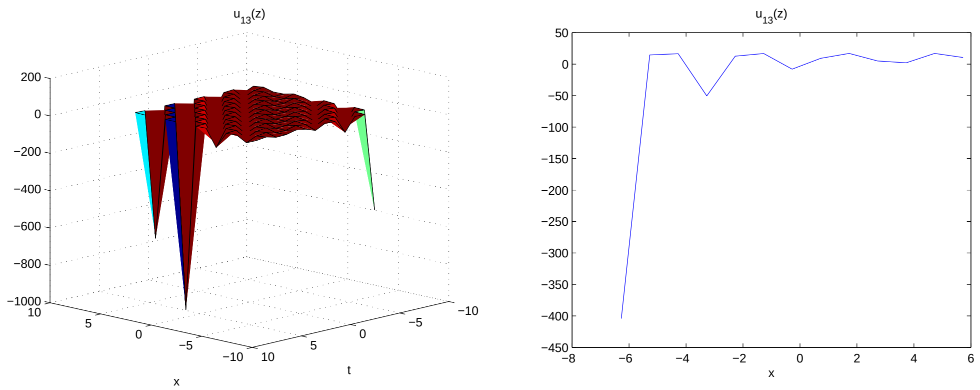

Figure 3. The 3D and 2D surfaces of $u_{13}(z)$ by considering the values $\gamma = 4$, $\mu = 5$, $k = 1$, $r = 1$, $c = 1$, $b = 1$, $a = -295$ and $t = 0$ for the 2D graphic.

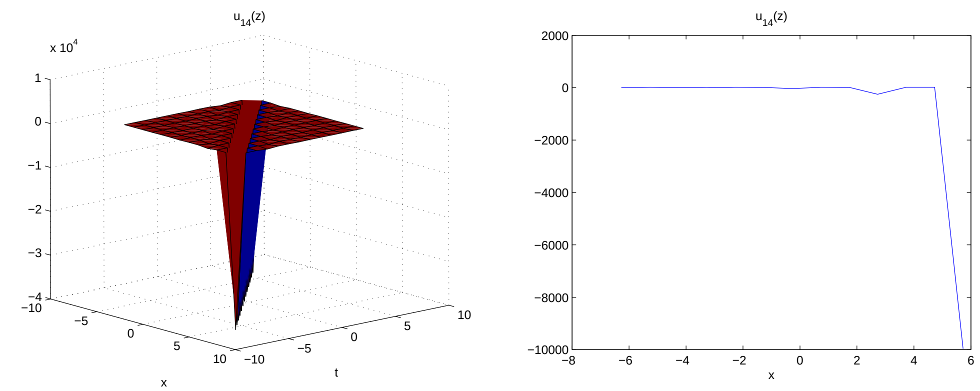

Figure 4. The 3D and 2D surfaces of $u_{14}(z)$ by considering the values $\gamma = 4$, $\mu = 5$, $k = 1$, $r = 1$, $c = 1$, $b = 1$, $a = -295$ and $t = 0$ for the 2D graphic.

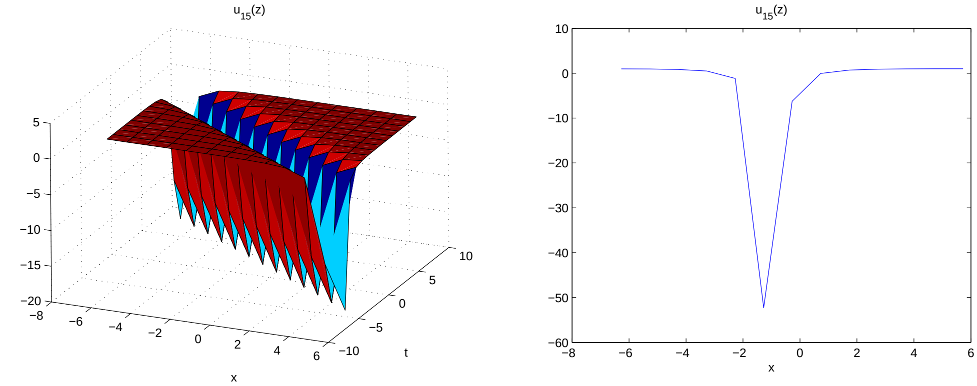

Figure 5. The 3D and 2D surfaces of $u_{15}(z)$ by considering the values $\gamma = 1$, $\mu = 0$, $k = 1$, $r = 1$, $c = 1$, $b = 1$, $a = -20$ and $t = 0$ for the 2D graphic.

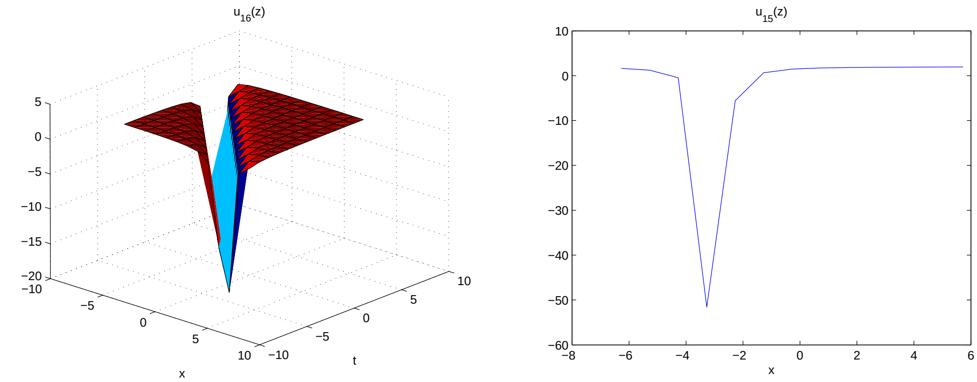

Figure 6. The 3D and 2D surfaces of $u_{16}(z)$ by considering the values $\gamma = 1$, $\mu = \frac{1}{4}$, $k = 1$, $r = 1$, $c = 1$, $b = 1$, $a = -30$ and $t = 0$ for the 2D graphic.

4. The extended complex method

Step 1. Substitute the transformation $I : u(x,t) \to U(z)\,, (x,t) \to z$ into a nonlinear PDE to yield an ODE

$$G(U, U', U'', \cdots) = 0. \tag{4.1}$$

Step 2. Determination of the weak $\langle p, q \rangle$ condition.

Assume that the meromorphic solutions U of Eq (4.1) have at least one pole and let $q,\ p \in \mathbb{N}$. Substitute the Laurent series

$$U(z) = \sum_{k=-q}^{\infty} T_k z^k, T_{-q} \neq 0, q > 0, \tag{4.2}$$

into Eq (4.1) to determine p distinct Laurent principal parts

$$\sum_{k=-q}^{-1} T_k z^k,$$

then we say that the weak $\langle p, q \rangle$ condition of Eq (4.1) holds.

It is know that Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ has double periods and satisfies:

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

and it admit an addition formula [35] as follows:

$$\wp(z - z_0) = -\wp(z) + \frac{1}{4}\left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}\right]^2 - \wp(z_0).$$

Step 3. Substituting the indeterminate forms

$$U(z) = \sum_{i=1}^{h-1}\sum_{j=2}^{q}\frac{(-1)^j\beta_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\left(\frac{1}{4}\left(\frac{\wp'(z) + D_i}{\wp(z) - C_i}\right)^2 - \wp(z)\right) + \sum_{i=1}^{h-1}\frac{\beta_{-i1}}{2}\frac{\wp'(z) + D_i}{\wp(z) - C_i}$$

$$+ \sum_{j=2}^{q}\frac{(-1)^j\beta_{-hj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z) + \beta_0, \tag{4.3}$$

$$U(z) = \sum_{i=1}^{h}\sum_{j=1}^{q}\frac{\beta_{ij}}{(z - z_i)^j} + \beta_0, \tag{4.4}$$

$$U(e^{\alpha z}) = \sum_{i=1}^{h}\sum_{j=1}^{q}\frac{\beta_{ij}}{(e^{\alpha z} - e^{\alpha z_i})^j} + \beta_0, \tag{4.5}$$

into Eq (4.1) respectively yields a set of algebraic equations, and then solving these equations, we achieve elliptic function solutions, simply periodic solutions and rational function solutions with a pole at $z = 0$, in which $D_i^2 = 4C_i^3 - g_2C_i - g_3$, β_{-ij} are determined by (4.2), and $\sum_{i=1}^{h}\beta_{-i1} = 0$, and $R(z)$, $R(e^{\alpha z})$ $(\alpha \in \mathbb{C})$ have $h(\leq p)$ distinct poles of multiplicity q.

Step 4. Derive the meromorphic solutions at arbitrary pole, and insert the inverse transform I^{-1} back to the meromorphic solutions to obtain exact solutions of the given PDE.

5. Utilization of the extended complex method to the KdV-Sawada-Kotera equation

Inserting (4.2) into Eq.(3.1) yields

$$T_{-2} = -4k^2, T_{-1} = 0, T_0 = -\frac{a}{15b}, T_1 = 0, T_2 = \frac{5rb - ka^2}{300k^3b^2}, \cdots$$

and

$$T_{-2} = -2k^2, T_{-1} = 0, T_0 = -\frac{a}{15b}, T_1 = 0, T_2 = \frac{ka^2 - 5rb}{150k^3b^2}, \cdots$$

Therefore we know that $p = 2,\ q = 2$, then the weak $\langle 2, 2\rangle$ condition of Eq (3.1) hold.

By the weak $\langle 2, 2\rangle$ condition and (4.3), we have the form of the elliptic solutions of Eq (3.1)

$$U_{10}(z) = \beta_{-2}\wp(z) + \beta_{20},$$

with pole at $z = 0$.

Substituting $U_{10}(z)$ into Eq (3.1) yields

$$\sum_{i=1}^{4} c_{1i}\wp^{i-1}(z) = 0, \tag{5.1}$$

where

$$c_{11} = -12\,k^5 b\beta_{-2} g_3 - \frac{15}{2}\,k^3 b\beta_{-2}\beta_{20} g_2 + 15\,kb\beta_{20}^3 + 3\,ka\beta_{20}^2 - \frac{1}{2}\,ak^3\beta_{-2} g_2 + r\beta_{20} + \zeta,$$

$$c_{12} = -18\,k^5 b\beta_{-2} g_2 - \frac{15}{2}\,k^3 b\beta_{-2}^2 g_2 + 45\,kb\beta_{-2}\beta_{20}^2 + 6\,ka\beta_{-2}\beta_{20} + r\beta_{-2},$$

$$c_{13} = 90\,k^3 b\beta_{-2}\beta_{20} + 6\,ak^3\beta_{-2} + 45\,bk\beta_{-2}^2\beta_{20} + 3\,ak\beta_{-2}^2,$$

$$c_{14} = 120\,bk^5\beta_{-2} + 90\,bk^3\beta_{-2}^2 + 15\,bk\beta_{-2}^3.$$

Equate the coefficients of all powers of $\wp(z)$ in Eq (5.1) to zero to achieve one set of algebraic equations:

$$c_{11} = 0, c_{12} = 0, c_{13} = 0, c_{14} = 0.$$

Solve the above equations, then

$$\beta_{-2} = -4k^2, \beta_{20} = -\frac{a}{15b}, g_2 = \frac{a^2k - 5\,br}{60\,b^2k^5}, g_3 = -\frac{2\,a^3k + 225\,\zeta\,b^2 - 15\,abr}{10800\,b^3k^7},$$

and

$$\beta_{-2} = -2k^2, \beta_{20} = -\frac{a}{15b}, g_2 = \frac{5\,br - a^2k}{15\,b^2k^5}, g_3 = -\frac{2\,a^3k + 225\,\zeta\,b^2 - 15\,abr}{5400\,b^3k^7},$$

then

$$U_{11,0}(z) = -4k^2\wp(z) - \frac{a}{15b},$$

and

$$U_{12,0}(z) = -2k^2\wp(z) - \frac{a}{15b}.$$

Thus, elliptic solutions of Eq (3.1) with arbitrary pole are

$$U_{11}(z) = -4k^2\wp(z - z_0) - \frac{a}{15b},$$

and

$$U_{12}(z) = -2k^2\wp(z - z_0) - \frac{a}{15b},$$

where $z_0 \in \mathbb{C}$.

Use the addition formula to $U_{11}(z)$ and $U_{12}(z)$, then

$$U_{11}(z) = 4k^2\wp(z) - k^2\left(\frac{\wp'(z) + D}{\wp(z) - C}\right)^2 + \frac{60k^2bC - a}{15b},$$

and

$$U_{12}(z) = 2k^2\wp(z) - \frac{k^2}{2}\left(\frac{\wp'(z) + D}{\wp(z) - C}\right)^2 + \frac{30k^2bC - a}{15b},$$

where $C^2 = 4D^3 - g_2 D - g_3$. $g_2 = \frac{a^2k-5\,b\mu}{60\,b^2k^5}$, $g_3 = -\frac{2\,a^3k+225\zeta\,b^2-15\,ab\mu}{10800\,b^3k^7}$ in the former case, $g_2 = \frac{5\,br-a^2k}{15\,b^2k^5}$, $g_3 = -\frac{2\,a^3k+225\,\zeta\,b^2-15\,abr}{5400\,b^3k^7}$ in the latter case.

By (4.4) and the weak $\langle 2, 2\rangle$ condition, we have the indeterminate form of rational solutions

$$U_{20}(z) = \frac{\beta_{12}}{z^2} + \frac{\beta_{11}}{z} + \beta_{10},$$

with pole at $z = 0$.

Substituting $U_{20}(z)$ into Eq (3.1) yields

$$\sum_{i=1}^{7} c_{2i} z^{i-7} = 0, \tag{5.2}$$

where

$$c_{21} = 120\,bk^5\beta_{12} + 90\,bk^3\beta_{12}{}^2 + 15\,bk\beta_{12}{}^3,$$
$$c_{22} = 24\,bk^5\beta_{11} + 120\,bk^3\beta_{11}\beta_{12} + 45\,bk\beta_{11}\beta_{12}{}^2,$$
$$c_{23} = 90\,bk^3\beta_{10}\beta_{12} + 30\,bk^3\beta_{11}{}^2 + 6\,ak^3\beta_{12} + 45\,bk\beta_{10}\beta_{12}{}^2 + 45\,bk\beta_{11}{}^2\beta_{12} + 3\,ak\beta_{12}{}^2,$$
$$c_{24} = 30\,bk^3\beta_{10}\beta_{11} + 2\,ak^3\beta_{11} + 90\,bk\beta_{10}\beta_{11}\beta_{12} + 15\,bk\beta_{11}{}^3 + 6\,ak\beta_{11}\beta_{12},$$
$$c_{25} = 45\,bk\beta_{10}{}^2\beta_{12} + 45\,bk\beta_{10}\beta_{11}{}^2 + 6\,ak\beta_{10}\beta_{12} + 3\,ak\beta_{11}{}^2 + r\beta_{12},$$
$$c_{26} = 45\,bk\beta_{10}{}^2\beta_{11} + 6\,ak\beta_{10}\beta_{11} + r\beta_{11},$$
$$c_{27} = 15\,kb\beta_{10}{}^3 + 3\,ka\beta_{10}{}^2 + r\beta_{10} + \zeta.$$

Equate the coefficients of all powers of z in Eq (5.2) to zero to achieve a system of algebraic equations:

$$c_{21} = 0, c_{22} = 0, c_{23} = 0, c_{24} = 0, c_{25} = 0, c_{26} = 0, c_{27} = 0.$$

Solving the above equations, we get

$$\beta_{12} = -4k^2, \beta_{11} = 0, \beta_{10} = -\frac{a}{15b},$$

and

$$\beta_{12} = -2k^2, \beta_{11} = 0, \beta_{10} = -\frac{a}{15b},$$

then

$$U_{21,0}(z) = -\frac{4k^2}{z^2} - \frac{a}{15b},$$

and

$$U_{22,0}(z) = -\frac{2k^2}{z^2} - \frac{a}{15b},$$

where $r = \frac{ka^2}{5b}$, $\zeta = \frac{ka^3}{225\,b^2}$.

Insert $U(z) = R(\eta)$ into Eq (3.1) to yield

$$k^5 b\alpha^4(R^{(4)}\eta^4 + 6R'''\eta^3 + 7R''\eta^2 + R'\eta) + 15k^3 b\alpha^2 R(\eta R' + \eta^2 R'')$$

$$+15kbR^3+3kaR^2+ak^3\alpha^2(\eta R'+\eta^2R'')+rR+\zeta=0, \tag{5.3}$$

where $\eta=e^{\alpha z}$ ($\alpha\in\mathbb{C}$).

Substituting

$$U_{30}(z)=\frac{b_{12}}{(e^{\alpha z}-1)^2}+\frac{b_{11}}{e^{\alpha z}-1}+b_{10},$$

into the Eq (5.3), we have

$$\sum_{i=1}^{7}c_{3i}\alpha^2e^{(7-i)\alpha z}(e^{\alpha z}-1)^{-6}=0, \tag{5.4}$$

in which

$$c_{31}=15\,bkb_{10}^3+3\,akb_{10}^2+rb_{10}+\zeta,$$

$$c_{32}=\alpha^4bk^5b_{11}+15\,\alpha^2bk^3b_{10}b_{11}+a\alpha^2k^3b_{11}-90\,bkb_{10}^3+45\,bkb_{10}^2b_{11}-18\,akb_{10}^2$$
$$+6\,akb_{10}b_{11}-6\,rb_{10}+rb_{11}-6\,\zeta,$$

$$c_{33}=10\,\alpha^4bk^5b_{11}+16\,\alpha^4bk^5b_{12}-30\,\alpha^2bk^3b_{10}b_{11}+60\,\alpha^2bk^3b_{10}b_{12}+15\,\alpha^2bk^3b_{11}^2$$
$$-2\,a\alpha^2k^3b_{11}+4\,a\alpha^2k^3b_{12}+225\,bkb_{10}{}^3-225\,bkb_{10}{}^2b_{11}+45\,bkb_{10}{}^2b_{12}+45\,bkb_{10}b_{11}^2$$
$$+45\,akb_{10}^2-30\,akb_{10}b_{11}+6\,akb_{10}b_{12}+3\,kab_{11}^2+15\,rb_{10}-5\,rb_{11}+rb_{12}+15\,\zeta,$$

$$c_{34}=66\,\alpha^4bk^5b_{12}-90\,\alpha^2bk^3b_{10}b_{12}-15\,\alpha^2bk^3b_{11}{}^2+75\,\alpha^2bk^3b_{11}b_{12}-6\,a\alpha^2k^3b_{12}$$
$$-300\,bkb_{10}^3+450\,bkb_{10}^2b_{11}-180\,bkb_{10}^2b_{12}-180\,bkb_{10}b_{11}^2+90\,bkb_{10}b_{11}b_{12}+15\,kbb_{11}^3$$
$$+60\,akb_{10}b_{11}-24\,akb_{10}b_{12}-12\,kab_{11}^2+6\,akb_{11}b_{12}-20\,rb_{10}+10\,rb_{11}-4\,rb_{12}-20\,\zeta$$
$$-60\,akb_{10}^2,$$

$$c_{35}=-10\,\alpha^4bk^5b_{11}+36\,\alpha^4bk^5b_{12}+30\,\alpha^2bk^3b_{10}b_{11}-15\,\alpha^2bk^3b_{11}^2-30\,\alpha^2bk^3b_{11}b_{12}$$
$$+60\,\alpha^2bk^3b_{12}^2+2\,a\alpha^2k^3b_{11}+225\,bkb_{10}^3-450\,bkb_{10}^2b_{11}+270\,bkb_{10}^2b_{12}+270\,bkb_{10}b_{11}^2$$
$$-270\,bkb_{10}b_{11}b_{12}+45\,bkb_{10}b_{12}^2-45\,kbb_{11}^3+45\,bkb_{11}^2b_{12}+45\,akb_{10}^2-60\,akb_{10}b_{11}$$
$$+36\,akb_{10}b_{12}+18\,kab_{11}^2-18\,akb_{11}b_{12}+3\,kab_{12}^2+15\,rb_{10}-10\,rb_{11}+6\,rb_{12}+15\,\zeta,$$

$$c_{36}=-\alpha^4bk^5b_{11}+2\,\alpha^4bk^5b_{12}-15\,\alpha^2bk^3b_{10}b_{11}+30\,\alpha^2bk^3b_{10}b_{12}+15\,\alpha^2bk^3b_{11}^2$$
$$-45\,\alpha^2bk^3b_{11}b_{12}+30\,\alpha^2bk^3b_{12}^2-a\alpha^2k^3b_{11}+2\,a\alpha^2k^3b_{12}-90\,bkb_{10}^3+18\,akb_{11}b_{12}$$
$$+225\,bkb_{10}^2b_{11}-180\,bkb_{10}^2b_{12}-180\,bkb_{10}b_{11}^2+270\,bkb_{10}b_{11}b_{12}+45\,kbb_{11}^3-6\,rb_{10}$$
$$-90\,bkb_{11}^2b_{12}+45\,bkb_{11}b_{12}^2-18\,akb_{10}^2+30\,akb_{10}b_{11}-24\,akb_{10}b_{12}-6\,kab_{12}^2+5\,rb_{11}$$
$$-12\,kab_{11}^2-90\,bkb_{10}b_{12}^2-4\,rb_{12}-6\,\zeta,$$

$$c_{37} = 15\,bkb_{10}^3 - 45\,bkb_{10}^2 b_{11} + 45\,bkb_{10}^2 b_{12} + 45\,bkb_{10} b_{11}^2 - 90\,bkb_{10} b_{11} b_{12} + 45\,bkb_{10} b_{12}^2$$
$$-15\,kbb_{11}^3 + 45\,bkb_{11}^2 b_{12} - 45\,bkb_{11} b_{12}^2 + 15\,kbb_{12}^3 + 3\,akb_{10}{}^2 - 6\,akb_{10} b_{11} + 6\,akb_{10} b_{12}$$
$$+3\,kab_{11}^2 - 6\,akb_{11} b_{12} + 3\,kab_{12}^2 + rb_{10} - rb_{11} + rb_{12} + \zeta.$$

Equate the coefficients of all powers about $e^{\alpha z}$ in Eq (5.4) to zero to achieve a system of algebraic equations:

$$c_{31} = 0, c_{32} = 0, c_{33} = 0, c_{34} = 0, c_{35} = 0, c_{36} = 0, c_{37} = 0.$$

Solving the above equations, we get

$$b_{12} = -4k^2\alpha^2, b_{11} = -4k^2\alpha^2, b_{10} = -\frac{5k^2 b\alpha^2 + a}{15b},$$
$$r = \frac{\left(-5\,\alpha^4 b^2 k^4 + a^2\right) k}{5b}, \zeta = \frac{\left(10\,\alpha^4 b^2 k^4 - 5\,a\alpha^2 bk^2 + a^2\right) k \left(5\,\alpha^2 bk^2 + a\right)}{225\,b^2},$$

and

$$b_{12} = -2k^2\alpha^2, b_{11} = -2k^2\alpha^2, b_{10} = -\frac{5k^2 b\alpha^2 + a}{15b},$$
$$r = \frac{\left(-5\,\alpha^4 b^2 k^4 + a^2\right) k}{5b}, \zeta = \frac{\left(10\,\alpha^4 b^2 k^4 - 5\,a\alpha^2 bk^2 + a^2\right) k \left(5\,\alpha^2 bk^2 + a\right)}{225\,b^2}.$$

So simply periodic solutions of Eq (3.1) with pole at $z = 0$ are

$$U_{31,0}(z) = -\frac{4k^2\alpha^2}{(e^{\alpha z} - 1)^2} - \frac{4k^2\alpha^2}{(e^{\alpha z} - 1)} - \frac{5k^2 b\alpha^2 + a}{15b}$$
$$= -\frac{4k^2\alpha^2 e^{\alpha z}}{(e^{\alpha z} - 1)^2} - \frac{5k^2 b\alpha^2 + a}{15b}$$
$$= -k^2\alpha^2 \coth^2\frac{\alpha z}{2} + \frac{10k^2 b\alpha^2 - a}{15b},$$

and

$$U_{32,0}(z) = -\frac{2k^2\alpha^2}{(e^{\alpha z} - 1)^2} - \frac{2k^2\alpha^2}{(e^{\alpha z} - 1)} - \frac{5k^2 b\alpha^2 + a}{15b}$$
$$= -\frac{2k^2\alpha^2 e^{\alpha z}}{(e^{\alpha z} - 1)^2} - \frac{5k^2 b\alpha^2 + a}{15b}$$
$$= -\frac{k^2\alpha^2}{2} \coth^2\frac{\alpha z}{2} + \frac{5k^2 b\alpha^2 - 2a}{30b}.$$

Similar to $U_{30}(z)$, we substitute

$$U_{40}(z) = \frac{b_{12}}{(e^{\alpha z} + 1)^2} + \frac{b_{11}}{e^{\alpha z} + 1} + b_{10},$$

into the Eq (5.3) to yield

$$b_{12} = -4k^2\alpha^2, b_{11} = 4k^2\alpha^2, b_{10} = -\frac{5k^2b\alpha^2 + a}{15b},$$

$$r = \frac{\left(-5\,\alpha^4b^2k^4 + a^2\right)k}{5b}, \zeta = \frac{\left(10\,\alpha^4b^2k^4 - 5\,a\alpha^2bk^2 + a^2\right)k\left(5\,\alpha^2bk^2 + a\right)}{225\,b^2},$$

and

$$b_{12} = -2k^2\alpha^2, b_{11} = 2k^2\alpha^2, b_{10} = -\frac{5k^2b\alpha^2 + a}{15b},$$

$$r = \frac{\left(-5\,\alpha^4b^2k^4 + a^2\right)k}{5b}, \zeta = \frac{\left(10\,\alpha^4b^2k^4 - 5\,a\alpha^2bk^2 + a^2\right)k\left(5\,\alpha^2bk^2 + a\right)}{225\,b^2},$$

then

$$\begin{aligned} U_{41,0}(z) &= -\frac{4k^2\alpha^2}{(e^{\alpha z}+1)^2} + \frac{4k^2\alpha^2}{(e^{\alpha z}+1)} - \frac{5k^2b\alpha^2 + a}{15b} \\ &= \frac{4k^2\alpha^2e^{\alpha z}}{(e^{\alpha z}+1)^2} - \frac{5k^2b\alpha^2 + a}{15b} \\ &= -k^2\alpha^2\tanh^2\frac{\alpha z}{2} - \frac{a - 10k^2b\alpha^2}{15b}, \end{aligned}$$

and

$$\begin{aligned} U_{42,0}(z) &= -\frac{2k^2\alpha^2}{(e^{\alpha z}+1)^2} + \frac{2k^2\alpha^2}{(e^{\alpha z}+1)} - \frac{5k^2b\alpha^2 + a}{15b} \\ &= \frac{2k^2\alpha^2e^{\alpha z}}{(e^{\alpha z}+1)^2} - \frac{5k^2b\alpha^2 + a}{15b} \\ &= -\frac{k^2\alpha^2}{2}\tanh^2\frac{\alpha z}{2} - \frac{2a - 5k^2b\alpha^2}{30b}. \end{aligned}$$

Substituting

$$U_{50}(z) = \frac{b_{14}}{(e^{\alpha z}-1)^2} + \frac{b_{13}}{(e^{\alpha z}+1)^2} + \frac{b_{12}}{e^{\alpha z}-1} + \frac{b_{11}}{e^{\alpha z}+1} + b_{10},$$

into the Eq (5.3) to yield

$$b_{14} - 4k^2\alpha^2, b_{13} = -4k^2\alpha^2, b_{12} = -4k^2\alpha^2, b_{11} = 4k^2\alpha^2, b_{10} = -\frac{2(5k^2b\alpha^2 + a)}{15b},$$

$$r = \frac{\left(-5\,\alpha^4b^2k^4 + a^2\right)k}{5b}, \zeta = \frac{\left(10\,\alpha^4b^2k^4 - 5\,a\alpha^2bk^2 + a^2\right)k\left(5\,\alpha^2bk^2 + a\right)}{225\,b^2},$$

and

$$b_{14} - 2k^2\alpha^2, b_{13} = -2k^2\alpha^2, b_{12} = -2k^2\alpha^2, b_{11} = 2k^2\alpha^2, b_{10} = -\frac{2(5k^2b\alpha^2 + a)}{15b},$$

$$r = \frac{\left(-5\,\alpha^4b^2k^4 + a^2\right)k}{5b}, \zeta = \frac{\left(10\,\alpha^4b^2k^4 - 5\,a\alpha^2bk^2 + a^2\right)k\left(5\,\alpha^2bk^2 + a\right)}{225\,b^2},$$

then

$$U_{51,0}(z) = -\frac{4k^2\alpha^2}{(e^{\alpha z}-1)^2} - \frac{4k^2\alpha^2}{(e^{\alpha z}+1)^2} - \frac{4k^2\alpha^2}{(e^{\alpha z}-1)} + \frac{4k^2\alpha^2}{(e^{\alpha z}+1)} - \frac{2(5k^2b\alpha^2+a)}{15b}$$
$$= -\frac{4k^2\alpha^2 e^{\alpha z}}{(e^{\alpha z}-1)^2} + \frac{4k^2\alpha^2 e^{\alpha z}}{(e^{\alpha z}+1)^2} - \frac{2(5k^2b\alpha^2+a)}{15b},$$

and

$$U_{52,0}(z) = -\frac{2k^2\alpha^2}{(e^{\alpha z}-1)^2} - \frac{2k^2\alpha^2}{(e^{\alpha z}+1)^2} - \frac{2k^2\alpha^2}{(e^{\alpha z}-1)} + \frac{2k^2\alpha^2}{(e^{\alpha z}+1)} - \frac{2(5k^2b\alpha^2+a)}{15b}$$
$$= -\frac{2k^2\alpha^2 e^{\alpha z}}{(e^{\alpha z}-1)^2} + \frac{2k^2\alpha^2 e^{\alpha z}}{(e^{\alpha z}+1)^2} - \frac{2(5k^2b\alpha^2+a)}{15b}.$$

Collecting meromorphic solutions of Eq (3.1) in above procedures, we have the following solutions with arbitrary pole:

$$(1)U_{11}(z) = 4k^2\wp(z) - k^2\left(\frac{\wp'(z)+D}{\wp(z)-C}\right)^2 + \frac{60k^2bC-a}{15b},$$
$$(2)U_{12}(z) = 2k^2\wp(z) - \frac{k^2}{2}\left(\frac{\wp'(z)+D}{\wp(z)-C}\right)^2 + \frac{30k^2bC-a}{15b},$$

where $C^2 = 4D^3 - g_2D - g_3$, $g_2 = \frac{a^2k-5b\mu}{60b^2k^5}$, $g_3 = -\frac{2a^3k+225\zeta b^2-15ab\mu}{10800b^3k^7}$ in the former case, $g_2 = \frac{5br-a^2k}{15b^2k^5}$, $g_3 = -\frac{2a^3k+225\zeta b^2-15abr}{5400b^3k^7}$ in the latter case;

$$(3)U_{21}(z) = -\frac{4k^2}{(z-z_0)^2} - \frac{a}{15b},$$
$$(4)U_{22}(z) = -\frac{2k^2}{(z-z_0)^2} - \frac{a}{15b},$$

where $r = \frac{ka^2}{5b}, \zeta = \frac{ka^3}{225b^2}$;

$$(5)U_{31}(z) = -k^2\alpha^2\coth^2\frac{\alpha(z-z_0)}{2} + \frac{10k^2b\alpha^2-a}{15b},$$
$$(6)U_{32}(z) = -\frac{k^2\alpha^2}{2}\coth^2\frac{\alpha(z-z_0)}{2} + \frac{5k^2b\alpha^2-2a}{30b},$$
$$(7)U_{41}(z) = -k^2\alpha^2\tanh^2\frac{\alpha(z-z_0)}{2} - \frac{a-10k^2b\alpha^2}{15b},$$
$$(8)U_{42}(z) = -\frac{k^2\alpha^2}{2}\tanh^2\frac{\alpha(z-z_0)}{2} - \frac{2a-5k^2b\alpha^2}{30b},$$
$$(9)U_{51}(z) = -\frac{4k^2\alpha^2e^{\alpha(z-z_0)}}{(e^{\alpha(z-z_0)}-1)^2} + \frac{4k^2\alpha^2e^{\alpha(z-z_0)}}{(e^{\alpha(z-z_0)}+1)^2} - \frac{2(5k^2b\alpha^2+a)}{15b},$$
$$(10)U_{52}(z) = -\frac{2k^2\alpha^2e^{\alpha(z-z_0)}}{(e^{\alpha(z-z_0)}-1)^2} + \frac{2k^2\alpha^2e^{\alpha(z-z_0)}}{(e^{\alpha(z-z_0)}+1)^2} - \frac{2(5k^2b\alpha^2+a)}{15b},$$

where $r = \frac{(-5\alpha^4b^2k^4+a^2)k}{5b}, \zeta = \frac{(10\alpha^4b^2k^4-5a\alpha^2bk^2+a^2)k(5\alpha^2bk^2+a)}{225b^2}$.

Figures 7–11 show the properties of the solutions.

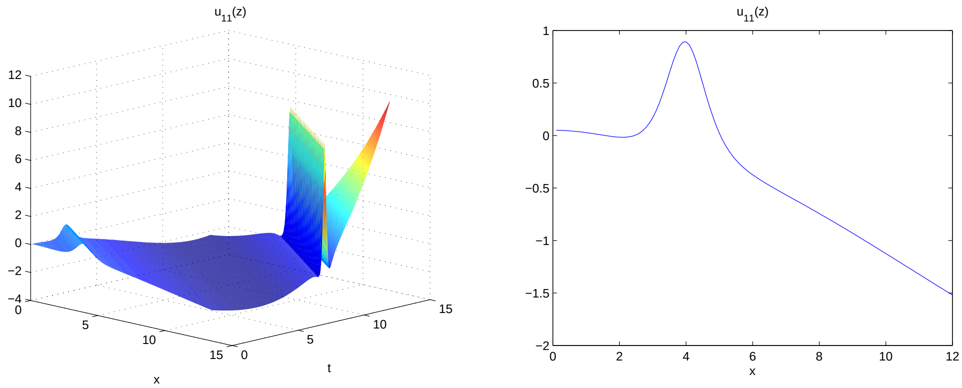

Figure 7. The 3D and 2D surfaces of $U_{11}(z)$ by considering the values $k = 0.28$, $r = 0.42$, $a = -3.79$, $b = 4.97$, $D = -0.032$, $\mu = 1.32$, $\zeta = -0.071$ and $t = 0$ for the 2D graphic.

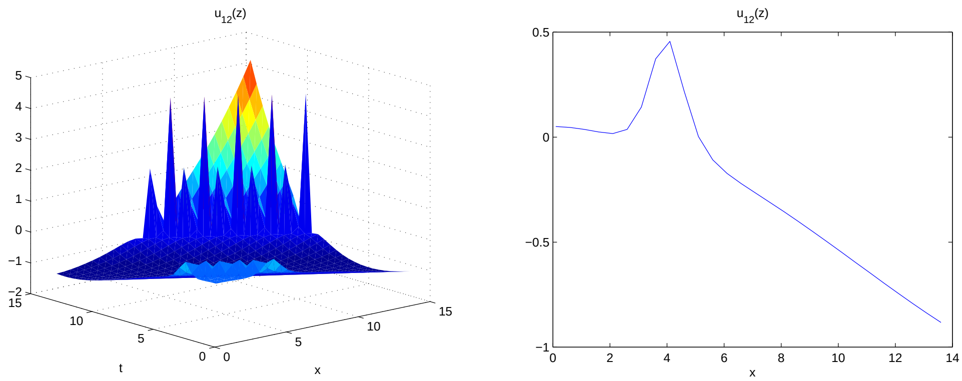

Figure 8. The 3D and 2D surfaces of $U_{12}(z)$ by considering the values$k = 0.28$, $r = 0.42$, $a = -3.79$, $b = 4.97$, $D = -0.032$, $\mu = 1.32$, $\zeta = -0.071$ and $t = 0$ for the 2D graphic.

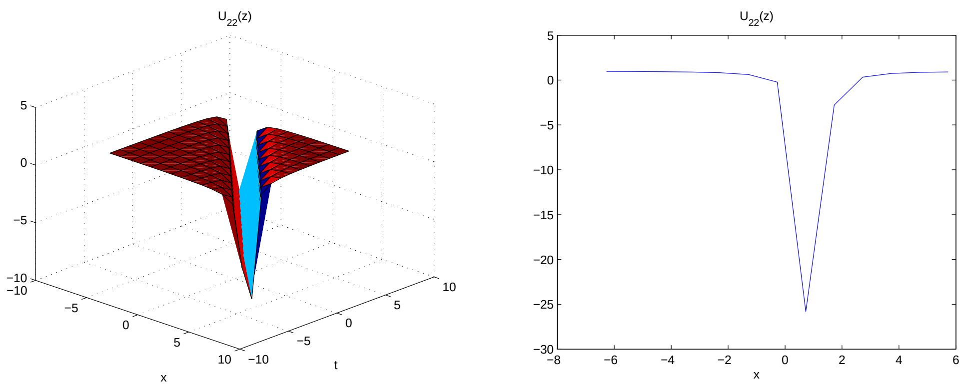

Figure 9. The 3D and 2D surfaces of $U_{22}(z)$ by considering the values $k = 1$, $r = 1$, $b = 1$, $a = -15$, $z_0 = -1$ and $t = 0$ for the 2D graphic.

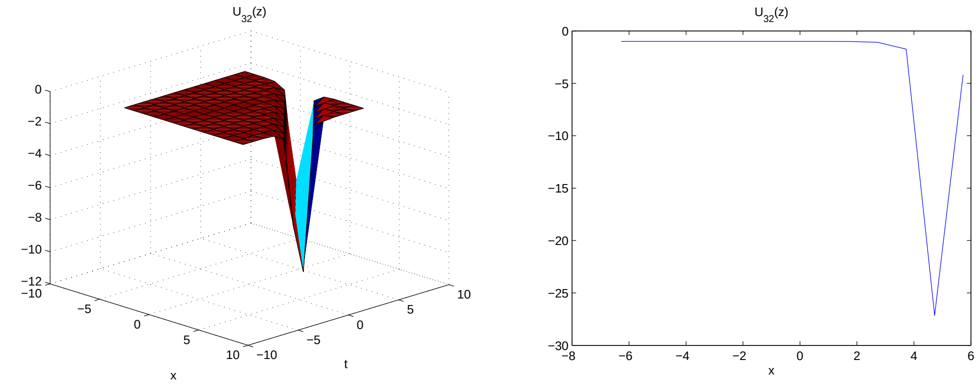

Figure 10. The 3D and 2D surfaces of $U_{32}(z)$ by considering the values $k = 1$, $r = 1$, $b = 1$, $a = -5$, $\alpha = 2$, $z_0 = -5$ and $t = 0$ for the 2D graphic.

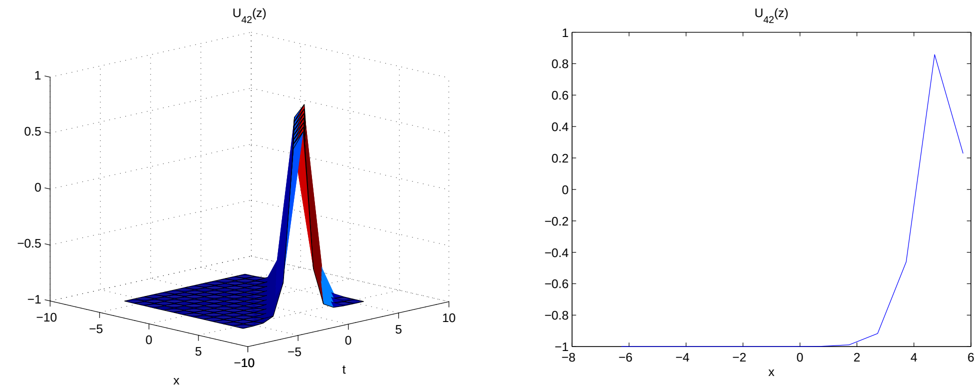

Figure 11. The 3D and 2D surfaces of $U_{42}(z)$ by considering the values $k = 1$, $r = 1$, $b = 1$, $a = -5$, $\alpha = 2$, $z_0 = -5$ and $t = 0$ for the 2D graphic.

6. Conclusions

In this paper, we derive meromorphic exact solutions to the KdV-Sawada-Kotera equation via two different systematic methods. Five types of solutions are constructed, including hyperbolic, trigonometric, exponential, elliptic and rational function solutions. Dynamic behaviors of these solutions are given by some graphs. Observing from the figures, we know that the obtained solutions are soliton solutions. Among of them, figures 3, 7 and 8 show multiple soliton solutions, and others show singular soliton solutions. The graphs of Weierstrass elliptic function solutions $U_{11}(z)$ and $U_{12}(z)$ are more interesting and have never been shown in other literatures. We can use the ideas of this study to other differential equations in complexity and nonlinear science.

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Conflict of interest

The authors declare no conflict of interest.

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