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Research article

On GT-convexity and related integral inequalities

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Abstract: In the paper, the authors introduce a new class of convex functions, GT-convex functions, establish some integral inequalities for GT-convex functions and for the product of two GT-convex functions, and give some applications to classical special means.

Keywords: GT-convexity; MT-convexity; GA-convexity; integral inequality; mean inequality
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1. Introduction

The definition of convexity is well known in the literature.

Definition 1.1. A function $G : I \subseteq \mathbb{R} \to \mathbb{R}$ defined on an interval $I \neq \emptyset$ is said to be convex if

$$G(\lambda x + (1-\lambda)y) \le \lambda G(x) + (1-\lambda)G(y)$$

is valid for all $x, y \in I$ and $\lambda \in [0,1]$.

Theorem 1.1. *Let $G : I \subseteq \mathbb{R} \to \mathbb{R}$ be convex and $x, y \in I$ with $x \neq y$. Then*

$$G\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y G(s)\,\mathrm{d}s \le \frac{G(x)+G(y)}{2}. \tag{1.1}$$

If G is concave, the above double inequality holds in the reverse way.

The double inequality (1.1) is called the Hermite–Hadamard integral inequality for convex functions.

In recent years, the Hermite–Hadamard integral inequality has been the subject of intensive research. Various improvements, generalizations, and variants of this inequality can be found in [1–13] and closely related references therein. In [14], Pachpatte established some integral inequalities for the product of two convex functions.

Theorem 1.2 ([14, Theorem 1]). *Suppose that $G, H : I \to (0,\infty)$ are convex, $x, y \in I$ with $x < y$, and $GH \in L_1[x,y]$. Then*

$$2G\Big(\frac{x+y}{2}\Big)H\Big(\frac{x+y}{2}\Big) - \frac{M(x,y)+2N(x,y)}{6} \le \frac{1}{y-x}\int_x^y G(s)H(s)\mathrm{d}s \le \frac{2M(x,y)+N(x,y)}{6}, \tag{1.2}$$

where

$$M(x,y) = G(x)H(x) + G(y)H(y) \quad \textit{and} \quad N(x,y) = G(x)H(y) + G(y)H(x). \tag{1.3}$$

Motivated by the inequality (1.2), several integral inequalities for the product of two convex functions were also established in the papers [15–17]. In [18], the double inequality (1.2) was applied to generalize and refine Young's integral inequality in terms of higher order derivatives.

In [19], Tunç and Yildrim defined the MT-convexity and obtained some new integral inequalities for MT-convex functions.

Definition 1.2 ([19]). Let $I \subseteq \mathbb{R}$. Then a nonnegative function $G : I \to [0,\infty)$ is said to be MT-convex, denoted by $G \in MT(I)$, if

$$G(\lambda x + (1-\lambda)y) \le \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}G(x) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}G(y)$$

validates for $x, y \in I$ and $\lambda \in (0,1)$.

Theorem 1.3 ([19]). *Let $G \in MT(I)$, $x, y \in I$ with $x < y$, and $G \in L_1[x,y]$. Then*

$$\frac{\pi}{2}G\Big(\frac{x+y}{2}\Big) \le G(x) + G(y), \quad G\Big(\frac{x+y}{2}\Big) \le \frac{1}{y-x}\int_x^y f(s)\mathrm{d}s,$$

and

$$\frac{2}{y-x}\int_x^y \nu(s)G(s)\mathrm{d}s \le \frac{G(x)+G(y)}{2}$$

where

$$\nu(s) = \frac{2\sqrt{(y-s)(s-x)}}{y-x}, \quad s \in [x,y].$$

In this paper, we will introduce a new class of convex functions, GT-convex functions, establish some new integral inequalities for GT-convex functions and for the product of two GT-convex functions, and apply these to classical special means.

2. GT-convexity and lemmas

In this section, we introduce the concept of GT-convexity and list several lemmas.

Definition 2.1. Suppose that $I \subseteq (0, \infty)$. A real-valued function $G : I \to \mathbb{R}$ is called GT-convex, denoted by $G \in GT(I)$, if

$$G(x^{\lambda}y^{1-\lambda}) \leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}G(x) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}G(y) \tag{2.1}$$

validates for all $x, y \in I$ and $\lambda \in (0, 1)$.

Remark 2.1. When $\lambda = \frac{1}{2}$, the inequality (2.1) reduces to

$$G(\sqrt{xy}) \leq \frac{G(x) + G(y)}{2}.$$

Proposition 2.1. *Let $I \subseteq (0, \infty)$ and $x, y \in I$ with $x < y$. If $H : [\ln x, \ln y] \to \mathbb{R}$ is MT-convex, then the function $G : [x, y] \to \mathbb{R}$, $G(t) = H(\ln t)$, is GT-convex.*

Proof. If $s, t \in [x, y]$ and $\lambda \in (0, 1)$, then

$$\begin{aligned} G(s^{\lambda}t^{1-\lambda}) &= H(\ln(s^{\lambda}t^{1-\lambda})) = H(\lambda \ln s + (1-\lambda)\ln t) \\ &\leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}H(\ln s) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}H(\ln t) = \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}G(s) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}G(t), \end{aligned}$$

which shows that G is GT-convex on $[x, y]$. □

Definition 2.2 ([20–22]). A function $G : I \subseteq [0, \infty) \to \mathbb{R}$ is said to be GA-convex if

$$G(x^{\lambda}y^{1-\lambda}) \leq \lambda G(x) + (1-\lambda)G(y)$$

validates for all $x, y \in I$ and $\lambda \in [0, 1]$.

Proposition 2.2. *All of GA-convex functions are GT-convex functions, but not conversely.*

Proof. If G is GA-convex, since $\lambda \leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}$ and $1 - \lambda \leq \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}$, it is easy to see

$$G(x^{\lambda}y^{1-\lambda}) \leq \lambda G(x) + (1-\lambda)G(y) \leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}G(x) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}G(y),$$

where $x, y \in I \subseteq (0, \infty)$ and $\lambda \in (0, 1)$. This means that each GA-convex function is GT-convex.

The function $G : (1, \infty) \to \mathbb{R}$, $G(x) = \ln^p x$, $p \in (0, \frac{1}{1000})$, is GT-convex, but it is not a GA-convex function. □

In order to prove our main results, the following lemma and definition are needed.

Lemma 2.1 ([23]). *Suppose $I \subseteq (0, \infty)$ and $x, y \in I$ with $x < y$. Let $G : [x, y] \to \mathbb{R}$ be a Lebesgue integrable function on $[x, y]$. Then, for any $\lambda \in [0, 1]$,*

$$\int_0^1 G(x^s y^{1-s})\mathrm{d}s = (1-\lambda)\int_0^1 G\Big(x^s(x^{\lambda}y^{1-\lambda})^{1-s}\Big)\mathrm{d}s + \lambda\int_0^1 G\Big((x^{\lambda}y^{1-\lambda})^s y^{1-s}\Big)\mathrm{d}s.$$

Definition 2.3 ([24]). Two functions $G, H : X \to \mathbb{R}$ are said to be similarly ordered if

$$[G(x) - G(y)][H(x) - H(y)] \geq 0$$

for every $x, y \in X$.

3. Some integral inequalities for GT-convex functions

We are in a position to establish some integral inequalities for GT-convex functions.

Theorem 3.1. *Suppose that $I \subseteq (0,\infty)$, $G \in GT(I)$, $x, y \in I$ with $x < y$, and $G \in L_1[x,y]$. Then*

$$G(\sqrt{xy}) \le \frac{1}{\ln y - \ln x}\int_x^y \frac{G(s)}{s}\mathrm{d}s \le \frac{\pi}{4}[G(x)+G(y)]. \tag{3.1}$$

Proof. Because G is GT-convex, for any $x, y \in I$ with $x < y$ and $\lambda \in (0,1)$, we have

$$G(x^\lambda y^{1-\lambda}) \le \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}G(x) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}G(y).$$

Integrating the above inequality over $\lambda \in (0,1)$ gives the right inequality of (3.1).

For $\lambda \in (0,1)$, we have

$$G(\sqrt{xy}) = G\left(\sqrt{x^\lambda y^{1-\lambda} x^{1-\lambda} y^\lambda}\right) \le \frac{1}{2}[G(x^\lambda y^{1-\lambda}) + G(x^{1-\lambda} y^\lambda)]. \tag{3.2}$$

Integrating the above inequality on $\lambda \in [0,1]$ results in

$$G(\sqrt{xy}) \le \frac{1}{2}\left[\int_0^1 G(x^\lambda y^{1-\lambda})\mathrm{d}\lambda + \int_0^1 G(x^{1-\lambda} y^\lambda)\mathrm{d}\lambda\right].$$

Using the fact that

$$\int_0^1 G(x^\lambda y^{1-\lambda})\mathrm{d}\lambda = \int_0^1 G(x^{1-\lambda} y^\lambda)\mathrm{d}\lambda$$

and replacing $x^\lambda y^{1-\lambda}$ by s lead to the left inequality of (3.1). □

Corollary 3.1. *Suppose that $I \subseteq (0,\infty)$, $G \in GT(I)$, $x, y \in I$ with $x < y$, and $G \in L_1[x,y]$. Then*

$$G(\sqrt{xy}) \le \int_0^1 G(x^s y^{1-s})\mathrm{d}s \le \frac{\pi}{4}[G(x)+G(y)]. \tag{3.3}$$

Theorem 3.2. *Suppose that $I \subseteq (0,\infty)$, $G \in GT(I)$, $x, y \in I$ with $x < y$, and $G \in L_1[x,y]$. Then*

$$\frac{\pi}{4}G(\sqrt{xy}) \le \frac{1}{\ln y - \ln x}\int_x^y \tau(s)\frac{G(s)}{s}\mathrm{d}s \le \frac{G(x)+G(y)}{2}, \tag{3.4}$$

where

$$\tau(s) = \frac{2\sqrt{(\ln y - \ln s)(\ln s - \ln x)}}{\ln y - \ln x}. \tag{3.5}$$

Proof. For $\lambda \in (0,1)$, multiplying on both sides of (3.2) by the quantity $2\sqrt{\lambda(1-\lambda)}$ arrives at

$$2\sqrt{\lambda(1-\lambda)}\,G(\sqrt{xy}) \le \sqrt{\lambda(1-\lambda)}\,[G(x^\lambda y^{1-\lambda}) + G(x^{1-\lambda} y^\lambda)]. \tag{3.6}$$

Integrating the inequality (3.6) on $\lambda \in (0,1)$ yields the first inequality in (3.4).

By the GT-convexity of G, we have

$$2\sqrt{\lambda(1-\lambda)}\,G(x^{\lambda}y^{1-\lambda}) \le \lambda G(x) + (1-\lambda)G(y)$$

and

$$2\sqrt{\lambda(1-\lambda)}\,G(x^{1-\lambda}y^{\lambda}) \le (1-\lambda)G(x) + \lambda G(y).$$

Adding these inequalities and integrating over $\lambda \in [0,1]$ reveal

$$\int_0^1 \sqrt{\lambda(1-\lambda)}\,[G(x^{\lambda}y^{1-\lambda}) + G(x^{1-\lambda}y^{\lambda})]\mathrm{d}\lambda \le \frac{G(x)+G(y)}{2}.$$

Therefore, making use of the fact that

$$\int_0^1 \sqrt{\lambda(1-\lambda)}\,G(x^{\lambda}y^{1-\lambda})\mathrm{d}\lambda = \int_0^1 \sqrt{\lambda(1-\lambda)}\,G(x^{1-\lambda}y^{\lambda})\mathrm{d}\lambda$$

and replacing $x^{\lambda}y^{1-\lambda}$ by s acquire the second inequality in (3.4). □

Corollary 3.2. *Suppose that $I \subseteq (0,\infty)$, $G \in GT(I)$, $x, y \in I$ with $x < y$, and $G \in L_1[x,y]$. Then*

$$\frac{\pi}{2}G(\sqrt{xy}) \le G(x) + G(y).$$

Theorem 3.3. *Suppose that $I \subseteq (0,\infty)$, $G \in GT(I)$, $x, y \in I$ with $x < y$, and $G \in L_1[x,y]$. Then, for any $\lambda \in (0,1)$,*

$$\begin{aligned} 2\sqrt{\lambda(1-\lambda)}\,G(\sqrt{xy}) &\le (1-\lambda)G\Big(x^{(1+\lambda)/2}y^{(1-\lambda)/2}\Big) + \lambda G\Big(x^{\lambda/2}y^{(2-\lambda)/2}\Big) \le \frac{1}{\ln y - \ln x}\int_x^y \frac{G(s)}{s}\mathrm{d}s \\ &\le \frac{\pi}{4}[(1-\lambda)G(x) + \lambda G(y) + G(x^{\lambda}y^{1-\lambda})] \le \frac{\pi}{8\sqrt{\lambda(1-\lambda)}}[G(x)+G(y)]. \end{aligned} \tag{3.7}$$

Proof. Using the inequality (3.3), for any $\lambda \in (0,1)$, we have

$$G\Big(\sqrt{xx^{\lambda}y^{1-\lambda}}\Big) \le \int_0^1 G\Big(x^s(x^{\lambda}y^{1-\lambda})^{1-s}\Big)\mathrm{d}s \le \frac{\pi}{4}[G(x) + G(x^{\lambda}y^{1-\lambda})]$$

which is equivalent to

$$G\Big(x^{(1+\lambda)/2}y^{(1-\lambda)/2}\Big) \le \int_0^1 G\Big(x^s(x^{\lambda}y^{1-\lambda})^{1-s}\Big)\mathrm{d}s \le \frac{\pi}{4}[G(x) + G(x^{\lambda}y^{1-\lambda})]. \tag{3.8}$$

Similarly, we obtain

$$G\Big(\sqrt{x^{\lambda}y^{1-\lambda}y}\Big) \le \int_0^1 G((x^{\lambda}y^{1-\lambda})^s y^{1-s})\mathrm{d}s \le \frac{\pi}{4}[G(x^{\lambda}y^{1-\lambda}) + G(y)]$$

which is equivalent to

$$G\Big(x^{\lambda/2}y^{(2-\lambda)/2}\Big) \le \int_0^1 G((x^{\lambda}y^{1-\lambda})^s y^{1-s})\mathrm{d}s \le \frac{\pi}{4}[G(x^{\lambda}y^{1-\lambda}) + G(y)]. \tag{3.9}$$

Multiplying on both sides of (3.8) and (3.9) by $1-\lambda$ and λ respectively, adding resulted inequalities, and employing Lemma 2.1 produce

$$
\begin{aligned}
&(1-\lambda)G\Big(x^{(1+\lambda)/2}y^{(1-\lambda)/2}\Big)+\lambda G\Big(x^{\lambda/2}y^{(2-\lambda)/2}\Big)\le\int_0^1 G(x^s y^{1-s})\mathrm{d}s\\
&\quad\le\frac{\pi}{4}(1-\lambda)[G(x)+G(x^\lambda y^{1-\lambda})]+\frac{\pi}{4}\lambda[G(x^\lambda y^{1-\lambda})+G(y)]=\frac{\pi}{4}[(1-\lambda)G(x)+\lambda G(y)+G(x^\lambda y^{1-\lambda})],
\end{aligned}
$$

which proves the second and third inequality in (3.7).

By the GT-convexity of G, we have

$$
\begin{aligned}
&(1-\lambda)G\Big(x^{(1+\lambda)/2}y^{(1-\lambda)/2}\Big)+\lambda G\Big(x^{\lambda/2}y^{(2-\lambda)/2}\Big)\\
&=2\sqrt{\lambda(1-\lambda)}\,\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}G\Big(x^{(1+\lambda)/2}y^{(1-\lambda)/2}\Big)+2\sqrt{\lambda(1-\lambda)}\,\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}G\Big(x^{\lambda/2}y^{(2-\lambda)/2}\Big)\\
&\ge 2\sqrt{\lambda(1-\lambda)}\,G\Big(\Big(x^{(1+\lambda)/2}y^{(1-\lambda)/2}\Big)^{1-\lambda}\Big(x^{\lambda/2}y^{(2-\lambda)/2}\Big)^{\lambda}\Big)=2\sqrt{\lambda(1-\lambda)}\,G(\sqrt{xy}),
\end{aligned}
$$

which proves the first inequality in (3.7).

Similarly, we obtain

$$
\begin{aligned}
&\frac{\pi}{4}[(1-\lambda)G(x)+\lambda G(y)+G(x^\lambda y^{1-\lambda})]\\
&\le\frac{\pi}{4}\Big[(1-\lambda)G(x)+\lambda G(y)+\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}G(x)+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}G(y)\Big]\\
&\le\frac{\pi}{4}\Big(\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\Big)[G(x)+G(y)]=\frac{\pi}{8\sqrt{\lambda(1-\lambda)}}[G(x)+G(y)],
\end{aligned}
$$

which proves the last inequality in (3.7). □

Corollary 3.3. *Under assumptions of Theorem 3.3, if taking* $\lambda=\frac{1}{2}$*, then*

$$
\begin{aligned}
G(\sqrt{xy})&\le\frac{1}{2}[G(x^{1/4}y^{3/4})+G(x^{3/4}y^{1/4})]\le\frac{1}{\ln y-\ln x}\int_x^y\frac{G(s)}{s}\mathrm{d}s\\
&\le\frac{\pi}{8}[G(x)+G(y)+2G(\sqrt{xy})]\le\frac{\pi}{4}[G(x)+G(y)].
\end{aligned}
$$

Theorem 3.4. *Suppose that* $I\subseteq(0,\infty)$*,* $G\in GT(I)$*,* $x,y\in I$ *with* $x<y$*, and* $G\in L_1[x,y]$*. Then*

$$
\begin{aligned}
\frac{1}{y-x}\int_x^y\tau(s)G(s)\mathrm{d}s&\le\frac{[\ln y-\ln I(x,y)]G(x)+[\ln I(x,y)-\ln x]G(y)}{\ln y-\ln x}\\
&=\frac{L(x,y)-x}{y-x}G(x)+\frac{y-L(x,y)}{y-x}G(y),
\end{aligned}
\tag{3.10}
$$

where $\tau(s)$ *is defined in* (3.5) *and the quantities*

$$
I(x,y)=\frac{1}{e}\Big(\frac{y^y}{x^x}\Big)^{1/(y-x)}\quad\text{and}\quad L(x,y)=\frac{y-x}{\ln y-\ln x}
$$

for $y>x>0$ *are the identric and logarithmic means.*

Proof. Because G is GT-convex, then $G \circ \exp$ is MT-convex. Thus, we have

$$
\begin{aligned}
G(s) = G \circ \exp(\ln s) &= G \circ \exp\left(\frac{(\ln y - \ln s)\ln x + (\ln s - \ln x)\ln y}{\ln y - \ln x}\right) \\
&\leq \frac{1}{2}\sqrt{\frac{\ln y - \ln s}{\ln s - \ln x}}\, G \circ \exp(\ln x) + \frac{1}{2}\sqrt{\frac{\ln s - \ln x}{\ln y - \ln s}}\, G \circ \exp(\ln y) \\
&= \frac{1}{2}\sqrt{\frac{\ln y - \ln s}{\ln s - \ln x}}\, G(x) + \frac{1}{2}\sqrt{\frac{\ln s - \ln x}{\ln y - \ln s}}\, G(y)
\end{aligned}
$$

for any $s \in [x, y]$. This can be rearranged as

$$
\frac{2\sqrt{(\ln y - \ln s)(\ln s - \ln x)}}{\ln y - \ln x} G(s) \leq \frac{\ln y - \ln s}{\ln y - \ln x} G(x) + \frac{\ln s - \ln x}{\ln y - \ln x} G(y). \tag{3.11}
$$

Integrating (3.11) gives

$$
\frac{1}{y - x}\int_x^y \tau(s) G(s)\mathrm{d}s \leq \frac{(\ln y - \frac{1}{y-x}\int_x^y \ln s\mathrm{d}s)G(x) + (\frac{1}{y-x}\int_x^y \ln s\mathrm{d}s - \ln x)G(y)}{\ln y - \ln x}.
$$

By virtue of the integral representation

$$
\frac{1}{y - x}\int_x^y \ln s\mathrm{d}s = \ln I(x, y),
$$

we obtain the first inequality in (3.10).

Now we observe that

$$
\frac{\ln y - \ln I(x, y)}{\ln y - \ln x} = \frac{\ln y - \frac{y\ln y - x\ln x}{y-x} + 1}{\ln y - \ln x} = \frac{y - x - x(\ln y - \ln x)}{(y - x)(\ln y - \ln x)} = \frac{L(x, y) - x}{y - x}.
$$

Similarly,

$$
\frac{\ln I(x, y) - \ln x}{\ln y - \ln x} = \frac{y - L(x, y)}{y - x}.
$$

Consequently, the last part of (3.10) is proved. □

Theorem 3.5. *Suppose that* $I \subseteq (0, \infty)$, $G, H \in GT(I)$, $x, y \in I$ *with* $x < y$, *and* $GH \in L_1[x, y]$. *If* G, H *are nonnegative, then*

$$
\frac{1}{\ln y - \ln x}\int_x^y \frac{\tau^2(s)}{s^2} G(s)H(s)\mathrm{d}s \leq \frac{2M(x, y) + N(x, y)}{6},
$$

where $\tau(s)$ *is defined in* (3.5) *and* $M(x, y)$ *and* $N(x, y)$ *are defined in* (1.3).

Proof. Because $G, H \in GT(I)$, then, for any $\lambda \in (0, 1)$, we have

$$
G(x^\lambda y^{1-\lambda}) \leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}} G(x) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}} G(y) \quad \text{and} \quad H(x^\lambda y^{1-\lambda}) \leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}} H(x) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}} H(y).
$$

Since G, H are nonnegative, we obtain

$$G(x^\lambda y^{1-\lambda})H(x^\lambda y^{1-\lambda}) \leq \frac{\lambda}{4(1-\lambda)}G(x)H(x) + \frac{1-\lambda}{4\lambda}G(y)H(y) + \frac{1}{4}[G(x)H(y) + G(y)H(x)],$$

that is,

$$G(x^\lambda y^{1-\lambda})H(x^\lambda y^{1-\lambda}) \leq \frac{\lambda^2 G(x)H(x) + (1-\lambda)^2 G(y)H(y) + \lambda(1-\lambda)[G(x)H(y) + G(y)H(x)]}{4\lambda(1-\lambda)}. \quad (3.12)$$

Integrating on both sides yields

$$\frac{1}{\ln y - \ln x}\int_x^y \frac{(\ln y - \ln s)(\ln s - \ln x)}{(\ln y - \ln x)^2}\frac{G(s)H(s)}{s^2}\mathrm{d}s$$
$$\leq \frac{1}{24}[2G(x)H(x) + 2G(y)H(y) + G(x)H(y) + G(y)H(x)]$$

which completes the proof. □

Corollary 3.4. *If choosing $\lambda = \frac{1}{2}$ in the inequality* (3.12)*, then*

$$G(\sqrt{xy})H(\sqrt{xy}) \leq \frac{1}{4}[G(x)H(x) + G(y)H(y) + G(x)H(y) + G(y)H(x)].$$

Theorem 3.6. *Suppose that $I \subseteq (0, \infty)$, $G, H \in GT(I)$, $x, y \in I$ with $x < y$, and $GH \in L_1[x, y]$. If G, H are nonnegative and similarly ordered, then*

$$\frac{1}{\ln y - \ln x}\int_x^y \frac{\tau^2(s)}{s^2}G(s)H(s)\mathrm{d}s \leq \frac{1}{2}[G(x)H(x) + G(y)H(y)],$$

where $\tau(s)$ is defined in (3.5).

Proof. Because G, H are similarly ordered, nonnegative, and GT-convex, then, for any $\lambda \in (0, 1)$, we have

$$G(x^\lambda y^{1-\lambda})H(x^\lambda y^{1-\lambda}) \leq \frac{\lambda}{4(1-\lambda)}G(x)H(x) + \frac{1-\lambda}{4\lambda}G(y)H(y) + \frac{1}{4}[G(x)H(y) + G(y)H(x)],$$

that is,

$$4\lambda(1-\lambda)G(x^\lambda y^{1-\lambda})H(x^\lambda y^{1-\lambda}) \leq \lambda G(x)H(x) + (1-\lambda)G(y)H(y).$$

Integrating on both sides leads to

$$\frac{4}{\ln y - \ln x}\int_x^y \frac{(\ln y - \ln s)(\ln s - \ln x)}{(\ln y - \ln x)^2}\frac{G(s)H(s)}{s^2}\mathrm{d}s \leq \frac{1}{2}[G(x)H(x) + G(y)H(y)]$$

which completes the proof. □

Theorem 3.7. *Suppose $I \subseteq (0, \infty)$, $G, H \in GT(I)$, $x, y \in I$ with $x < y$, and $GH \in L_1[x, y]$. If G, H are nonnegative, then*

$$\frac{H(x)}{\ln y - \ln x}\int_x^y \frac{\ln y - \ln s}{\ln y - \ln x}\frac{\tau(s)G(s)}{s}\mathrm{d}s + \frac{H(y)}{\ln y - \ln x}\int_x^y \frac{\ln s - \ln x}{\ln y - \ln x}\frac{\tau(s)G(s)}{s}\mathrm{d}s$$

$$+\frac{G(x)}{\ln y-\ln x}\int_x^y \frac{\ln y-\ln s}{\ln y-\ln x}\frac{\tau(s)H(s)}{s}\mathrm{d}s+\frac{G(y)}{\ln y-\ln x}\int_x^y \frac{\ln s-\ln x}{\ln y-\ln x}\frac{\tau(s)H(s)}{s}\mathrm{d}s$$
$$\leq \frac{1}{\ln y-\ln x}\int_x^y \frac{\tau^2(s)}{s^2}G(s)H(s)\mathrm{d}s+\frac{2M(x,y)+N(x,y)}{24},$$

where $\tau(s)$ *is defined in* (3.5) *and* $M(x,y)$ *and* $N(x,y)$ *are defined in* (1.3).

Proof. Because $G,H\in GT(I)$, then, for any $\lambda\in(0,1)$, we have

$$G(x^\lambda y^{1-\lambda})\leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}G(x)+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}G(y)\quad\text{and}\quad H(x^\lambda y^{1-\lambda})\leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}H(x)+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}H(y).$$

Since G,H are nonnegative, we obtain

$$G(x^\lambda y^{1-\lambda})\Big[\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}H(x)+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}H(y)\Big]+H(x^\lambda y^{1-\lambda})\Big[\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}G(x)+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}G(y)\Big]$$
$$\leq G(x^\lambda y^{1-\lambda})H(x^\lambda y^{1-\lambda})+\Big[\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}G(x)+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}G(y)\Big]\Big[\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}H(x)+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}H(y)\Big],$$

that is,

$$\frac{H(x)\sqrt{\lambda}}{2\sqrt{1-\lambda}}G(x^\lambda y^{1-\lambda})+\frac{H(y)\sqrt{1-\lambda}}{2\sqrt{\lambda}}G(x^\lambda y^{1-\lambda})+\frac{G(x)\sqrt{\lambda}}{2\sqrt{1-\lambda}}H(x^\lambda y^{1-\lambda})+\frac{G(y)\sqrt{1-\lambda}}{2\sqrt{\lambda}}H(x^\lambda y^{1-\lambda})$$
$$\leq G(x^\lambda y^{1-\lambda})H(x^\lambda y^{1-\lambda})+\frac{\lambda}{4(1-\lambda)}G(x)H(x)+\frac{1-\lambda}{4\lambda}G(y)H(y)+\frac{1}{4}[G(x)H(y)+G(y)H(x)]. \tag{3.13}$$

Multiplying on both sides of (3.13) by $\lambda(1-\lambda)$ and integrating according to $\lambda\in(0,1)$ arrive at

$$H(x)\int_0^1 \lambda\sqrt{\lambda}\sqrt{1-\lambda}\,G(x^\lambda y^{1-\lambda})\mathrm{d}\lambda+H(y)\int_0^1(1-\lambda)\sqrt{\lambda}\sqrt{1-\lambda}\,G(x^\lambda y^{1-\lambda})\mathrm{d}\lambda$$
$$+G(x)\int_0^1 \lambda\sqrt{\lambda}\sqrt{1-\lambda}\,H(x^\lambda y^{1-\lambda})\mathrm{d}\lambda+G(y)\int_0^1(1-\lambda)\sqrt{\lambda}\sqrt{1-\lambda}\,H(x^\lambda y^{1-\lambda})\mathrm{d}\lambda$$
$$\leq \int_0^1 2\lambda(1-\lambda)G(x^\lambda y^{1-\lambda})H(x^\lambda y^{1-\lambda})\mathrm{d}\lambda+\frac{1}{2}\Big[G(x)H(x)\int_0^1\lambda^2\mathrm{d}\lambda$$
$$+G(y)H(y)\int_0^1(1-\lambda)^2\mathrm{d}\lambda\Big]+\frac{1}{2}[G(x)H(y)+G(y)H(x)]\int_0^1\lambda(1-\lambda)\mathrm{d}\lambda$$
$$=\int_0^1 2\lambda(1-\lambda)G(x^\lambda y^{1-\lambda})H(x^\lambda y^{1-\lambda})\mathrm{d}\lambda+\frac{1}{12}\{2G(x)H(x)+2G(y)H(y)+G(x)H(y)+G(y)H(x)\}.$$

By substitution of $s=x^\lambda y^{1-\lambda}$, we complete the proof of Theorem 3.7. □

Corollary 3.5. *In Theorem 3.7, if G and H are similarly ordered, then*

$$\frac{H(x)}{\ln y-\ln x}\int_x^y \frac{\ln y-\ln s}{\ln y-\ln x}\frac{\tau(s)G(s)}{s}\mathrm{d}s+\frac{H(y)}{\ln y-\ln x}\int_x^y \frac{\ln s-\ln x}{\ln y-\ln x}\frac{\tau(s)G(s)}{s}\mathrm{d}s$$
$$+\frac{G(x)}{\ln y-\ln x}\int_x^y \frac{\ln y-\ln s}{\ln y-\ln x}\frac{\tau(s)H(s)}{s}\mathrm{d}s+\frac{G(y)}{\ln y-\ln x}\int_x^y \frac{\ln s-\ln x}{\ln y-\ln x}\frac{\tau(s)H(s)}{s}\mathrm{d}s$$

$$\leq \frac{1}{\ln y - \ln x}\int_x^y \frac{\tau^2(s)}{s^2}G(s)H(s)\mathrm{d}s + \frac{G(x)H(x)+G(y)H(y)}{8},$$

where $\tau(s)$ is defined in (3.5).

Theorem 3.8. *Suppose that $I \subseteq (0,\infty)$, $G, H \in GT(I)$, $x, y \in I$ with $x < y$, and $GH \in L_1[x,y]$. If G, H are nonnegative, then*

$$G(\sqrt{xy})H(\sqrt{xy}) \leq \frac{3}{8}[M(x,y)+N(x,y)],$$

where $M(x,y)$ and $N(x,y)$ are defined in (1.3).

Proof. Because G, H are nonnegative and GT-convex, then, for any $\lambda \in (0,1)$, we obtain

$$\begin{aligned} G(\sqrt{xy}) = G\Big(\sqrt{x^\lambda y^{1-\lambda}x^{1-\lambda}y^\lambda}\Big) &\leq \frac{1}{2}[G(x^\lambda y^{1-\lambda})+G(x^{1-\lambda}y^\lambda)] \\ &\leq \frac{1}{2}\Big(\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\Big)[G(x)+G(y)]. \end{aligned} \tag{3.14}$$

Similarly, it follows that

$$H(\sqrt{xy}) \leq \frac{1}{2}\Big(\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\Big)[H(x)+H(y)]. \tag{3.15}$$

Multiplying (3.14) and (3.15) reaveals

$$\begin{aligned} G(\sqrt{xy})H(\sqrt{xy}) &\leq \frac{1}{16}\Big(\frac{\sqrt{\lambda}}{\sqrt{1-\lambda}}+\frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\Big)^2[G(x)+G(y)][H(x)+H(y)] \\ &= \frac{1}{16}\frac{1}{\lambda(1-\lambda)}[G(x)+G(y)][H(x)+H(y)]. \end{aligned} \tag{3.16}$$

Integrating on both sides of (3.16) with $\lambda \in (0,1)$ completes the proof of Theorem 3.8. □

Corollary 3.6. *In Theorem 3.8, if G and H are similarly ordered, then*

$$G(\sqrt{xy})H(\sqrt{xy}) \leq \frac{3}{4}[G(x)H(x)+G(y)H(y)].$$

Theorem 3.9. *Suppose that $I \subseteq (0,\infty)$, $G, H \in GT(I)$, $x, y \in I$ with $x < y$, and $GH \in L_1[x,y]$. If G, H are nonnegative, then*

$$\begin{aligned} &G(\sqrt{xy})[H(x)+H(y)]+H(\sqrt{xy})[G(x)+G(y)] \\ &\qquad \leq \frac{16}{3\pi}G(\sqrt{xy})H(\sqrt{xy})+\frac{2}{\pi}[G(x)+G(y)][H(x)+H(y)]. \end{aligned}$$

Proof. For $\lambda \in (0,1)$, by virtue of inequalities (3.14) and (3.15), it follows that

$$\begin{aligned} &\frac{G(\sqrt{xy})}{2}\Big(\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\Big)[H(x)+H(y)]+\frac{H(\sqrt{xy})}{2}\Big(\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\Big)[G(x)+G(y)] \\ &\qquad \leq G(\sqrt{xy})H(\sqrt{xy})+\frac{1}{16}\Big(\frac{\sqrt{\lambda}}{\sqrt{1-\lambda}}+\frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\Big)^2[G(x)+G(y)][H(x)+H(y)], \end{aligned}$$

that is,

$$\begin{aligned}&\frac{1}{4}G(\sqrt{xy})[H(x)+H(y)](\lambda\sqrt{\lambda}\sqrt{1-\lambda}+(1-\lambda)\sqrt{1-\lambda}\sqrt{\lambda})\\&+\frac{1}{4}H(\sqrt{xy})[G(x)+G(y)](\lambda\sqrt{\lambda}\sqrt{1-\lambda}+(1-\lambda)\sqrt{1-\lambda}\sqrt{\lambda})\\&\le\lambda(1-\lambda)G(\sqrt{xy})H(\sqrt{xy})+\frac{1}{16}[G(x)+G(y)][H(x)+H(y)].\end{aligned}\tag{3.17}$$

Integrating the inequality (3.17) with $\lambda\in(0,1)$ gives

$$\begin{aligned}&\frac{\pi}{32}\{G(\sqrt{xy})[H(x)+H(y)]+H(\sqrt{xy})[G(x)+G(y)]\}\\&\qquad\le\frac{G(\sqrt{xy})H(\sqrt{xy})}{6}+\frac{1}{16}[G(x)+G(y)][H(x)+H(y)]\end{aligned}$$

which completes the proof. □

Corollary 3.7. *In Theorem 3.9, if G and H are similarly ordered, then*

$$\frac{G(x)+G(y)}{G(\sqrt{xy})}+\frac{H(x)+H(y)}{H(\sqrt{xy})}\le\frac{16}{3\pi}+\frac{4}{\pi}\frac{G(x)H(x)+G(y)H(y)}{G(\sqrt{xy})H(\sqrt{xy})}.$$

4. Applications to some special means

For real numbers $x,y>0$, the arithmetic mean and the p-logarithmic mean are respectively defined [25, 26] by

$$A=A(x,y)=\frac{x+y}{2}\quad\text{and}\quad L_p=L_p(x,y)=\begin{cases}\left[\dfrac{y^{p+1}-x^{p+1}}{(p+1)(y-x)}\right]^{1/p}, & x\neq y;\\ x, & x=y.\end{cases}$$

Applying the GT-convex function $G:(1,\infty)\to\mathbb{R}$, $G(x)=\ln^p x$ for $p\in(0,\frac{1}{1000})$ to Theorem 3.1, Theorem 3.2, and Corollary 3.3 straightforwardly derives the following inequalities involving the arithmetic mean A and the p-logarithmic mean L_p.

Theorem 4.1. *Let* $1<x<y$ *and* $p\in(0,\frac{1}{1000})$. *Then*

$$[A(\ln x,\ln y)]^p\le[L_p(\ln x,\ln y)]^p\le\frac{\pi}{2}A(\ln^p x,\ln^p y),$$

$$\frac{\pi}{4}[A(\ln x,\ln y)]^p\le\frac{\pi\ln^p y}{4}F\left(-p,\frac{3}{2},3,1-\frac{\ln x}{\ln y}\right)\le A(\ln^p x,\ln^p y),$$

and

$$\begin{aligned}[A(\ln x,\ln y)]^p&\le\frac{[A(\ln x,3\ln y)]^p+[A(3\ln x,\ln y)]^p}{2^{p+1}}\le[L_p(\ln x,\ln y)]^p\\&\le\frac{\pi}{4}\{A(\ln^p x,\ln^p y)+[A(\ln x,\ln y)]^p\}\le\frac{\pi}{2}A(\ln^p x,\ln^p y),\end{aligned}$$

where $F(\alpha,\beta,\gamma,x)$ *is the hypergeometric function which can be represented by*

$$F(\alpha,\beta,\gamma,x)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1\frac{t^{\beta-1}(1-t)^{\gamma-\beta-1}}{(1-xt)^{\alpha}}\mathrm{d}t.$$

Remark 4.1. Although a list of published papers which deal with this subject is quite long, for the list of references in this paper to be much complete, basing on opinions and suggestions of anonymous referees, we would like to recommend recently published and closely related articles [27–31] to interested readers.

5. Conclusions

In this paper, we introduce a new class of convex functions, GT-convex functions, in Definition 2.1, establish some integral inequalities for GT-convex functions in Theorems 3.1 to 3.4 and in Corollaries 3.1 to 3.3, establish some integral inequalities for the product of two GT-convex functions in Theorems 3.5 to 3.7 and in Corollary 3.5, derive some inequalities for the product of two GT-convex functions in Theorems 3.8 and 3.9 and in Corollaries 3.4, 3.6, and 3.7, and give some applications to classical special means in Theorem 4.1.

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Conflict of interest

The authors declare that they have no conflict of interest.

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