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Research article

A modified characteristics projection finite element method for unsteady incompressible Magnetohydrodynamics equations

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Abstract: This paper provides a modified characteristics projection finite element method for the unsteady incompressible magnetohydrodynamics(MHD) equations. In this method, modified characteristics finite element method and the projection method will be combined for solving the unsteady incompressible MHD equations. Both the stability and the optimal error estimates both in L^2 and H^1 norms for the modified characteristics projection finite element method will be shown. In order to demonstrate the effectiveness of our method, we will present some numerical results at the end.

Keywords: unsteady incompressible MHD equations; finite element method; projection method; error estimates; modified characteristics projection method
Mathematics Subject Classification: 76D05, 35Q30, 65M60,65N30

1. Introduction

Magnetohydrodynamics(MHD) is a subject that studies the motion of conductive fluids in magnetic fields. It is mainly used in astrophysics, controlled thermonuclear reactions and industry. The system of MHD equations is a coupled equation system of Navier-Stokes equations of fluid mechanics and Maxwell's equations of electrodynamics (see [1, 2]). In this paper, we consider the unsteady incompressible MHD equations as follows:

$$\begin{cases} u_t + (u \cdot \nabla)u - \nu \triangle u + \mu B \times \mathrm{curl} B + \nabla p = f, & x \in \Omega, \\ \nabla \cdot u = 0, & x \in \Omega, \\ B_t + Rm^{-1}\mathrm{curl}\mathrm{curl} B - \mathrm{curl}(u \times B) = g, & x \in \Omega, \\ \nabla \cdot B = 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where u is velocity, ν kinematic viscosity, μ magnetic permeability, p hydrodynamic pressure, B magnetic field, f body force, and g applied current. The coefficients are the magnetic Reynolds number Rm. As in [3], we assume that Ω is a convex domain in R^d, $d = 2, 3$ and $t \in [0, T]$. Here, $u_t = \frac{\partial u}{\partial t}$, $B_t = \frac{\partial B}{\partial t}$. The term $u \times B$ explains the effect of hydrodynamic flow on current. The magnetic Reynolds number Rm is moderate or low (from 10^{-2} to 1) (see [4]). For the purposes of simplicity, we choose $Rm = 1$ in our numerical analysis.

System (1.1) is considered in conjunction with the following initial boundary conditions

$$
\begin{aligned}
&u(x,0) = u_0(x), \quad B(x,0) = B_0(x), \quad \forall x \in \Omega, \\
&u = 0, \quad B \cdot \mathbf{n} = 0, \quad \mathbf{n} \times \mathrm{curl} B = 0, \quad \forall x \in \partial\Omega, \quad t > 0,
\end{aligned}
$$

where $\mathbf{n}$ denotes the outer unit normal of $\partial\Omega$.

Over that last few decades, there has been a lot of works devoted on the numerical solution of MHD flows. In [5], some mathematical equations related to the MHD equations have been studied. In [6], a weighted regularization approach was applied to incompressible MHD problems. Reference [7, 8] decoupled the linear MHD problem involving conducting and insulating regions. A mixed finite element method for stationary incompressible MHD equations was given in [9]. In order to avoid in-sup condition, Gerbeau [10] presented a stabilized finite element method and Salah et al. [11] proposed a Galerkin least-square method. In [12], a two-level finite element method with a stabilizing sub-grid for the incompressible MHD equations has been shown.

Recently, a convergence analysis of three finite element iterative methods for 2D/3D stationary incompressible MHD equations has been given by Dong and He (see [13]). In order to continue the in-depth explore of three-dimensional incompressible MHD system, an unconditional convergence of the Euler semi-implicit scheme for the three-dimensional incompressible MHD equations was studied by He [14]. In [15], a two-level Newton iteration method for 2D/3D steady incompressible MHD equation was proposed by Dong and He. In addition, Yuksel and Ingram have discussed the full discretization of Grank-Nicolson scheme at small magnetic Reynolds numbers (see [16]). The defect correction finite element method for the MHD equations was shown by Si et. al. [17–19]. In [20], a decoupling penalty finite element method for the stationary incompressible MHD equation was presented. To further study this method, we can refer to [21–23].

In this paper, a modified characteristics projection finite element method for the unsteady incompressible magnetohydrodynamics(MHD) equations will be given. In this method, modified characteristics finite element method and the projection method will be combined for solving the incompressible MHD equations. Both the stability and the optimal error estimates both in L^2 and H^1 norms for the modified characteristics projection finite element method will be derived. In order to demonstrate the effectiveness of our method, we will present some numerical results at the end of the manuscript. The contents of this paper are divided into sections as follows. To obtain the unconditional stability and optimal error estimates, in section 2, we introduce some notations and present a modified characteristics projection algorithm for the MHD equations. Then, in section 3, we give stability and error analysis and show that this method has optimal convergence order by mathematical induction. In order to demonstrate the effectiveness of our method, some numerical results are presented in the last section.

2. Notation and preliminaries

In this section, we introduce some notations. We employ the standard Sobolev spaces $H^k(\Omega) = W^{k,2}$, for nonnegative k with norm $\|v\|_k = (\sum_{|\beta|=0}^{k} \|D^\beta v\|_0^2)^{1/2}$. For vector-value function, we use the Sobolev spaces $\mathrm{H}^k(\Omega) = (H^k(\Omega))^d$ with norm $\|\mathrm{v}\|_k = (\sum_{i=1}^{d} \|v_i\|_k^2)^{1/2}$(see [24, 25]). For simplicity, we set:

$$\begin{aligned}
&X = H_0^1(\Omega)^2, X_0 = \{v \in X : \nabla \cdot v = 0\},\\
&M = L_0^2(\Omega) = \left\{q \in L^2(\Omega)^d : \int_\Omega q(x)dx = 0\right\},\\
&H = \{v \in L^2(\Omega)^d : \nabla \cdot v = 0\},\\
&W = \{\psi \in (H_1(\Omega))^d : \psi \cdot n|_{\partial\Omega} = 0\}, W_0 = \{\psi \in W : \nabla \cdot \psi = 0\},
\end{aligned}$$

which are equipped with the norms

$$\|v\|_X = \|\nabla v\|_0, \ \ \|q\|_M = \|q\|_0, \ \ \|\psi\|_W = \|\psi\|_1.$$

In this paper, we will use the following equality

$$\mathrm{curl}(\psi \times B) = B \cdot \nabla\psi - \psi \cdot \nabla B, \ \ \forall \psi, B \in W_0.$$

Then, we recall the following Poincaré-Friedrichs inequality in W: there holds

$$\|c\|_0 \leq C\|\mathrm{curl} c\|_0, \ \ \forall c \in W,$$

with a constant $C > 0$ solely depending on the domain Ω; (see [26]). Here, we define the following trilinear form $a_1(\cdot,\cdot,\cdot)$ as follows, for all $u \in V$, $w, v \in X$,

$$a_1(u, v, w) = (u \cdot \nabla w + \frac{1}{2}(\nabla \cdot u)w, v) = \frac{1}{2}(u \cdot \nabla w, v) - \frac{1}{2}(u \cdot \nabla v, w)$$

and the properties of the trilinear form $a_1(\cdot,\cdot,\cdot)$ [27] are helpful in our analysis

$$\begin{aligned}
&a_1(u, v, w) = -a_1(u, w, v), \ \ u \in X_0, v, w \in X,\\
&|a_1(u, v, w)| \leq \frac{N}{2}\|u\|_0(\|\nabla v\|_0\|w\|_{L^\infty} + \|v\|_{L^6}\|\nabla w\|_{L^3}), \ \ u \in L^2(\Omega)^2, \ v \in X, w \in L^\infty(\Omega)^2 \cap X,\\
&|a_1(u, v, w)| \leq \frac{N}{2}(\|u\|_{L^\infty}\|\nabla v\|_0 + \|\nabla u\|_{L^3}\|v\|_{L^6})\|w\|_0, \ \ u \in L^\infty(\Omega)^2 \cap X, \ v \in X, w \in L^2(\Omega)^2,\\
&\|v\|_0 \leq \gamma_0\|\nabla v\|_0, \|v\|_{L^6} \leq C\|\nabla v\|_0, \|w\|_{L^4} \leq C_0\|\nabla w\|_0, \ \ \forall w \in X,
\end{aligned}$$

where $N > 0$ is a constant, γ_0 (only dependent on Ω) is a positive constant, and C_0 (only dependent on Ω) is an embedding constant of $H^1(\Omega) \hookrightarrow L^4(\Omega)$ ($\hookrightarrow$ denotes the continuous embedding).

Furthermore, we define the Stokes operator $\mathcal{A} : D(\mathcal{A}) \longrightarrow \tilde{H}$ such that $\mathcal{A} = -P\triangle$, where $D(\mathcal{A}) = H^2(\Omega)^d \cap X_0$ and P is an L^2-projection from $L^2(\Omega)^d$ to $\tilde{H} = \{v \in L^2(\Omega)^d; \nabla \cdot v = 0, v \cdot n|_{\partial\Omega} = 0\}$.

Next, we recall a discrete version of Gronwall's inequality established in [14].

Lemma 2.1 Let C_0, a_n, b_n and d_n, for integers $n \geq 0$, be the non-negative numbers such that

$$a_m + \tau \sum_{n=1}^{m} b_n \leq \tau \sum_{n=1}^{m-1} d_n a_n + C_0 \quad m \geq 1. \tag{2.1}$$

Then

$$a_m + \tau \sum_{n=1}^{m} b_n \leq C_0 \exp\left(\tau \sum_{n=1}^{m-1} d_n\right) \quad m \geq 1. \tag{2.2}$$

Throughout this paper, we make the following assumptions on the prescribed date for the problem (1.1), which satisfies the regularity of the data needed for our main results.

Assumption (A1)(see [27,28]): Assume that the boundary of Ω is smooth so that the unique solution (v, q) of the steady Stokes problem

$$-\Delta v + \nabla q = f,\ \nabla \cdot v = 0\ \ in\ \Omega,\ v|_{\partial\Omega} = 0,$$

for prescribed $f \in L^2(\Omega)^3$ satisfies

$$\|v\|_2 + \|q\|_1 \leq c\|f\|_0,$$

and Maxwell's equations

$$\text{curlcurl}B = g,\quad \nabla \cdot B = 0\ in\ \Omega,\ n \times \text{curl}B = 0,\ B \cdot \nabla n\ on\ \Omega,$$

for the prescribed $g \in L^2(\Omega)^3$ admits a unique solution $B \in W_0$ which satisfies

$$\|B\|_2 \leq c\|g\|_0.$$

Assumption (A2): The initial data u_0, B_0, f satisfy

$$\|\mathcal{A}u_0\|_0 + \|B_0\|_2 + \sup_{0\leq t\leq T} \|f\|_0 + \|f_t\|_0 \leq C. \tag{2.3}$$

Assumption (A3): We assume that the MHD problem admits a unique local strong solution (u, p, B) on $(0, T)$ such that

$$\begin{aligned}&\sup_{0\leq t\leq T} \|\mathcal{A}u_t\|_0 + \|B_t\|_2 + \|p\|_1 + \|u_t\|_0 + \|B_t\|_0\\&+ \int_0^T (\|u_t\|_1^2 + \|B_t\|_1^2 + \|p_t\|_1^2 + \|u_{tt}\|_0^2 + \|B_{tt}\|_0^2)dt \leq C.\end{aligned} \tag{2.4}$$

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0, T]$ with $t_n = n\tau$, and N being a positive integer. Set $u^n = (\cdot, t_n)$, For any sequence of functions $\{f^n\}_{n=0}^N$, we define

$$D_\tau f^n = \frac{f^n - f^{n-1}}{\tau}.$$

At the initial time step, we start with $U^0 = \tilde{U}^0 = u_0, B^0 = \tilde{B}^0 = b_0$ and an arbitrary $P^0 \in M \cap H^1(\Omega)$. Then, we can give

$$\nabla P^0 = f^0 - u_0 \cdot \nabla u_0 + \nu\Delta u_0 - \mu B^0 \times \text{curl}B^0.$$

Then, we give the algorithm as follows

Algorithm 2.1 (Time-discrete modified characteristics projection finite element method)
Step 1. Solve B^{n+1}, for instance

$$\frac{B^{n+1}-\hat{B}^n}{\tau}+\mathrm{curlcurl}B^{n+1}-(B^n\cdot\nabla)U^n=g^{n+1}, \tag{2.5}$$
$$B^{n+1}\cdot\mathbf{n}=0,\quad \mathbf{n}\times\mathrm{curl}B^{n+1}=0,\ \text{on}\ \partial\Omega, \tag{2.6}$$

where

$$\hat{l}^n=\begin{cases} l^n(\hat{x}), & \hat{x}=x-U^n\tau\in\Omega,\\ 0, & \text{othrewise.}\end{cases}$$

Step 2. Find $\tilde{U}^{n+1}$, such that

$$\frac{\tilde{U}^{n+1}-\hat{U}^n}{\tau}-\nu\Delta\tilde{U}^{n+1}+\nabla P^n+\mu B^n\times\mathrm{curl}B^{n+1}=f^{n+1}, \tag{2.7}$$
$$\tilde{U}^{n+1}=0,\ \text{on}\ \partial\Omega. \tag{2.8}$$

Step 3. Calculate (U^{n+1},P^{n+1}), for example

$$\frac{U^{n+1}-\tilde{U}^{n+1}}{\tau}+\nabla(P^{n+1}-P^n)=0, \tag{2.9}$$
$$\nabla\cdot U^{n+1}=0, \tag{2.10}$$
$$U^{n+1}=0\ \text{on}\ \partial\Omega,. \tag{2.11}$$

The corresponding weak form is
Step 1. Solve B^{n+1}, for instance

$$\left(\frac{B^{n+1}-\hat{B}^n}{\tau},\psi\right)+(\mathrm{curl}B^{n+1},\mathrm{curl}\psi)-((B^n\cdot\nabla)U^n,\psi)=(g^{n+1},\psi),\quad\forall\psi\in W. \tag{2.12}$$

Step 2. Find $\tilde{U}^{n+1}$, such that

$$\begin{aligned}&\left(\frac{\tilde{U}^{n+1}-\hat{U}^n}{\tau},v\right)+\nu(\nabla\tilde{U}^{n+1},\nabla v)+(\nabla P^n,v)\\&+\mu(B^n\cdot\nabla v,B^{n+1})-\mu(v\cdot\nabla B^n,B^n)=(f^{n+1},v),\quad\forall v\in X,\end{aligned} \tag{2.13}$$

Step 3. Calculate (U^{n+1},P^{n+1}), for example

$$\left(\frac{U^{n+1}-\tilde{U}^{n+1}}{\tau},v\right)+(\nabla(P^{n+1}-P^n),v)=0,\quad\forall v\in X, \tag{2.14}$$
$$(\nabla\cdot U^{n+1},\phi)=0,\quad\forall\phi\in M. \tag{2.15}$$

Remark 2.1 As we all know $(B^n\times\mathrm{curl}B^{n+1},v)=(B^n\cdot\nabla v,B^{n+1})-(v\cdot\nabla B^n,B^{n+1})$. In order to improve our algorithm, we change it as $(B^n\cdot\nabla v,B^{n+1})-(v\cdot\nabla B^n,B^n)$.

We denote T_h be a regular and quasi-uniform partition of the domain Ω into the triangles for $d = 2$ or tetrahedra for $d = 3$ with diameters by a real positive parameter $h(h \to 0)$. The finite element pair (X_h, M_h, W_h) is constructed based on T_h.

Let $W_{0n}^h = X_h \times W_h$. There exits a constant $\beta_0 > 0$ (only dependent on Ω) such that

$$\sup_{0 \neq (\mathbf{v},\psi) \in W_{0n}^h} \frac{d(\mathbf{v}, q)}{\|\nabla \mathbf{v}\|_0} \geq \beta_0 \|q\|_0, \quad \forall q \in M_h. \tag{2.16}$$

There exits a mapping $r_h \in \mathcal{L}(H^2(\Omega)^2 \cap V, X_h)$ satisfying

$$(\nabla \cdot (\mathbf{v} - r_h \mathbf{v}), q) = 0, \|\nabla(\mathbf{v} - r_h \mathbf{v})\|_0 \leq Ch\|\mathbf{v}\|_2, \quad \forall \mathbf{v} \in H^2(\Omega)^2 \cap V, \quad q \in M_h,$$

and a L^2-projection operator $\rho_h : M \to M_h$ satisfying

$$\|q - \rho_h q\|_0 \leq Ch\|q\|_1, \quad \forall q \in H^1(\Omega) \cap M,$$

and a mapping $R_h \in \mathcal{L}(H^2(\Omega)^2 \cap V_n, W_h)$ satisfying

$$(\nabla \times R_h\Phi, \nabla \times \Psi) + (\nabla \cdot R_h\Phi, \nabla \cdot \Psi) = (\nabla \times \Phi, \nabla \times \Psi) + (\nabla \cdot \Phi, \nabla\Psi) = (\nabla \times \Phi, \nabla \times \Psi), \quad \forall \Psi \in W_h,$$
$$\|\Phi - R_h\Phi\|_0 + h\|\Phi - R_h\Phi\|_1 \leq Ch^2\|\Phi\|_2, \quad \forall H^2(\Omega) \cap V_h.$$

Here, we define the discrete Stokes operator $A_h = -P_h \triangle_h$ and $\triangle_h$ (see [15] and the references therein) defined by

$$-(\triangle_h u_h, v_h) = (\nabla u_h, \nabla v_h), \forall u_h, v_h \in V_h.$$

Meanwhile, we define the discrete operator $A_{1h}B_h = R_{1h}(\nabla_h \times \nabla \times B_h + \nabla_h \nabla \cdot B_h) \in W_h$ (see [15]) as follows

$$(A_{1h}B_h, \Psi) = (A_{1h}^{\frac{1}{2}}B_h, A_{1h}^{\frac{1}{2}}B_h) = (\nabla \times B_h, \nabla \times \Psi) + (\nabla \cdot B_h, \nabla \cdot \Psi), \forall B_h, \Psi \in W_h.$$

We define the Stokes projection $(R_h(u, p), Q_h(u, P)) : (X, M) \to (X_h, M_h)$ by

$$\nu(R_h(u, p) - u, \nabla v_h) - (Q_h(u, p) - p, \nabla \cdot v_h) = 0, \tag{2.17}$$
$$(\nabla \cdot (R_h(u, p) - u), \phi_h) = 0. \tag{2.18}$$

By classical FEM theory ([25, 29, 30]), we have the following results.

Lemma 2.2 Assume that $u \in H_0^1(\Omega)^d \cap H^{r+1}(\Omega)^d$ and $p \in L_0^2(\Omega) \cap H^r(\Omega)$. Then,

$$\|R_h(u, p) - u\|_0 + h(\|\nabla(R_h(u, p) - u)\|_0 + \|Q_h(u, p) - p\|_0) \leq Ch^{r+1}(\|u\|_{r+1} + \|p\|_r), \tag{2.19}$$

and

$$\|R_h(u, p)\|_{L^\infty} \leq C(\|u\|_2 + \|p\|_1). \tag{2.20}$$

Lemma 2.3 If $(u, p) \in W^{2,k}(\Omega)^d \times W^{1,k}(\Omega)$ for $k > d$,

$$\|\nabla R_h(u, p)\|_{L^\infty} \leq C(\|u\|_{W^{1,\infty}} + \|p\|_{L^\infty}). \tag{2.21}$$

Algorithm 2.2 (The fully discrete modified characteristics projection finite element method)
Step 1. Solve B_h^{n+1}, for instance

$$\left(\frac{B_h^{n+1}-\hat{B}_h^n}{\tau},\psi_h\right)+(\mathrm{curl}B_h^{n+1},\mathrm{curl}\psi_h)-((B_h^n\cdot\nabla)u_h^n,\psi_h)=(g^{n+1},\psi_h),\quad\forall\psi_h\in W_h. \tag{2.22}$$

where

$$\hat{l}_h^n=\begin{cases}l_h^n(\hat{x}), & \hat{x}=x-u_h^n\tau\in\Omega,\\ 0, & \text{othrewise.}\end{cases}$$

Step 2. Find $\tilde{U}^{n+1}$, such that

$$\begin{aligned}&\left(\frac{\tilde{u}_h^{n+1}-\hat{u}_h^n}{\tau},v_h\right)+\nu(\nabla\tilde{u}_h^{n+1},\nabla v_h)+(\nabla p_h^n,v_h)\\&+\mu(B_h^n\cdot\nabla v_h,B_h^{n+1})-\mu(v_h\cdot\nabla B_h^n,B_h^n)=(f^{n+1},v_h),\quad\forall v_h\in X_h.\end{aligned} \tag{2.23}$$

Step 3. Calculate (u_h^{n+1},p_h^{n+1}), for example

$$\left(\frac{u_h^{n+1}-\tilde{u}_h^{n+1}}{\tau},v_h\right)+(\nabla(p_h^{n+1}-p_h^n),v_h)=0,\quad\forall v_h\in X_h, \tag{2.24}$$

$$(\nabla\cdot u_h^{n+1},\phi_h)=0,\quad\forall\phi_h\in M_h. \tag{2.25}$$

3. Unconditional stability and convergence

3.1. Error estimation for the time-discrete method

Here, we make use of the following notation:

$$\begin{aligned}&e^n=u^n-U^n,\quad\tilde{e}=u^n-\tilde{U}^n,\quad n=0,1,...,N,\\&\varepsilon^n=b^n-B^n,\quad\hat{\varepsilon}^n=b^n-\hat{B}^n,\quad n=0,1,...,N,\\&\xi^n=p^n-P^n,\quad n=0,1,...,N.\end{aligned}$$

Lemma 3.1 (see [31]). Assume that g_1,g_2,ρ are three functions defined in Ω and $\rho|_{\partial\Omega}=0$. If

$$\tau(\|g_1\|_{W^{1,\infty}}+\|g_2\|_{W^{1,\infty}})\le\frac{1}{2},$$

then

$$\begin{aligned}&\|\rho(x-g_1(x)\tau)-\rho(x-g_2(x)\tau)\|_{L^q}\le C\tau\|\rho\|_{W^{1,q_2}}\|g_1-g_2\|_{L^{q_1}},\\&\|\rho(x-g_1(x)\tau)-\rho(x-g_2(x)\tau)\|_{-1}\le C\tau\|\rho\|_0\|g_1-g_2\|_{W^{1,\infty}},\end{aligned}$$

where $1/q_1+1/q_2=1/q, 1<q<\infty$.

The following theorem gives insight on the error estimate between the time-discrete solution and the solution of the time-dependent MHD system (1.1).

Theorem 3.1 Suppose that assumption A2–A3 are satisfied, there exists a positive $C > 0$ such that

$$
\begin{aligned}
&\max_{0\le n\le m}(\|e^{m+1}\|_0^2+\mu\|\varepsilon^{m+1}\|_0^2)+\sum_{n=0}^{m}(\|e^{n+1}-e^n\|_0^2+\mu\|\varepsilon^{n+1}-\varepsilon^n\|_0^2),\\
&+\tau\sum_{n=0}^{m}\left(\nu\|\nabla\tilde{e}^{n+1}\|_0^2+\mu\|\mathrm{curl}\varepsilon^{n+1}\|_0^2\right)\le C\tau^2,\\
&\max_{0\le n\le m}\left(\nu\|e^{m+1}\|_1^2+\mu\|\mathrm{curl}\varepsilon^{m+1}\|_0^2\right)+\tau\sum_{n=1}^{m}\left(\nu\|e^{n+1}-e^n\|_1^2+\mu\|\mathrm{curl}\varepsilon^{n+1}-\mathrm{curl}\varepsilon^n\|_0^2\right)\\
&+\tau\sum_{n=1}^{m}(\|D_\tau e^{n+1}\|_0^2+\mu\|D_\tau\varepsilon^{n+1}\|_0^2)\le C\tau^2,\\
&\tau\sum_{n=0}^{m}(\|e^{n+1}\|_2^2+\|\tilde{e}^{n+1}\|_2^2+\|\varepsilon^{n+1}\|_2^2)\le C\tau^2,\\
&\tau\sum_{n=0}^{m}(\|D_\tau U^{n+1}\|_2^2+\|D_\tau\tilde{U}^{n+1}\|_2^2+\|D_\tau P^{n+1}\|_1^2+\|D_\tau B^{n+1}\|_2^2+\|U^{n+1}\|_{W^{2,d^*}}^2\\
&+\|B^{n+1}\|_{W^{2,d^*}}^2)+\max_{0\le n\le m}(\|U^{n+1}\|_2+\|\tilde{U}^{n+1}\|_2+\|P^{n+1}\|_1+\|B^{n+1}\|_2)\le C.
\end{aligned}
$$

Proof. Firstly, we prove the following inequality

$$\|e^m\|_2+\tau^{3/4}\|U^m\|_{W^{2,d^*}}\le 1, \tag{3.1}$$

$$\|\varepsilon^m\|_2+\tau^{3/4}\|B^m\|_{W^{2,d^*}}\le 1, \tag{3.2}$$

by mathematical induction for $m = 0, 1, ..., N$.

Since $U^0 = u^0, B^0 = b^0$ the above inequality hold for $m = 0$.

We assume that (3.1) and (3.2) hold for $m \le n$ for some integer $n \ge 0$. By Sobolev embedding theory, we get

$$\|U^m\|_{L^\infty}\le\|u^m\|_{L^\infty}+C\|e^m\|_2\le C, \tag{3.3}$$

$$\|B^m\|_{L^\infty}\le\|b^m\|_{L^\infty}+C\|\varepsilon^m\|_2\le C, \tag{3.4}$$

and

$$\tau\|U^m\|_{W^{1,\infty}}\le 1/4, \tag{3.5}$$

$$\tau\|B^m\|_{W^{1,\infty}}\le 1/4, \tag{3.6}$$

when $\tau \le \tau_1$ for some positive constant τ_1.

To prove (3.1) and (3.2) for $m = n + 1$, we rewrite (1.1) by

$$D_\tau u^{n+1}-\nu\Delta u^{n+1}+\nabla p^{n+1}=f^{n+1}-\frac{u^n-\bar{u}^n}{\tau}+R_{tr1}^{n+1}-\mu b^{n+1}\times\mathrm{curl}b^{n+1}, \tag{3.7}$$

$$\nabla \cdot u^{n+1} = 0, \tag{3.8}$$

$$D_\tau b^{n+1} + \text{curlcurl} b^{n+1} = g^{n+1} - \frac{b^n - \bar{b}^n}{\tau} + R_{tr2}^{n+1} + b^{n+1} \cdot \nabla u^{n+1}. \tag{3.9}$$

where $u^{n+1} = U(t_{n+1}), b^{n+1} = B(t_{n+1})$, and

$$\bar{l}^n = \begin{cases} l^n(\bar{x}), & \bar{x} = x - u^n\tau \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\begin{aligned} R_{tr1}^{n+1} &= \frac{u^{n+1} - \bar{u}^n}{\tau} - u_t^{n+1} - (u^{n+1} \cdot \nabla) u^{n+1}, \\ R_{tr2}^{n+1} &= \frac{b^{n+1} - \bar{b}^n}{\tau} - b_t^{n+1} - (u^{n+1} \cdot \nabla) b^{n+1}, \end{aligned}$$

define the truncation error. We can refer to ([31]), there holds that

$$\tau \sum_{m=1}^{N} \|R_{tr1}^m\|_0^2 \leq C\tau^2, \qquad \tau \sum_{m=1}^{N} \|R_{tr2}^m\|_0^2 \leq C\tau^2. \tag{3.10}$$

The corresponding weak form is

$$\begin{aligned} &(D_\tau u^{n+1}, v) + \nu(\nabla u^{n+1}, \nabla v) - (p^{n+1}, \nabla \cdot v) \\ =&(f^{n+1}, v) - \left(\frac{u^n - \bar{u}^n}{\tau}, v\right) + (R_{tr1}^{n+1}, v) - \mu(b^{n+1} \times \text{curl} b^{n+1}, v), \end{aligned} \tag{3.11}$$

$$(\nabla \cdot u^{n+1}, \phi) = 0, \tag{3.12}$$

$$\begin{aligned} &(D_\tau b^{n+1}, \psi) + (\text{curl} b^{n+1}, \text{curl} \psi) \\ =&(g^{n+1}, \psi) - \left(\frac{b^n - \bar{b}^n}{\tau}, \psi\right) + (R_{tr2}^{n+1}, \psi) + (b^{n+1} \cdot \nabla(u^{n+1} - u^n), \psi) + (b^{n+1} \cdot \nabla u^n, \psi). \end{aligned} \tag{3.13}$$

Combining (2.13) and (2.14), we obtain

$$\begin{aligned} &\left(\frac{U^{n+1} - \hat{U}^n}{\tau}, v\right) + \nu(\nabla \tilde{U}^{n+1}, \nabla v) - (P^{n+1}, \nabla \cdot v) + \mu(B^n \cdot \nabla v, B^{n+1}) \\ &- \mu(v \cdot \nabla B^n, B^n) = (f^{n+1}, v). \end{aligned} \tag{3.14}$$

Then, we rewrite (3.14) and (2.12) as

$$\begin{aligned} &(D_\tau U^{n+1}, v) + \nu(\nabla \tilde{U}^{n+1}, \nabla v) - (P^{n+1}, \nabla \cdot v) \\ =&(f^{n+1}, v) - \left(\frac{U^n - \hat{U}^n}{\tau}, v\right) - \mu(B^n \cdot \nabla v, B^{n+1}) + \mu(v \cdot \nabla B^n, B^n), \end{aligned} \tag{3.15}$$

and

$$(D_\tau B^{n+1}, \psi) + (\mathrm{curl} B^{n+1}, \mathrm{curl}\psi) = (g^{n+1}, \psi) + (B^n \cdot \nabla U^n, \psi) - \left(\frac{B^n - \hat{B}^n}{\tau}, \psi\right). \tag{3.16}$$

Subtracting (3.15) and (3.16) from (3.11) and (3.13), respectively, leads to

$$\begin{aligned}
&(D_\tau e^{n+1}, v) + \nu(\nabla \tilde{e}^{n+1}, \nabla v) - (\xi^{n+1}, \nabla \cdot v) \\
=& -\left(\frac{u^n - \bar{u}^n - (U^n - \hat{U}^n)}{\tau}, v\right) + (R_{tr1}^{n+1}, v) - \mu((b^{n+1} - b^n) \cdot \nabla v - v \cdot \nabla(b^{n+1} - b^n), b^{n+1}) \\
&+ \mu(v \cdot \nabla b^n, b^{n+1} - b^n) + \mu(v \cdot \nabla \varepsilon^n, b^n) + \mu(v \cdot \nabla B^n, \varepsilon^n) - \mu(\varepsilon^n \cdot \nabla v, b^{n+1}) \\
&- \mu(B^n \cdot \nabla v, \varepsilon^{n+1}),
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&(D_\tau \varepsilon^{n+1}, \psi) + (\mathrm{curl}\varepsilon^{n+1}, \mathrm{curl}\psi) \\
=& -\left(\frac{b^n - \bar{b}^n - (B^n - \hat{B}^n)}{\tau}, \psi\right) + (R_{tr2}^{n+1}, \psi) \\
&+ (b^{n+1} \cdot \nabla(u^{n+1} - u^n) + (b^{n+1} - b^n) \cdot \nabla u^n + \varepsilon^n \cdot \nabla u^n + B^n \cdot \nabla e^n, \psi).
\end{aligned} \tag{3.18}$$

Testing (2.9) by τe^{n+1}, since $\nabla \cdot e^{n+1} = 0$, we arrive at

$$(e^{n+1}, \tilde{e}^{n+1}) = (e^{n+1}, e^{n+1}). \tag{3.19}$$

Testing (2.9) by τe^n, we get

$$(e^n, \tilde{e}^{n+1}) = (e^n, e^{n+1}). \tag{3.20}$$

Subtracting (3.20) from (3.19), we can obtain

$$(e^{n+1} - e^n, \tilde{e}^{n+1}) = (e^{n+1} - e^n, e^{n+1}). \tag{3.21}$$

Let $v = 2\tau\tilde{e}^{n+1}$ in (3.17), we obtain

$$\begin{aligned}
&\|e^{n+1}\|_0^2 - \|e^n\|_0^2 + \|e^{n+1} - e^n\|_0^2 + 2\nu\tau\|\nabla\tilde{e}^{n+1}\|_0^2 - 2\tau(\xi^{n+1}, \nabla \cdot \tilde{e}^{n+1}) \\
=& -2((e^n - (\bar{u}^n - \hat{U}^n), \tilde{e}^{n+1}) + 2\tau(R_{tr1}^{n+1}, \tilde{e}^{n+1}) - 2\mu\tau((b^{n+1} - b^n) \cdot \nabla\tilde{e}^{n+1} \\
&- \tilde{e}^{n+1} \cdot \nabla(b^{n+1} - b^n), b^n) + 2\mu\tau(\tilde{e}^{n+1} \cdot \nabla b^n, b^{n+1} - b^n) + 2\mu\tau(\tilde{e}^{n+1} \cdot \nabla\varepsilon^n, b^n) \\
&+ 2\mu\tau(\tilde{e}^{n+1} \cdot \nabla B^n, \varepsilon^n) - 2\mu\tau(\varepsilon^n \cdot \nabla\tilde{e}^{n+1}, b^{n+1}) - 2\mu\tau(B^n \cdot \nabla\tilde{e}^{n+1}, \varepsilon^{n+1}).
\end{aligned} \tag{3.22}$$

Let $\psi = 2\mu\tau\varepsilon^{n+1}$ in (3.18), we get

$$\begin{aligned}
&\mu\|\varepsilon^{n+1}\|_0^2 - \mu\|\varepsilon^n\|_0^2 + \mu\|\varepsilon^{n+1} - \varepsilon^n\|_0^2 + 2\mu\tau\|\mathrm{curl}\varepsilon^{n+1}\|_0^2 \\
=& -2\mu(\varepsilon^n - (\bar{b}^n - \hat{B}^n), \varepsilon^{n+1}) + 2\mu\tau(R_{tr2}^{n+1}, \varepsilon^{n+1}) \\
&+ 2\mu\tau(b^{n+1} \cdot \nabla(u^{n+1} - u^n) + (b^{n+1} - b^n) \cdot \nabla u^n + \varepsilon^n \cdot \nabla u^n + B^n \cdot \nabla e^n, \varepsilon^{n+1}).
\end{aligned} \tag{3.23}$$

Taking sum of (3.22) and (3.23) yields

$$\begin{aligned}
&\|e^{n+1}\|_0^2 - \|e^n\|_0^2 + \mu\|\varepsilon^{n+1}\|_0^2 - \mu\|\varepsilon^n\|_0^2 + \|e^{n+1} - e^n\|_0^2 + \mu\|\varepsilon^{n+1} - \varepsilon^n\|_0^2 + 2\nu\tau\|\nabla\tilde{e}^{n+1}\|_0^2 \\
&+ 2\mu\tau\|\mathrm{curl}\varepsilon^{n+1}\|_0^2 - 2\tau(\xi^{n+1}, \nabla\cdot\tilde{e}^{n+1}) \\
=& - 2(e^n - \hat{e}^n, \tilde{e}^{n+1}) - 2(\hat{u}^n - \bar{u}^n, \tilde{e}^{n+1}) + 2\tau(R_{tr1}^{n+1}, \tilde{e}^{n+1}) - 2\mu\tau((b^{n+1} - b^n)\cdot\nabla\tilde{e}^{n+1}, b^n) \\
&+ 2\mu\tau(\tilde{e}^{n+1}\cdot\nabla(b^{n+1} - b^n), b^n) + 2\mu\tau(\tilde{e}^{n+1}\cdot\nabla b^n, b^{n+1} - b^n) + 2\mu\tau(\tilde{e}^{n+1}\cdot\nabla\varepsilon^n, b^n) \\
&+ 2\mu\tau(\tilde{e}^{n+1}\cdot\nabla B^n, \varepsilon^n) - 2\mu\tau(\varepsilon^n\cdot\nabla\tilde{e}^{n+1}, b^{n+1}) - 2\mu\tau(B^n\cdot\nabla\tilde{e}^{n+1}, \varepsilon^{n+1}) - 2\mu(\varepsilon^n - \hat{\varepsilon}^n, \varepsilon^{n+1}) \\
&- 2\mu(\hat{b}^n - \bar{b}^n, \varepsilon^{n+1}) + 2\mu\tau(R_{tr2}^{n+1}, \varepsilon^{n+1}) + 2\mu\tau(b^{n+1}\cdot\nabla(u^{n+1} - u^n), \varepsilon^{n+1}) \\
&+ 2\mu\tau((b^{n+1} - b^n)\cdot\nabla u^n, \varepsilon^{n+1}) + 2\mu\tau(\varepsilon^n\cdot\nabla u^n\varepsilon^{n+1}) + 2\mu\tau(B^n\cdot\nabla e^n, \varepsilon^{n+1}).
\end{aligned} \tag{3.24}$$

From (2.9), we have

$$\tilde{e}^{n+1} = e^{n+1} + \tau\nabla(\xi^{n+1} - \xi^n) - \tau\nabla(p^{n+1} - p^n).$$

Then, we can obtain

$$\begin{aligned}
&- 2\tau(\xi^{n+1}, \nabla\cdot\tilde{e}^{n+1}) \\
=& - 2\tau(\xi^{n+1}, \nabla\cdot e^{n+1}) + 2\tau^2(\nabla\xi^{n+1}, \nabla(\xi^{n+1} - \xi^n)) - 4\tau^2(\nabla\xi^{n+1}, \nabla(p^{n+1} - p^n)) \\
=&\tau^2(\|\nabla\xi^{n+1}\|_0^2 - \|\nabla\xi^n\|_0^2 + \|\nabla\xi^{n+1} - \nabla\xi^n\|_0^2) - 2\tau^2(\nabla\xi^{n+1}, \nabla(p^{n+1} - p^n)) \\
\geq&\tau^2(\|\nabla\xi^{n+1}\|_0^2 - \|\nabla\xi^n\|_0^2 + \|\nabla(\xi^{n+1} - \xi^n)\|_0^2) - C\tau^2\|\nabla\xi^{n+1}\|_0\|\nabla(p^{n+1} - p^n)\|_0 \\
=&\tau^2(\|\nabla\xi^{n+1}\|_0^2 - \|\nabla\xi^n\|_0^2 + \|\nabla(\xi^{n+1} - \xi^n)\|_0^2) - \tau(C\|\tau\nabla\xi^{n+1}\|_0\|\nabla(p^{n+1} - p^n)\|_0) \\
\geq&\tau^2(\|\nabla\xi^{n+1}\|_0^2 - \|\nabla\xi^n\|_0^2 + \|\nabla(\xi^{n+1} - \xi^n)\|_0^2) - \tau(C\tau^2\|\nabla\xi^{n+1}\|_0^2 + \frac{1}{4}\|\nabla(p^{n+1} - p^n)\|_0^2) \\
=&\tau^2(\|\nabla\xi^{n+1}\|_0^2 - \|\nabla\xi^n\|_0^2 + \|\nabla(\xi^{n+1} - \xi^n)\|_0^2) - C\tau^3\|\nabla\xi^{n+1}\|_0^2 + \frac{1}{4}\|\nabla(p^{n+1} - p^n)\|_0^2).
\end{aligned}$$

On the other hand, using Lemma 3.1, we deduce that

$$\begin{aligned}
2|(e^n - \hat{e}^n, \tilde{e}^{n+1})| &\leq 2\|e^n - \hat{e}^n\|_{-1}\|\nabla\tilde{e}^{n+1}\|_0 \\
&\leq C\tau\|e^n\|_0\|U^n\|_{W^{1,\infty}}\|\nabla\tilde{e}^{n+1}\|_0 \\
&\leq \frac{\nu\tau}{12}\|\nabla\tilde{e}^{n+1}\|_0^2 + C\tau\|e^n\|_0^2, \\
2|(\hat{u}^n - \bar{u}^n, \tilde{e}^{n+1})| &\leq C\|\hat{u}^n - \bar{u}^n\|_{L^{6/5}}\|\tilde{e}^{n+1}\|_{L^6} \\
&\leq C\tau\|\nabla u^n\|_{L^3}\|e^n\|_0\|\nabla\tilde{e}^{n+1}\|_0 \\
&\leq \frac{\nu\tau}{12}\|\nabla\tilde{e}^{n+1}\|_0^2 + C\tau\|e^n\|_0^2, \\
2\tau|(R_{tr1}^{n+1}, \tilde{e}^{n+1})| &\leq \|R_{tr1}^{n+1}\|_0\|\tilde{e}^{n+1}\|_0 \\
&\leq \frac{\nu\tau}{12}\|\nabla\tilde{e}^{n+1}\|_0^2 + C\tau\|R_{tr1}^{n+1}\|_0^2, \\
2\mu\tau|((b^{n+1} - b^n)\cdot\nabla\tilde{e}^{n+1}, b^{n+1})| &\leq C\tau\|b^{n+1} - b^n\|_0\|\nabla\tilde{e}^{n+1}\|_0\|b^{n+1}\|_2 \\
&\leq \frac{\nu\tau}{12}\|\nabla\tilde{e}^{n+1}\|_0^2 + C\tau^3\|b_t^{n+1}\|_0^2\|b^{n+1}\|_2^2, \\
2\mu\tau|(\tilde{e}^{n+1}\cdot\nabla(b^{n+1} - b^n), b^{n+1})| &\leq C\tau\|\tilde{e}^{n+1}\|_{L^6}\|\nabla b^{n+1}\|_{L^3}\|b^{n+1} - b^n\|_0
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\nu\tau}{12}\|\nabla\tilde{e}^{n+1}\|_0^2 + C\tau^3\|b_t^{n+1}\|_0^2\|b^{n+1}\|_2^2,\\
2\mu\tau|(\tilde{e}^{n+1}\cdot\nabla b^n, b^{n+1}-b^n)| &\leq C\tau\|\tilde{e}^{n+1}\|_{L^6}\|\nabla b^{n+1}\|_{L^3}\|b^{n+1}-b^n\|_0\\
&\leq \frac{\nu\tau}{12}\|\nabla\tilde{e}^{n+1}\|_0^2 + C\tau^3\|b_t^{n+1}\|_0^2\|b^{n+1}\|_2^2,\\
2\mu\tau|(\tilde{e}^{n+1}\cdot\nabla\varepsilon^n, b^n)| &\leq C\tau\|\nabla\tilde{e}^{n+1}\|_0\|b^n\|_2\|\varepsilon^n\|_0\\
&\leq \frac{\nu\tau}{12}\|\nabla\tilde{e}^{n+1}\|_0^2 + C\tau\|\varepsilon^n\|_0^2\|b^n\|_2^2,\\
2\mu\tau|(\tilde{e}^{n+1}\cdot\nabla B^n, \varepsilon^n)| &\leq C\tau\|\nabla\tilde{e}^{n+1}\|_0\|B^n\|_{L^\infty}\|\varepsilon^n\|_0\\
&\leq \frac{\nu\tau}{12}\|\nabla\tilde{e}^{n+1}\|_0^2 + C\tau\|\varepsilon^n\|_0^2\|B^n\|_{L^\infty}^2,\\
2\mu\tau|(\varepsilon^n\cdot\nabla\tilde{e}^{n+1}, b^{n+1})| &\leq C\tau\|\varepsilon^n\|_0\|\nabla\tilde{e}^{n+1}\|_0\|b^{n+1}\|_{L^\infty}\\
&\leq \frac{\nu\tau}{12}\|\nabla\tilde{e}^{n+1}\|_0^2 + C\tau\|\varepsilon^n\|_0^2\|b^{n+1}\|_2^2,\\
2\mu\tau|(B^n\cdot\nabla\tilde{e}^{n+1}, \varepsilon^{n+1})| &\leq C\tau\|B^n\|_{L^\infty}\|\nabla\tilde{e}^{n+1}\|_0\|\varepsilon^{n+1}\|_0\\
&\leq \frac{\nu\tau}{12}\|\nabla\tilde{e}^{n+1}\|_0^2 + C\tau\|\varepsilon^n\|_0^2,\\
2\mu\tau|(\varepsilon^n-\hat{\varepsilon}^n, \varepsilon^{n+1})| &\leq C\tau\|\varepsilon^n\|_0\|B^n\|_{W^{1,\infty}}\|\nabla\varepsilon^{n+1}\|_0\\
&\leq \frac{\mu\tau}{8}\|\mathrm{curl}\varepsilon^{n+1}\|_0^2 + C\tau\|\varepsilon^n\|_0^2,\\
2\mu\tau|(\hat{b}^n-\bar{b}^n, \varepsilon^{n+1})| &\leq C\tau\|\hat{b}^n-\bar{b}^n\|_{L^{6/5}}\|\varepsilon^{n+1}\|_{L^6}\\
&\leq C\tau\|\nabla b^n\|_{L^3}\|\varepsilon^n\|_0\|\nabla\varepsilon^{n+1}\|_0\\
&\leq \frac{\mu\tau}{8}\|\mathrm{curl}\varepsilon^{n+1}\|_0^2 + C\tau\|\varepsilon^n\|_0^2,\\
2\mu\tau|(R_{tr2}^{n+1}, \varepsilon^{n+1})| &\leq C\tau\|R_{tr2}^{n+1}\|_0\|\varepsilon^{n+1}\|_0\\
&\leq \frac{\mu\tau}{8}\|\mathrm{curl}\varepsilon^{n+1}\|_0^2 + C\tau\|R_{tr2}^{n+1}\|_0^2,\\
2\mu\tau|(b^{n+1}\cdot\nabla(u^{n+1}-u^n), \varepsilon^{n+1})| &\leq C\tau\|b^{n+1}\|_2\|u^{n+1}-u^n\|_0\|\mathrm{curl}\varepsilon^{n+1}\|_0\\
&\leq \frac{\mu\tau}{8}\|\mathrm{curl}\varepsilon^{n+1}\|_0^2 + C\tau\|u^{n+1}-u^n\|_0^2,\\
2\mu\tau|((b^{n+1}-b^n)\cdot\nabla u^n, \varepsilon^{n+1})| &\leq C\tau\|b^{n+1}-b^n\|_0\|\nabla u^n\|_{L^\infty}\|\varepsilon^{n+1}\|_0\\
&\leq \frac{\mu\tau}{8}\|\mathrm{curl}\varepsilon^{n+1}\|_0^2 + C\tau^3\|b_t^{n+1}\|_0^2\|\nabla u^n\|_{L^\infty}^2,\\
2\mu\tau|(\varepsilon^n\cdot\nabla u^n, \varepsilon^{n+1})| &\leq C\tau\|\varepsilon^n\|_0\|\nabla u^n\|_{L^3}\|\mathrm{curl}\varepsilon^{n+1}\|_0\\
&\leq \frac{\mu\tau}{8}\|\mathrm{curl}\varepsilon^{n+1}\|_0^2 + C\tau\|\varepsilon^n\|_0^2\|\nabla u^n\|_{L^3}^2,\\
2\mu\tau|(B^n\cdot\nabla e^n, \varepsilon^{n+1})| &\leq C\tau\|B^n\|_{L^\infty}\|e^n\|_0\|\mathrm{curl}\varepsilon^{n+1}\|_0\\
&\leq \frac{\mu\tau}{8}\|\mathrm{curl}\varepsilon^{n+1}\|_0^2 + C\tau\|e^n\|_0^2.
\end{aligned}
$$

Substituting these above inequality into (3.24), we obtain

$$
\begin{aligned}
&\|e^{n+1}\|_0^2 - \|e^n\|_0^2 + \mu\|\varepsilon^{n+1}\|_0^2 - \mu\|\varepsilon^n\|_0^2 + \tau^2\|\nabla\xi^{n+1}\|_0^2 - \tau^2\|\nabla\xi^n\|_0^2 + \|e^{n+1}-e^n\|_0^2\\
&+\mu\|\varepsilon^{n+1}-\varepsilon^n\|_0^2 + \tau^2\|\nabla(\xi^{n+1}-\xi^n)\|_0^2 + \nu\tau\|\nabla\tilde{e}^{n+1}\|_0^2 + \mu\tau\|\mathrm{curl}\varepsilon^{n+1}\|_0^2
\end{aligned}
$$

$$\leq C\tau(\|e^n\|_0^2 + \|\varepsilon^n\|_0^2 + \|\varepsilon^{n+1}\|_0^2) + C\tau(\|R_{tr1}^{n+1}\|_0^2 + \|R_{tr2}^{n+1}\|_0^2) + C\tau^3(\|\nabla\xi^{n+1}\|_0^2 + \|\nabla\xi^n\|_0^2) + C\tau^3. \tag{3.25}$$

Taking sum of the (3.25) for n from 0 to $m \leq N$ and using discrete Gronwall's inequality Lemma 2.1, we get

$$\max_{0\leq n\leq m}(\|e^{m+1}\|_0^2 + \mu\|\varepsilon^{m+1}\|_0^2 + \tau^2\|\nabla\xi^{m+1}\|_0^2) + \sum_{n=0}^{m}(\|e^{n+1} - e^n\|_0^2 + \mu\|\varepsilon^{n+1} - \varepsilon^n\|_0^2) + \tau\sum_{n=0}^{m}\left(\nu\|\nabla\tilde{e}^{n+1}\|_0^2 + \mu\|\mathrm{curl}\varepsilon^{n+1}\|_0^2\right) \leq C\tau^2. \tag{3.26}$$

Owing to (3.21), we have

$$(\nabla(e^{n+1} - e^n), \nabla\tilde{e}^{n+1}) = (\nabla(e^{n+1} - e^n), \nabla e^{n+1}). \tag{3.27}$$

To acquire the H^1 estimates, we take $v = 2\tau D_\tau e^{n+1}$ in (3.17) to get

$$\begin{aligned}
&\nu(\|e^{n+1}\|_1^2 - \|e^n\|_1^2 + \|e^{n+1} - e^n\|_1^2) + 2\tau\|D_\tau e^{n+1}\|_0^2 \\
=& -2((e^n - (\bar{u}^n - \hat{U}^n), D_\tau e^{n+1}) + 2\tau(R_{tr1}^{n+1}, D_\tau e^{n+1}) \\
&- 2\mu\tau((b^{n+1} - b^n)\cdot\nabla D_\tau e^{n+1} - D_\tau e^{n+1}\cdot\nabla(b^{n+1} - b^n), b^{n+1}) \\
&+ 2\mu\tau(D_\tau e^{n+1}\cdot\nabla b^n, b^{n+1} - b^n) + 2\mu\tau(D_\tau e^{n+1}\cdot\nabla\varepsilon^n, b^n) \\
&+ 2\mu\tau(D_\tau e^{n+1}\cdot\nabla B^n, \varepsilon^n) - 2\mu\tau(\varepsilon^n\cdot\nabla D_\tau e^{n+1}, b^{n+1}) \\
&- 2\mu\tau(B^n\cdot\nabla D_\tau e^{n+1}, \varepsilon^{n+1}).
\end{aligned} \tag{3.28}$$

Then, we take $\psi = 2\mu\tau D_\tau\varepsilon^{n+1}$ in (3.18) to get

$$\begin{aligned}
&\mu(\|\mathrm{curl}\varepsilon^{n+1}\|_0^2 - \|\mathrm{curl}\varepsilon^n\|_0^2 + \|\mathrm{curl}\varepsilon^{n+1} - \mathrm{curl}\varepsilon^n\|_0^2) + 2\mu\tau\|D_\tau\varepsilon^{n+1}\|_0^2 \\
=& -2\mu(\varepsilon^n - (\bar{b}^n - \hat{B}^n), D_\tau\varepsilon^{n+1}) + 2\mu\tau(R_{tr2}^{n+1}, D_\tau\varepsilon^{n+1}) \\
&+ 2\mu\tau(b^{n+1}\cdot\nabla(u^{n+1} - u^n) + (b^{n+1} - b^n)\cdot\nabla u^n + \varepsilon^n\cdot\nabla u^n + B^n\cdot\nabla e^n, D_\tau\varepsilon^{n+1}).
\end{aligned} \tag{3.29}$$

Combining (3.28) and (3.29), we obtain

$$\begin{aligned}
&\nu(\|e^{n+1}\|_1^2 - \|e^n\|_1^2 + \|e^{n+1} - e^n\|_1^2) + \mu(\|\mathrm{curl}\varepsilon^{n+1}\|_0^2 - \|\mathrm{curl}\varepsilon^n\|_0^2 + \|\mathrm{curl}\varepsilon^{n+1} - \mathrm{curl}\varepsilon^n\|_0^2) \\
&+ 2\tau\|D_\tau e^{n+1}\|_0^2 + 2\nu\tau\|D_\tau\varepsilon^{n+1}\|_0^2 \\
=& -2(e^n - \hat{e}^n, D_\tau e^{n+1}) - 2(\hat{u}^n - \bar{u}^n, D_\tau e^{n+1}) + 2\tau(R_{tr1}^{n+1}, D_\tau e^{n+1}) \\
&- 2\mu\tau((b^{n+1} - b^n)\cdot\nabla D_\tau e^{n+1}, b^{n+1}) + 2\mu\tau(D_\tau e^{n+1}\cdot\nabla(b^{n+1} - b^n), b^{n+1}) \\
&+ 2\mu\tau(D_\tau e^{n+1}\cdot\nabla b^n, b^{n+1} - b^n) + 2\mu\tau(D_\tau e^{n+1}\cdot\nabla\varepsilon^n, b^n) + 2\mu\tau(D_\tau e^{n+1}\cdot\nabla B^n, \varepsilon^n) \\
&- 2\mu\tau(\varepsilon^n\cdot\nabla D_\tau e^{n+1}, b^{n+1}) - 2\mu\tau(B^n\cdot\nabla D_\tau e^{n+1}, \varepsilon^{n+1}) - 2\mu(\varepsilon^n - \hat{\varepsilon}^n, \varepsilon^{n+1}) \\
&- 2\mu(\hat{b}^n - \bar{b}^n, D_\tau\varepsilon^{n+1}) + 2\mu\tau(R_{tr2}^{n+1}, D_\tau\varepsilon^{n+1}) + 2\mu\tau(b^{n+1}\cdot\nabla(u^{n+1} - u^n), D_\tau\varepsilon^{n+1}) \\
&+ 2\mu\tau((b^{n+1} - b^n)\cdot\nabla u^n, D_\tau\varepsilon^{n+1}) + 2\mu\tau(\varepsilon^n\cdot\nabla u^n, D_\tau\varepsilon^{n+1}) + 2\mu\tau(B^n\cdot\nabla e^n, D_\tau\varepsilon^{n+1}).
\end{aligned} \tag{3.30}$$

Similarly, by Lemma 3.1, we have

$$2|(e^n - \hat{e}^n, D_\tau e^{n+1})| \leq 2\|e^n\|_{W^{1,2}}\|U^n\|_{L^\infty}\|D_\tau e^{n+1}\|_0$$

$$
\begin{aligned}
&\leq \frac{\tau}{16}\|D_\tau e^{n+1}\|_0^2 + C\tau\|e^n\|_1^2,\\
2|(\hat{u}^n - \bar{u}^n, D_\tau e^{n+1})| &\leq C\|u^n\|_{W^{1,4}}\|e^n\|_{L^4}\|D_\tau e^{n+1}\|_0\\
&\leq C\|u^n\|_{W^{1,4}}\|e^n\|_1\|D_\tau e^{n+1}\|_0\\
&\leq \frac{\tau}{16}\|D_\tau e^{n+1}\|_0^2 + C\tau\|e^n\|_1^2,\\
2\tau|(R_{tr1}^{n+1}, D_\tau e^{n+1})| &\leq \|R_{tr1}^{n+1}\|_0\|D_\tau e^{n+1}\|_0\\
&\leq \frac{\tau}{16}\|D_\tau e^{n+1}\|_0^2 + C\tau\|R_{tr1}^{n+1}\|_0^2,\\
2\mu\tau|((b^{n+1} - b^n)\cdot\nabla D_\tau e^{n+1}, b^{n+1})| &\leq C\tau\|\nabla(b^{n+1} - b^n)\|_0\|D_\tau e^{n+1}\|_0\|b^{n+1}\|_2\\
&\leq \frac{\tau}{16}\|D_\tau e^{n+1}\|_0^2 + C\tau^3\|\nabla b_t^{n+1}\|_0^2\|b^{n+1}\|_2^2,\\
2\mu\tau|(D_\tau e^{n+1}\cdot\nabla(b^{n+1} - b^n), b^{n+1})| &\leq C\tau\|D_\tau e^{n+1}\|_0\|\nabla(b^{n+1} - b^n)\|_0\|b^{n+1}\|_2\\
&\leq \frac{\tau}{16}\|D_\tau e^{n+1}\|_0^2 + C\tau^3\|b_t^{n+1}\|_0^2\|b^{n+1}\|_2^2,\\
2\mu\tau|(D_\tau e^{n+1}\cdot\nabla b^n, b^{n+1} - b^n)| &\leq C\tau\|D_\tau e^{n+1}\|_0\|\nabla b^n\|_{L^\infty}\|b^{n+1} - b^n\|_0\\
&\leq \frac{\tau}{16}\|D_\tau e^{n+1}\|_0^2 + C\tau^3\|\nabla b_t^{n+1}\|_0^2\|\nabla b^n\|_{L^\infty}^2,\\
2\mu\tau|(D_\tau e^{n+1}\cdot\nabla\varepsilon^n, b^n)| &\leq C\tau\|D_\tau e^{n+1}\|_0\|\nabla\varepsilon^n\|_0\|b^n\|_{L^\infty}\\
&\leq \frac{\tau}{16}\|D_\tau e^{n+1}\|_0^2 + C\tau\|\mathrm{curl}\varepsilon^n\|_0^2\|b^n\|_{L^\infty}^2,\\
2\mu\tau|(D_\tau e^{n+1}\cdot\nabla B^n, \varepsilon^n)| &\leq C\tau\|D_\tau e^{n+1}\|_0\|\nabla B^n\|_{L^3}\|\varepsilon^n\|_{L^6}\\
&\leq \frac{\tau}{16}\|D_\tau e^{n+1}\|_0^2 + C\tau\|\nabla\varepsilon^n\|_0^2\|\nabla B^n\|_{L^3}^2,\\
&\leq \frac{\tau}{16}\|D_\tau e^{n+1}\|_0^2 + C\tau\|\mathrm{curl}\varepsilon^n\|_0^2\|\nabla B^n\|_{L^3}^2,\\
2\mu\tau|(\varepsilon^n\cdot\nabla D_\tau e^{n+1}, b^{n+1})| &\leq C\tau\|\nabla\varepsilon^n\|_0\|D_\tau e^{n+1}\|_0\|b^{n+1}\|_2\\
&\leq \frac{\tau}{16}\|D_\tau e^{n+1}\|_0^2 + C\tau\|\mathrm{curl}\varepsilon^n\|_0^2\|b^{n+1}\|_2^2,\\
2\mu\tau|(B^n\cdot\nabla D_\tau e^{n+1}, \varepsilon^{n+1})| &\leq C\tau\|B^n\|_{L^\infty}\|\nabla\varepsilon^{n+1}\|_0\|D_\tau e^{n+1}\|_0\\
&\leq \frac{\tau}{16}\|D_\tau e^{n+1}\|_0^2 + C\tau\|\mathrm{curl}\varepsilon^{n+1}\|_0^2\|B^n\|_{L^\infty}^2,\\
2\mu\tau|(\varepsilon^n - \hat{\varepsilon}^n, D_\tau\varepsilon^{n+1})| &\leq C\tau\|\varepsilon^n\|_{W^{1,2}}\|B^n\|_{L^\infty}\|D_\tau\varepsilon^{n+1}\|_0\\
&\leq \frac{\mu\tau}{8}\|D_\tau\varepsilon^{n+1}\|_0^2 + C\tau\|\mathrm{curl}\varepsilon^n\|_0^2,\\
2\mu\tau|(\hat{b}^n - \bar{b}^n, D_\tau\varepsilon^{n+1})| &\leq C\tau\|\nabla b^n\|_{L^4}\|\varepsilon^n\|_{L^4}\|D_\tau\varepsilon^{n+1}\|_0\\
&\leq \frac{\mu\tau}{8}\|D_\tau\varepsilon^{n+1}\|_0^2 + C\tau\|\mathrm{curl}\varepsilon^n\|_0^2,\\
2\mu\tau|(R_{tr2}^{n+1}, D_\tau\varepsilon^{n+1})| &\leq C\tau\|R_{tr2}^{n+1}\|_0\|D_\tau\varepsilon^{n+1}\|_0\\
&\leq \frac{\mu\tau}{8}\|D_\tau\varepsilon^{n+1}\|_0^2 + C\tau\|R_{tr2}^{n+1}\|_0^2,\\
2\mu\tau|(b^{n+1}\cdot\nabla(u^{n+1} - u^n), D_\tau\varepsilon^{n+1})| &\leq C\tau\|b^{n+1}\|_2\|\nabla(u^{n+1} - u^n)\|_0\|D_\tau\varepsilon^{n+1}\|_0\\
&\leq \frac{\mu\tau}{8}\|D_\tau\varepsilon^{n+1}\|_0^2 + C\tau\|\nabla(u^{n+1} - u^n)\|_0^2,
\end{aligned}
$$

$$\begin{aligned}
2\mu\tau|((b^{n+1}-b^n)\cdot\nabla u^n, D_\tau\varepsilon^{n+1})| &\le C\tau\|b^{n+1}-b^n\|_0\|\nabla u^n\|_{L^\infty}\|D_\tau\varepsilon^{n+1}\|_0\\
&\le \frac{\mu\tau}{8}\|D_\tau\varepsilon^{n+1}\|_0^2 + C\tau^3\|b_t^{n+1}\|_0^2\|\nabla u^n\|_{L^\infty}^2,\\
2\mu\tau|(\varepsilon^n\cdot\nabla u^n, D_\tau\varepsilon^{n+1})| &\le C\tau\|\varepsilon^n\|_0\|\nabla u^n\|_{L^\infty}\|D_\tau\varepsilon^{n+1}\|_0\\
&\le \frac{\mu\tau}{8}\|D_\tau\varepsilon^{n+1}\|_0^2 + C\tau\|\text{curl}\varepsilon^n\|_0^2\|\nabla u^n\|_{L^\infty}^2,\\
2\mu\tau|(B^n\cdot\nabla e^n, D_\tau\varepsilon^{n+1})| &\le C\tau\|B^n\|_{L^\infty}\|\nabla e^n\|_0\|D_\tau\varepsilon^{n+1}\|_0\\
&\le \frac{\mu\tau}{8}\|D_\tau\varepsilon^{n+1}\|_0^2 + C\tau\|\nabla e^n\|_0^2\|B^n\|_{L^\infty}^2.
\end{aligned}$$

Substituting these above inequality into (3.30), we obtain

$$\begin{aligned}
&\nu(\|e^{n+1}\|_1^2 - \|e^n\|_1^2 + \|e^{n+1}-e^n\|_1^2) + \mu(\|\text{curl}\varepsilon^{n+1}\|_0^2 - \|\text{curl}\varepsilon^n\|_0^2 + \|\text{curl}\varepsilon^{n+1} - \text{curl}\varepsilon^n\|_0^2)\\
&+\tau\|D_\tau e^{n+1}\|_0^2 + \mu\tau\|D_\tau\varepsilon^{n+1}\|_0^2\\
\le& C\tau(\|e^n\|_1^2 + \|\text{curl}\varepsilon^{n+1}\|_0^2 + \|\text{curl}\varepsilon^n\|_0^2) + C\tau(\|R_{tr1}^{n+1}\|_0^2 + \|R_{tr2}^{n+1}\|_0^2) + C\tau^3. \qquad (3.31)
\end{aligned}$$

Taking sum of the (3.31) for n from 0 to $m \le N$ and using discrete Gronwall's inequality Lemma 2.1, we arrive at

$$\begin{aligned}
&\max_{0\le n\le m}\left(\nu\|e^{m+1}\|_1^2 + \mu\|\text{curl}\varepsilon^{m+1}\|_0^2\right) + \tau\sum_{n=1}^{m}\left(\nu\|e^{n+1}-e^n\|_1^2 + \mu\|\text{curl}\varepsilon^{n+1} - \text{curl}\varepsilon^n\|_0^2\right)\\
&+\tau\sum_{n=1}^{m}(\|D_\tau e^{n+1}\|_0^2 + \mu\|D_\tau\varepsilon^{n+1}\|_0^2) \le C\tau^2. \qquad (3.32)
\end{aligned}$$

Furthermore, the standard result for (3.17) and (3.18) with $p = 2$, respectively, leads to

$$\begin{aligned}
&\|\tilde{e}^{n+1}\|_2\\
\le& C\|D_\tau e^{n+1}\|_0 + \frac{C}{\tau}\|e^n - \hat{e}^n\|_0 + \frac{C}{\tau}\|\hat{U}^n - \bar{U}^n\|_0 + C\|R_{tr1}^{n+1}\|_0 + C\|b^{n+1}-b^n\|_0\|\nabla b^{n+1}\|_{L^\infty}\\
&+ C\|b^{n+1}-b^n\|_0\|\nabla b^{n+1}\|_{L^\infty} + C\|\nabla b^n\|_{L^\infty}\|b^{n+1}-b^n\|_0 + C\|\nabla\varepsilon^n\|_0\|b^n\|_{L^\infty}\\
&+ C\|B^n\|_{W^{1,\infty}}\|\varepsilon^n\|_0 + C\|\varepsilon^n\|_0\|\nabla b^{n+1}\|_{L^\infty} + C\|B^n\|_{L^\infty}\|\text{curl}\varepsilon^{n+1}\|_0 + \|\xi^{n+1}\|_1\\
\le& C\|D_\tau e^{n+1}\|_0 + C\|R_{tr1}^{n+1}\|_0 + \|\xi^{n+1}\|_1 + C\tau, \qquad (3.33)
\end{aligned}$$

$$\begin{aligned}
&\|\varepsilon^{n+1}\|_2\\
\le& C\|D_\tau\varepsilon^{n+1}\|_0 + \frac{C}{\tau}\|\varepsilon^n - \hat{\varepsilon}^n\|_0 + \frac{C}{\tau}\|\hat{b}^n - \bar{b}^n\|_0 + C\|R_{tr2}^{n+1}\|_0 + C\|b^{n+1}\|_{L^\infty}\|u^{n+1}-u^n\|_0\\
&+ C\|b^{n+1}-b^n\|_0\|\nabla u^n\|_{L^\infty} + C\|\varepsilon^n\|_0\|\nabla u^n\|_{L^\infty} + C\|B^n\|_{L^\infty}\|\nabla e^n\|_0\\
\le& C\|D_\tau e^{n+1}\|_0 + C\|R_{tr}^{n+1}\|_0 + C\tau. \qquad (3.34)
\end{aligned}$$

Thanks to (3.19), we can obtain

$$\begin{aligned}
&(\Delta e^{n+1}, \Delta e^{n+1}) = (\Delta\tilde{e}^{n+1}, \Delta e^{n+1}),\\
&\|e^{n+1}\|_2 \le \|\tilde{e}^{n+1}\|_2.
\end{aligned}$$

Then, we have

$$\tau \sum_{n=0}^{m} (\|e^{n+1}\|_2^2 + \|\tilde{e}^{n+1}\|_2^2 + \|\varepsilon^{n+1}\|_2^2) \le C\tau^2. \tag{3.35}$$

From (3.35), we can arrive at

$$\max_{0\le n\le m} \|U^{n+1}\|_2 \le \max_{0\le n\le m} (\|u^{n+1}\|_2 + \|e^{n+1}\|_2) \le C, \tag{3.36}$$

$$\max_{0\le n\le m} \|\tilde{U}^{n+1}\|_2 \le \max_{0\le n\le m} (\|u^{n+1}\|_2 + \|\tilde{e}^{n+1}\|_2) \le C, \tag{3.37}$$

$$\max_{0\le n\le m} \|P^{n+1}\|_1 \le \max_{0\le n\le m} (\|p^{n+1}\|_1 + \|\xi^{n+1}\|_1) \le C, \tag{3.38}$$

$$\max_{0\le n\le m} \|B^{n+1}\|_2 \le \max_{0\le n\le m} (\|b^{n+1}\|_2 + \|\varepsilon^{n+1}\|_2) \le C, \tag{3.39}$$

$$\tau \sum_{n=0}^{m} \|D_\tau U^{n+1}\|_2^2 \le 2\tau \sum_{n=0}^{m} (\|D_\tau u^{n+1}\|_2^2 + \|D_\tau e^{n+1}\|_2^2) \le C, \tag{3.40}$$

$$\tau \sum_{n=0}^{m} \|D_\tau \tilde{U}^{n+1}\|_2^2 \le 2\tau \sum_{n=0}^{m} (\|D_\tau u^{n+1}\|_2^2 + \|D_\tau \tilde{e}^{n+1}\|_2^2) \le C, \tag{3.41}$$

$$\tau \sum_{n=0}^{m} \|D_\tau P^{n+1}\|_1^2 \le 2\tau \sum_{n=0}^{m} (\|D_\tau p^{n+1}\|_1^2 + \|D_\tau \xi^{n+1}\|_1^2) \le C, \tag{3.42}$$

$$\tau \sum_{n=0}^{m} \|D_\tau B^{n+1}\|_2^2 \le 2\tau \sum_{n=0}^{m} (\|D_\tau b^{n+1}\|_2^2 + \|D_\tau \varepsilon^{n+1}\|_2^2) \le C. \tag{3.43}$$

From (2.7) and (2.5), we have

$$D_\tau U^{n+1} - \nu\Delta\tilde{U}^{n+1} + \nabla P^{n+1} = f^{n+1} - \frac{U^n - \hat{U}^n}{\tau} - \mu B^n \times \mathrm{curl} B^{n+1}, \tag{3.44}$$

and

$$D_\tau B^{n+1} + \mathrm{curlcurl} B^{n+1} = g^{n+1} + (B^n \cdot \nabla)U^n - \frac{B^{n+1} - \hat{B}^n}{\tau}. \tag{3.45}$$

By (2.9), we have

$$-\Delta U^{n+1} = \nabla \times \nabla \times \tilde{U}^{n+1}. \tag{3.46}$$

Considering the identity $\nabla^2\tilde{U}^{n+1} = \nabla\nabla\cdot\tilde{U}^{n+1} - \nabla\times\nabla\times\tilde{U}^{n+1}$ and (3.46), we can obtain

$$\Delta\tilde{U}^{n+1} = \nabla\nabla\cdot\tilde{U}^{n+1} - \Delta U^{n+1}. \tag{3.47}$$

Now, the standard result for the Stokes system (3.17) and (3.18) with $p = d^* > 2$, respectively, leads to

$$\begin{aligned}&\|U^{n+1}\|_{W^{2,d^*}} + \|P^{n+1} - \nabla\cdot\tilde{U}^{n+1}\|_{W^{1,d^*}}\\ \le& C\|D_\tau U^{n+1}\|_{L^{d^*}} + \frac{C}{\tau}\|\hat{U}^n - U^n\|_{L^{d^*}} + C\|f^{n+1}\|_{L^{d^*}} + C\|B^n\times \mathrm{curl}B^{n+1}\|_{L^{d^*}}\\ \le& C(\|D_\tau U^{n+1}\|_{L^{d^*}} + \|\nabla U^n\|_{L^{d^*}}\|U^n\|_{L^\infty} + \|f^{n+1}\|_{L^{d^*}} + \|B^n\|_{L^{d^*}}\|\mathrm{curl}B^{n+1}\|_{L^{d^*}}),\end{aligned} \tag{3.48}$$

$$\begin{aligned}&\|B^{n+1}\|_{W^{2,d^*}}\\ \le& C\|D_\tau B^{n+1}\|_{L^{d^*}} + C\|g^{n+1}\|_{L^{d^*}} + \frac{C}{\tau}\|\hat{B}^n - B^n\|_{L^{d*}} + C\|B^n\cdot\nabla U^n\|_{L^{d^*}}\\ \le& C(\|D_\tau B^{n+1}\|_{L^{d^*}} + \|g^{n+1}\|_{L^{d^*}} + \|\nabla B^n\|_{L^{d^*}}\|B^n\|_{L^\infty} + \|B^n\|_{L^{d^*}}\|\nabla U^n\|_{L^{d^*}}).\end{aligned} \tag{3.49}$$

By (3.48) and (3.49), we get

$$\tau\sum_{n=0}^{m}(\|U^{n+1}\|^2_{W^{2,d^*}} + \|B^{n+1}\|^2_{W^{2,d^*}}) \le C. \tag{3.50}$$

Thus, we can obtain

$$\|e^{n+1}\|_2 + \tau^{3/4}\|U^{n+1}\|_{W^{2,d^*}} \le 1, \tag{3.51}$$

$$\|\varepsilon^{n+1}\|_2 + \tau^{3/4}\|B^{n+1}\|_{W^{2,d^*}} \le 1. \tag{3.52}$$

The proof is complete.

□

3.2. *Error estimation for the fully discrete method*

For the sake of simplicity, we denote $R_h^n = R_h(U^n, P^n), Q_h^n = Q_h(U^n, P^n), R_{0h} = R_{0h}B^n, n = 1, 2, ..., N$, and $e_h^n = u_h^n - R_h^n, \tilde{e}_h^n = \tilde{u}_h^n - R_h^n, \varepsilon_h^n = B_h^n - R_{0h}^n, n = 1, 2, ..., N$, where $R_{0h} : L^2(\Omega)^3 \to W_h$ is the L^2-orthogonal projection, where

$$\|R_{0h}w - w\|_0 + h\|\nabla(R_{0h}w - w)\|_0 \le Ch^{r+1}\|w\|_{r+1}, \tag{3.53}$$

$$\|R_{0h}w\|_{L^\infty} \le C\|w\|_2, \tag{3.54}$$

$$\|\nabla R_{0h}w\|_{L^\infty} \le C\|w\|_{W^{1,\infty}}. \tag{3.55}$$

Theorem 3.2 Suppose that assumption A2–A3 are satisfied, there exists a positive $C > 0$ such that

$$\|u_h^{m+1}\|_0^2 + \mu\|B_h^{m+1}\|_0^2 + \sum_{n=0}^{m}(\|u_h^{n+1} - u_h^n\|_0^2 + \mu\|B_h^{n+1} - B_h^n\|_0^2)$$
$$+ \tau\sum_{n=0}^{m}\left(\nu\|\nabla\tilde{u}_h^{n+1}\|_0^2 + \mu\|\text{curl}B_h^{n+1}\|_0^2\right) \leq C,$$
$$\|u_h^{m+1}\|_1^2 + \mu\|\text{curl}B_h^{m+1}\|_0^2 + \sum_{n=0}^{m}(\|u_h^{n+1} - u_h^n\|_1^2 + \mu\|\text{curl}B_h^{n+1} - \text{curl}B_h^n\|_0^2)$$
$$+ \tau\sum_{n=0}^{m}\left(\nu\|\mathcal{A}u_h^{n+1}\|_0^2 + \mu\|\text{curl}B_h^{n+1}\|_0^2\right) \leq C,$$
$$\|e_h^{m+1}\|_0^2 + \mu\|\varepsilon_h^{m+1}\|_0^2 + \sum_{n=0}^{m}(\|e_h^{n+1} - e_h^n\|_0^2 + \mu\|\varepsilon_h^{n+1} - \varepsilon_h^n\|_0^2)$$
$$+ \tau\sum_{n=0}^{m}\left(\nu\|\nabla\tilde{e}_h^{n+1}\|_0^2 + \mu\|\text{curl}\varepsilon_h^{n+1}\|_0^2\right) \leq Ch^4,$$
$$\|u_h^{n+1}\|_{L^\infty}^2 + \mu\|B_h^{n+1}\|_{L^\infty}^2 + \tau\sum_{n=0}^{m}(\|u_h^{n+1}\|_{W^{1,\infty}}^2 + \mu\|B_h^{n+1}\|_{W^{1,\infty}}^2) \leq C.$$

Proof. We prove following estimates

$$\|e_h^m\|_0^2 + \mu\|\varepsilon_h^m\|_0^2 + \tau\sum_{m=0}^{n}\left(\nu\|\nabla e_h^m\|_0^2 + \mu\|\text{curl}\varepsilon_h^m\|_0^2\right) \leq Ch^4, \tag{3.56}$$

by mathematical induction for $m = 0, 1, ..., N$.

When $m = 0, u_h^0 = u_0, B_h^0 = B_0$. It is obvious (3.56) holds at the initial time step. We assume that (3.56) holds for $0 \leq m \leq n$ for some integer $n \geq 0$. By (2.20), (2.21), (3.54), (3.55), Theorem 3.1 and inverse inequality, we infer

$$\begin{aligned}\tau\|u_h^m\|_{W^{1,\infty}} \leq &(\|e_h^m\|_{W^{1,\infty}} + \|R_h^m\|_{W^{1,\infty}}) \leq C(h^{-\frac{d}{2}}\|\nabla e_h^m\|_0 + \|U^m\|_{W^{1,\infty}} + \|P^m\|_{L^\infty}) \\ \leq &\frac{1}{4},\end{aligned} \tag{3.57}$$

$$\begin{aligned}\tau\|B_h^m\|_{W^{1,\infty}} \leq &(\|\varepsilon_h^m\|_{W^{1,\infty}} + \|R_{0h}^m\|_{W^{1,\infty}}) \leq C(h^{-\frac{d}{2}}\|\nabla\varepsilon_h^m\|_0 + \|B^m\|_{W^{1,\infty}}) \\ \leq &\frac{1}{4},\end{aligned} \tag{3.58}$$

and

$$\tau\|u_h^m\|_{L^\infty} \leq (\|e_h^m\|_{L^\infty} + \|R_h^m\|_{L^\infty}) \leq C(h^{-\frac{d}{2}}\|e_h^m\|_0 + \|U^m\|_2 + \|P^m\|_1) \leq C, \tag{3.59}$$

$$\tau\|B_h^m\|_{L^\infty} \leq (\|\varepsilon_h^m\|_{L^\infty} + \|R_{0h}^m\|_{L^\infty}) \leq C(h^{-\frac{d}{2}}\|\varepsilon_h^m\|_0 + \|B^m\|_2) \leq C. \tag{3.60}$$

Combining (2.23) and (2.24), we obtain

$$\left(\frac{u_h^{n+1} - \hat{u}_h^n}{\tau}, v_h\right) + \nu(\nabla\tilde{u}_h^{n+1}, \nabla v_h) - (p_h^{n+1}, \nabla\cdot v_h) + \mu(B_h^n\cdot\nabla v_h, B_h^{n+1})$$

$$-\mu(v_h\cdot\nabla B_h^n,B_h^n)=(f^{n+1},v_h). \tag{3.61}$$

When $m=n+1$, letting $v_h=2\tau\tilde{u}_h^{n+1}$ in (3.61), we can obtain

$$\begin{aligned}&2(u_h^{n+1}-u_h^n+u_h^n-\hat{u}_h^n,\tilde{u}_h^{n+1})+2\nu\tau(\nabla\tilde{u}_h^{n+1},\nabla\tilde{u}_h^{n+1})-2\tau(p_h^{n+1},\nabla\cdot\tilde{u}_h^{n+1})\\&+2\mu\tau(B_h^n\cdot\nabla\tilde{u}_h^{n+1},B_h^{n+1})-2\mu\tau(\tilde{u}_h^{n+1}\cdot\nabla B_h^n,B_h^n)\\=&2\tau(f^{n+1},\tilde{u}_h^{n+1}).\end{aligned} \tag{3.62}$$

Letting $v_h=\tau u_h^{n+1}$ in (2.24), since $\nabla\cdot u_h^{n+1}=0$, it follows that

$$(u_h^{n+1},\tilde{u}_h^{n+1})=(u_h^{n+1},u_h^{n+1}). \tag{3.63}$$

Taking $v_h=\tau u_h^n$ in (2.24), we get

$$(u_h^n,\tilde{u}_h^{n+1})=(u_h^n,u_h^{n+1}). \tag{3.64}$$

Subtracting (3.64) from (3.63), we can obtain

$$(u_h^{n+1}-u_h^n,\tilde{u}_h^{n+1})=(u_h^{n+1}-u_h^n,u_h^{n+1}). \tag{3.65}$$

Using $2(a-b,a)=\|a\|_0^2-\|b\|_0^2+\|a-b\|_0^2$, leads to

$$\begin{aligned}&\|u_h^{n+1}\|_0^2-\|u_h^n\|_0^2+\|u_h^{n+1}-u_h^n\|_0^2+2\nu\tau\|\nabla\tilde{u}_h^{n+1}\|_0^2-2\tau(p_h^{n+1},\nabla\cdot\tilde{u}_h^{n+1})\\=&2\tau(f^{n+1},\tilde{u}_h^{n+1})+2(\hat{u}_h^n-u_h^n,\tilde{u}_h^{n+1})-2\mu\tau(B_h^n\cdot\nabla\tilde{u}_h^{n+1},B_h^{n+1}).\end{aligned} \tag{3.66}$$

Taking $v_h=\nabla p_h^{n+1}$ in (2.24), we deduce

$$\begin{aligned}&-2\tau(p_h^{n+1},\nabla\cdot\tilde{u}^{n+1})\\=&2\tau^2\|\nabla p_h^{n+1}\|_0^2-2\tau^2\|\nabla p_h^n\|_0^2+2\tau^2\|\nabla p_h^{n+1}-\nabla p_h^n\|_0^2.\end{aligned} \tag{3.67}$$

When $m=n+1$, letting $\psi_h=2\mu\tau B_h^{n+1}$ in (2.22), we can obtain

$$\begin{aligned}&2\mu(B_h^{n+1}-B_h^n+B_h^n-\hat{B}_h^n,B_h^{n+1})+2\mu\tau(\mathrm{curl}B_h^{n+1},\mathrm{curl}B_h^{n+1})\\&-2\mu\tau(B_h^n\cdot\nabla u_h^n,B_h^{n+1})=2\mu\tau(g^{n+1},B_h^{n+1}).\end{aligned} \tag{3.68}$$

Using $2(a-b,a)=\|a\|_0^2-\|b\|_0^2+\|a-b\|_0^2$, we have

$$\begin{aligned}&\mu\|B_h^{n+1}\|_0^2-\mu\|B_h^n\|_0^2+\mu\|B_h^{n+1}-B_h^n\|_0^2+2\mu\tau\|\mathrm{curl}B_h^{n+1}\|_0^2\\=&2\mu\tau(g^{n+1},B_h^{n+1})+2\mu\tau(B_h^n\cdot\nabla u_h^n,B_h^{n+1})+2\mu(\hat{B}_h^n-B_h^n,B_h^{n+1}).\end{aligned} \tag{3.69}$$

Taking sum of (3.66) and (3.69) yields

$$\begin{aligned}&\|u_h^{n+1}\|_0^2-\|u_h^n\|_0^2+\mu\|B_h^{n+1}\|_0^2-\mu\|B_h^n\|_0^2+2\tau^2\|\nabla p_h^{n+1}\|_0^2-2\tau^2\|\nabla p_h^n\|_0^2+\|u_h^{n+1}-u_h^n\|_0^2\\&+\mu\|B_h^{n+1}-B_h^n\|_0^2+2\tau^2\|\nabla p_h^{n+1}-\nabla p_h^n\|_0^2+2\nu\tau\|\nabla\tilde{u}_h^{n+1}\|_0^2+2\mu\tau\|\mathrm{curl}B_h^{n+1}\|_0^2\\=&2\tau(f^{n+1},\tilde{u}_h^{n+1})+2\mu\tau(g^{n+1},B_h^{n+1})+2(\hat{u}_h^n-u_h^n,\tilde{u}_h^{n+1})-2\mu\tau(B^n\cdot\nabla\tilde{u}_h^{n+1},B_h^{n+1})\\&+2\mu(\hat{B}_h^n-B_h^n,B_h^{n+1})+2\mu\tau(B_h^n\cdot\nabla u_h^n,B_h^{n+1}).\end{aligned} \tag{3.70}$$

Then, we can obtain

$$\begin{aligned}
2\tau|(f^{n+1},\tilde{u}_h^{n+1})| \leq& C\tau\|f^{n+1}\|_0\|\tilde{u}_h^{n+1}\|_0\\
\leq&\frac{\nu\tau}{4}\|\nabla\tilde{u}_h^{n+1}\|_0^2+C\tau\|f^{n+1}\|_0^2,\\
2\mu\tau|(g^{n+1},B_h^{n+1})| \leq& C\tau\|g^{n+1}\|_0\|B_h^{n+1}\|_0\\
\leq&\frac{\mu\tau}{4}\|\mathrm{curl}B_h^{n+1}\|_0^2+C\tau\|g^{n+1}\|_0^2,\\
2|(\hat{u}_h^n-u_h^n,\tilde{u}_h^{n+1})| \leq& C\tau\|\hat{u}_h^n-u_h^n\|_{6/5}\|\tilde{u}_h^{n+1}\|_{L^6}\\
\leq& C\tau\|\nabla u_h^n\|_{L^3}\|u_h^n\|_0\|\nabla\tilde{u}_h^{n+1}\|_0\\
\leq&\frac{\nu\tau}{4}\|\nabla\tilde{u}_h^{n+1}\|_0^2+C\tau\|u_h^n\|_0^2,\\
2\mu\tau|(B^n\cdot\nabla\tilde{u}_h^{n+1},B_h^{n+1})| \leq& C\tau\|B^n\|_{L^\infty}\|\nabla\tilde{u}_h^{n+1}\|_0\|B_h^{n+1}\|_0\\
\leq&\frac{\nu\tau}{4}\|\nabla\tilde{u}_h^{n+1}\|_0^2+C\tau\|B_h^{n+1}\|_0^2,\\
2\mu|(\hat{B}_h^n-B_h^n,B_h^{n+1})| \leq& C\tau\|\hat{B}_h^n-B_h^n\|_{-1}\|B_h^{n+1}\|_1\\
\leq& C\tau\|B_h^n\|_0\|B_h^n\|_{W^{1,\infty}}\|\mathrm{curl}B_h^{n+1}\|_0\\
\leq&\frac{\mu\tau}{4}\|\mathrm{curl}B_h^{n+1}\|_0^2+C\tau\|B_h^n\|_0^2\|B_h^n\|_{W^{1,\infty}}^2,\\
2\mu\tau|(B^n\cdot\nabla u_h^n,B_h^{n+1})| \leq& C\tau\|B^n\|_{L^\infty}\|u_h^n\|_0\|\mathrm{curl}B_h^{n+1}\|_0\\
\leq&\frac{\mu\tau}{4}\|\mathrm{curl}B_h^{n+1}\|_0^2+C\tau\|u_h^n\|_0^2.
\end{aligned}$$

Substituting these above estimates into (3.70), we obtain

$$\begin{aligned}
&\|u_h^{n+1}\|_0^2-\|u_h^n\|_0^2+\mu\|B_h^{n+1}\|_0^2-\mu\|B_h^n\|_0^2+\tau^2\|\nabla p_h^{n+1}\|_0^2-\tau^2\|\nabla p_h^n\|_0^2+\|u_h^{n+1}-u_h^n\|_0^2\\
&+\mu\|B_h^{n+1}-B_h^n\|_0^2+2\tau^2\|\nabla p_h^{n+1}-\nabla p_h^n\|_0^2+\nu\tau\|\nabla\tilde{u}_h^{n+1}\|_0^2+\mu\tau\|\mathrm{curl}B_h^{n+1}\|_0^2\\
\leq& C\tau\|f^{n+1}\|_0^2+C\tau\|g^{n+1}\|_0^2+C\tau\|u_h^n\|_0^2+C\tau\|B_h^n\|_0^2.
\end{aligned}\tag{3.71}$$

Taking sum of the (3.71) for n from 0 to $m\leq N$ and using discrete Gronwall's inequality Lemma 2.1, we get

$$\begin{aligned}
&\|u_h^{m+1}\|_0^2+\mu\|B_h^{m+1}\|_0^2+\tau^2\|\nabla p_h^{m+1}\|_0^2+\sum_{n=0}^{m}(\|u_h^{n+1}-u_h^n\|_0^2+\mu\|B_h^{n+1}-B_h^n\|_0^2)\\
&+\tau\sum_{n=0}^{m}\left(\nu\|\nabla\tilde{u}_h^{n+1}\|_0^2+\mu\|\mathrm{curl}B_h^{n+1}\|_0^2\right)\leq C.
\end{aligned}$$

Letting $v_h=2\tau\mathcal{A}u_h^{n+1}$ in (3.61), we obtain

$$\begin{aligned}
&2(u_h^{n+1}-u_h^n+u_h^n-\hat{u}_h^n,\mathcal{A}u_h^{n+1})+2\nu\tau(\mathcal{A}\tilde{u}_h^{n+1},\mathcal{A}u_h^{n+1})\\
&+2\mu\tau(B_h^n\cdot\nabla\mathcal{A}u_h^{n+1},B_h^{n+1})-2\mu\tau(\mathcal{A}u_h^{n+1}\cdot\nabla B_h^n,B_h^n)\\
=&2\tau(f^{n+1},\mathcal{A}u_h^{n+1}).
\end{aligned}\tag{3.72}$$

From (3.63), we arrive at

$$(\mathcal{A}\tilde{u}_h^{n+1},\mathcal{A}u_h^{n+1})=(\mathcal{A}u_h^{n+1},\mathcal{A}u_h^{n+1}),$$

$$(\mathcal{A}\tilde{u}_h^{n+1}, \mathcal{A}u_h^{n+1}) = ||\mathcal{A}u_h^{n+1}||_0^2.$$

Using $2(a-b,a) = ||a||_0^2 - ||b||_0^2 + ||a-b||_0^2$, leads to

$$\begin{aligned} &||u_h^{n+1}||_1^2 - ||u_h^n||_1^2 + ||u_h^{n+1} - u_h^n||_1^2 + 2\nu\tau||\mathcal{A}u_h^{n+1}||_0^2 \\ =&2\tau(f^{n+1}, \mathcal{A}u_h^{n+1}) + 2(\hat{u}_h^n - u_h^n, \mathcal{A}u_h^{n+1}) - 2\mu\tau(B_h^n \cdot \nabla\mathcal{A}u_h^{n+1}, B_h^{n+1}). \end{aligned} \tag{3.73}$$

Letting $\psi_h = 2\mu\tau\text{curlcurl}B_h^{n+1}$ in (2.22), we can obtain

$$\begin{aligned} &\mu||\text{curl}B_h^{n+1}||_0^2 - \mu||\text{curl}B_h^n||_0^2 + \mu||\text{curl}B_h^{n+1} - \text{curl}B_h^n||_0^2 + 2\mu\tau||\text{curlcurl}B_h^{n+1}||_0^2 \\ =&2\mu\tau(g^{n+1}, \text{curlcurl}B_h^{n+1}) + 2\mu\tau(B_h^n \cdot \nabla u_h^n, \text{curlcurl}B_h^{n+1}) + 2\mu(\hat{B}_h^n - B_h^n, \text{curlcurl}B_h^{n+1}). \end{aligned} \tag{3.74}$$

Taking sum of (3.73) and (3.74) yields

$$\begin{aligned} &||u_h^{n+1}||_1^2 - ||u_h^n||_1^2 + \mu||\text{curl}B_h^{n+1}||_0^2 - \mu||\text{curl}B_h^n||_0^2 + ||u_h^{n+1} - u_h^n||_1^2 + \mu||\text{curl}B_h^{n+1} - \text{curl}B_h^n||_0^2 \\ &+ 2\nu\tau||\mathcal{A}u_h^{n+1}||_0^2 + 2\mu\tau||\text{curlcurl}B_h^{n+1}||_0^2 \\ =&2\tau(f^{n+1}, \mathcal{A}u_h^{n+1}) + 2(\hat{u}_h^n - u_h^n, \mathcal{A}u_h^{n+1}) + 2\tau(g^{n+1}, \text{curlcurl}B_h^{n+1}) - 2\mu\tau(B_h^n \cdot \nabla\mathcal{A}u_h^{n+1}, B_h^{n+1}) \\ &+ 2\mu\tau(B_h^n \cdot \nabla u_h^n, \text{curlcurl}B_h^{n+1}) + 2\mu(\hat{B}_h^n - B_h^n, \text{curlcurl}B_h^{n+1}). \end{aligned} \tag{3.75}$$

Then, we get

$$\begin{aligned} 2\tau|(f^{n+1}, \mathcal{A}u_h^{n+1})| &\le C\tau||f^{n+1}||_0||\mathcal{A}u_h^{n+1}||_0 \\ &\le \frac{\nu\tau}{6}||\mathcal{A}u_h^{n+1}||_0^2 + C\tau||f^{n+1}||_0^2, \\ 2\mu\tau|(g^{n+1}, \text{curlcurl}B_h^{n+1})| &\le C\tau||g^{n+1}||_0||\text{curlcurl}B_h^{n+1}||_0 \\ &\le \frac{\mu\tau}{4}||\text{curlcurl}B_h^{n+1}||_0^2 + C\tau||g^{n+1}||_0^2, \\ 2|(\hat{u}_h^n - u_h^n, \mathcal{A}u_h^{n+1})| &\le C\tau||u_h^n||_{W^{1,4}}||u_h^n||_{L^4}||\mathcal{A}u_h^{n+1}||_0 \\ &\le \frac{\nu\tau}{6}||\mathcal{A}u_h^{n+1}||_0^2 + C\tau||u_h^n||_1^2, \\ 2\mu\tau|(B_h^n \cdot \nabla\mathcal{A}u_h^{n+1}, B_h^{n+1})| &\le C\tau||B_h^n||_{W^{1,\infty}}||\mathcal{A}u_h^{n+1}||_0||B_h^{n+1}||_0 \\ &\le \frac{\nu\tau}{6}||\mathcal{A}u_h^{n+1}||_0^2 + C\tau||\text{curl}B_h^{n+1}||_0^2, \\ 2\mu\tau|(B_h^n \cdot \nabla u_h^n, \text{curlcurl}B_h^{n+1})| &\le C\tau||B_h^n||_{L^\infty}||\nabla u_h^n||_0||\text{curlcurl}B_h^{n+1}||_0 \\ &\le \frac{\mu\tau}{4}||\text{curlcurl}B_h^{n+1}||_0^2 + C\tau||\nabla u_h^n||_0^2, \\ 2\mu|(\hat{B}_h^n - B_h^n, \text{curlcurl}B_h^{n+1})| &\le C\tau||B_h^n||_{W^{1,4}}||B_h^n||_{L^4}||\text{curlcurl}B_h^{n+1}||_0 \\ &\le \frac{\mu\tau}{4}||\text{curlcurl}B_h^{n+1}||_0^2 + C\tau||B_h^n||_{W^{1,4}}^2||B_h^n||_{L^4}^2 \\ &\le \frac{\mu\tau}{4}||\text{curlcurl}B_h^{n+1}||_0^2 + C\tau||\text{curl}B_h^n||_0^2. \end{aligned}$$

Substituting these above estimates into (3.75), we obtain

$$||u_h^{n+1}||_1^2 - ||u_h^n||_1^2 + \mu||\text{curl}B_h^{n+1}||_0^2 - \mu||\text{curl}B_h^n||_0^2 + ||u_h^{n+1} - u_h^n||_1^2 + \mu||\text{curl}B_h^{n+1} - \text{curl}B_h^n||_0^2$$

$$
\begin{aligned}
&+\nu\tau\|\mathcal{A}u_h^{n+1}\|_0^2+\mu\tau\|\mathrm{curlcurl}B_h^{n+1}\|_0^2\\
\le& C\tau\|f^{n+1}\|_0^2+C\tau\|g^{n+1}\|_0^2+C\tau\|u_h^n\|_1^2+C\tau(\|\mathrm{curl}B_h^{n+1}\|_0^2+\|\mathrm{curl}B_h^n\|_0^2). \qquad (3.76)
\end{aligned}
$$

Taking sum of the (3.76) for n from 0 to $m\le N$ and using discrete Gronwall's inequality Lemma 2.1, we get

$$
\begin{aligned}
&\|u_h^{m+1}\|_1^2+\mu\|\mathrm{curl}B_h^{m+1}\|_0^2+\sum_{n=0}^{m}(\|u_h^{n+1}-u_h^n\|_1^2+\mu\|\mathrm{curl}B_h^{n+1}-\mathrm{curl}B_h^n\|_0^2)\\
&+\tau\sum_{n=0}^{m}\left(\nu\|\mathcal{A}u_h^{n+1}\|_0^2+\mu\|\mathrm{curlcurl}B_h^{n+1}\|_0^2\right)\le C.
\end{aligned}
$$

Then, we rewrite (3.61) and (2.22) as

$$
\begin{aligned}
&(D_\tau u_h^{n+1},v_h)+\nu(\nabla\tilde{u}_h^{n+1},\nabla v_h)-(P_h^{n+1},\nabla\cdot v_h)\\
=&(f^{n+1},v_h)-\left(\frac{u_h^n-\hat{u}_h^n}{\tau},v_h\right)-\mu(B_h^n\cdot\nabla v_h,B_h^{n+1})+\mu(v_h\cdot\nabla B_h^n,B_h^n), \qquad (3.77)
\end{aligned}
$$

and

$$
(D_\tau B_h^{n+1},\psi_h)+(\mathrm{curl}B_h^{n+1},\mathrm{curl}\psi_h)=(g^{n+1},\psi_h)+(B_h^n\cdot\nabla u_h^n,\psi_h)-\left(\frac{B_h^n-\hat{B}_h^n}{\tau},\psi_h\right). \qquad (3.78)
$$

Subtracting (3.15) and (3.16) from (3.77) and (3.78), respectively, leads to

$$
\begin{aligned}
&(D_\tau e_h^{n+1},v_h)+\nu(\nabla\tilde{e}_h^{n+1},\nabla v_h)-(p_h^{n+1}-Q_h^{n+1},\nabla\cdot v_h)\\
=&(D_\tau(U^{n+1}-R_h^{n+1}),v_h)+\left(\frac{\hat{u}_h^n-u_h^n}{\tau},v_h\right)+\left(\frac{U^n-\hat{U}^n}{\tau},v_h\right)-\mu(\varepsilon_h^n\cdot\nabla v_h,B_h^{n+1})\\
&-\mu((R_{0h}^n-B^n)\cdot\nabla v_h,B_h^{n+1})-\mu(B^n\cdot\nabla v_h,\varepsilon_h^{n+1})-\mu(B^n\cdot\nabla v_h,R_{0h}^{n+1}-B^{n+1})\\
&+\mu(v_h\cdot\nabla\varepsilon_h^n,B_h^n)+\mu(v_h\cdot\nabla B^n,\varepsilon_h^n)-\mu(v_h\cdot\nabla B^n,R_{0h}^n-B^n)\\
&+\mu(v_h\cdot\nabla(R_{0h}^n-B^n),B_h^n), \qquad (3.79)
\end{aligned}
$$

and

$$
\begin{aligned}
&(D_\tau\varepsilon_h^{n+1},\psi_h)+(\mathrm{curl}\varepsilon_h^{n+1},\mathrm{curl}\psi_h^n)\\
=&(D_\tau(B^{n+1}-R_{0h}^{n+1}),\psi_h)+\left(\frac{B_h^n-\hat{B}_h^n}{\tau},\psi_h\right)+\left(\frac{\hat{B}^n-B^n}{\tau},\psi_h\right)+(\varepsilon_h^n\cdot\nabla u_h^n,\psi_h)\\
&+((R_{0h}^n-B^n)\cdot\nabla u_h^n,\psi_h)+(B^n\cdot\nabla(R_h^n-U^n),\psi_h)+(B^n\cdot\nabla e_h^n,\psi_h). \qquad (3.80)
\end{aligned}
$$

Taking $v_h=2\tau e_h^{n+1}$ in (2.24), we can obtain

$$
\begin{aligned}
&(e_h^{n+1},\tilde{e}_h^{n+1})=(e_h^{n+1},e_h^{n+1}),\\
&(D_\tau e_h^{n+1},\tilde{e}_h^{n+1})=(D_\tau e_h^{n+1},e_h^{n+1}),\\
&(\nabla e_h^{n+1},\nabla\tilde{e}_h^{n+1})=(\nabla e_h^{n+1},\nabla e_h^{n+1})=\|\nabla e_h^{n+1}\|_0^2.
\end{aligned}
$$

Let $\theta^n = U^n - R_h^n, \eta^n = B^n - R_{0h}^n$. By taking $v_h = 2\tau e_h^{n+1}$ in (3.79), we deduce

$$\begin{aligned}
&\|e_h^{n+1}\|_0^2 - \|e_h^n\|_0^2 + \|e_h^{n+1} - e_h^n\|_0^2 + 2\nu\tau\|\nabla e_h^{n+1}\|_0^2\\
=&2\tau(D_\tau U^{n+1} - R_h(D_\tau U^{n+1}, D_\tau P^{n+1}), e_h^{n+1}) + 2(\hat{e}_h^n - e_h^n, e_h^{n+1}) + 2(\theta^n - \hat{\theta}^n, e_h^{n+1})\\
&- 2\mu\tau(\varepsilon_h^n \cdot \nabla e_h^{n+1}, B_h^{n+1}) - 2\mu\tau(\eta^n \cdot \nabla e_h^{n+1}, B_h^{n+1}) - 2\mu\tau(B^n \cdot \nabla e_h^{n+1}, \varepsilon_h^{n+1})\\
&- 2\mu\tau(B^n \cdot \nabla e_h^{n+1}, \eta^{n+1}) + 2\mu\tau(e_h^{n+1} \cdot \nabla \varepsilon_h^n, B_h^n) + 2\mu\tau(e_h^{n+1} \cdot \nabla B^n, \varepsilon_h^n)\\
&+ 2\mu\tau(e_h^{n+1} \cdot \nabla B^n, \eta^n) + 2\mu\tau(e_h^{n+1} \cdot \nabla \eta^n, B_h^n). \qquad (3.81)
\end{aligned}$$

By taking $\psi_h = 2\mu\tau\varepsilon_h^{n+1}$ in (3.80), we have

$$\begin{aligned}
&\mu\|\varepsilon_h^{n+1}\|_0^2 - \mu\|\varepsilon_h^n\|_0^2 + \mu\|\varepsilon_h^{n+1} - \varepsilon_h^n\|_0^2 + 2\mu\tau\|\mathrm{curl}\varepsilon_h^{n+1}\|_0^2\\
=&2\tau(D_\tau B^{n+1} - R_{0h} D_\tau B^{n+1}), \varepsilon_h^{n+1}) + 2\mu(\varepsilon_h^n - \hat{\varepsilon}_h^n, \varepsilon_h^{n+1}) + 2\mu(\eta^n - \hat{\eta}^n, \varepsilon_h^{n+1})\\
&+ 2\mu\tau(\varepsilon_h^n \cdot \nabla u_h^n, \varepsilon_h^{n+1}) - 2\mu\tau(\eta^n \cdot \nabla u_h^n, \varepsilon_h^{n+1}) + 2\mu\tau(B^n \cdot \nabla(R_h^n - U^n), \varepsilon_h^{n+1})\\
&+ 2\mu\tau(B^n \cdot \nabla e_h^n, \varepsilon_h^{n+1}). \qquad (3.82)
\end{aligned}$$

Combining (3.81) and (3.82), we obtain

$$\begin{aligned}
&\|e_h^{n+1}\|_0^2 - \|e_h^n\|_0^2 + \mu\|\varepsilon_h^{n+1}\|_0^2 - \mu\|\varepsilon_h^n\|_0^2 + \|e_h^{n+1} - e_h^n\|_0^2 + S\|\varepsilon_h^{n+1} - \varepsilon_h^n\|_0^2\\
&+ 2\nu\tau\|\nabla e_h^{n+1}\|_0^2 + 2\mu\tau\|\mathrm{curl}\varepsilon_h^{n+1}\|_0^2\\
=&2\tau(D_\tau U^{n+1} - R_h(D_\tau U^{n+1}, D_\tau P^{n+1}), e_h^{n+1}) + 2(\hat{e}_h^n - e_h^n, e_h^{n+1}) + 2(\theta^n - \hat{\theta}^n, e_h^{n+1})\\
&- 2\mu\tau(\varepsilon_h^n \cdot \nabla e_h^{n+1}, B_h^{n+1}) - 2\mu\tau(\eta^n \cdot \nabla e_h^{n+1}, B_h^{n+1}) - 2\mu\tau(B^n \cdot \nabla e_h^{n+1}, B_h^{n+1})\\
&- 2\mu\tau(B^n \cdot \nabla e_h^{n+1}, \eta^{n+1}) + 2\mu\tau(e_h^{n+1} \cdot \nabla \varepsilon_h^n, B_h^n) + 2\mu\tau(e_h^{n+1} \cdot \nabla B^n, \varepsilon_h^n) + 2\mu\tau(e_h^{n+1} \cdot \nabla B^n, \eta^n)\\
&+ 2\mu\tau(e_h^{n+1} \cdot \nabla \eta^n, B_h^n) + 2\tau(D_\tau B^{n+1} - R_{0h} D_\tau B^{n+1}), \varepsilon_h^{n+1}) + 2\mu(\varepsilon_h^n - \hat{\varepsilon}_h^n, \varepsilon_h^{n+1})\\
&+ 2\mu(\eta^n - \hat{\eta}^n, \varepsilon_h^{n+1}) + 2\mu\tau(\varepsilon_h^n \cdot \nabla u_h^n, \varepsilon_h^{n+1}) - 2\mu\tau(\eta^n \cdot \nabla u_h^n, \varepsilon_h^{n+1})\\
&+ 2\mu\tau(B^n \cdot \nabla(R_h^n - U^n), \varepsilon_h^{n+1}) - 2\mu\tau(B^n \cdot \nabla e_h^n, B_h^{n+1}). \qquad (3.83)
\end{aligned}$$

Due to (2.19) and Theorem 3.1, we can get $\|\theta^n\|_0 \le Ch^2(\|U^n\|_2 + \|P^n\|_1) \le Ch^2$, $\|\eta^n\|_0 \le Ch^2\|B^n\|_2$. By Lemma 3.1, there holds that

$$\begin{aligned}
2\tau|(D_\tau U^{n+1} - R_h(D_\tau U^{n+1}, D_\tau P^{n+1}), e_h^{n+1})| &\le C\tau h^2\|e_h^{n+1}\|_0(\|D_\tau U^{n+1}\|_2 + D_\tau P^{n+1}\|_1)\\
&\le \frac{\nu\tau}{20}\|\nabla e_h^{n+1}\|_0^2 + C\tau h^4(\|D_\tau U^{n+1}\|_2^2 + D_\tau P^{n+1}\|_1^2),\\
2|(\hat{e}_h^n - e_h^n, e_h^{n+1})| &\le C\|e_h^n - \hat{e}_h^n\|_{-1}\|\nabla e_h^{n+1}\|_0\\
&\le C\tau\|\nabla e_h^n\|_0\|u_h^n\|_{W^{1,\infty}}\|\nabla e_h^{n+1}\|_0\\
&\le \frac{\nu\tau}{20}\|\nabla e_h^{n+1}\|_0^2 + C\tau\|e_h^n\|_0^2,\\
2|(\theta^n - \hat{\theta}^n, e_h^{n+1})| &\le C\|\theta^n - \hat{\theta}^n\|_{-1}\|\nabla e_h^{n+1}\|_0\\
&\le C\tau\|\theta^n\|_0\|u_h^n\|_{W^{1,\infty}}\|\nabla e_h^{n+1}\|_0\\
&\le \frac{\nu\tau}{20}\|\nabla e_h^{n+1}\|_0^2 + C\tau\|\theta^n\|_0^2,\\
2\mu\tau|(\varepsilon_h^n \cdot \nabla e_h^{n+1}, B_h^{n+1})| &\le C\tau\|\varepsilon_h^n\|_0\|\nabla e_h^{n+1}\|_0\|B_h^{n+1}\|_{L^\infty}\\
&\le \frac{\nu\tau}{20}\|\nabla e_h^{n+1}\|_0^2 + C\tau\|\varepsilon_h^n\|_0^2,
\end{aligned}$$

$$
\begin{aligned}
2\mu\tau|(\eta^n \cdot \nabla e_h^{n+1}, B_h^{n+1})| &\leq C\tau\|\eta^n\|_0\|\nabla e_h^{n+1}\|_0\|B_h^{n+1}\|_{L^\infty} \\
&\leq \frac{\nu\tau}{20}\|\nabla e_h^{n+1}\|_0^2 + C\tau\|\eta^n\|_0^2, \\
2\mu\tau|(B^n \cdot \nabla e_h^{n+1}, \varepsilon_h^{n+1})| &\leq C\tau\|B^n\|_{L^\infty}\|\nabla e_h^{n+1}\|_0\|\varepsilon_h^{n+1}\|_0 \\
&\leq \frac{\nu\tau}{20}\|\nabla e_h^{n+1}\|_0^2 + C\tau\|\varepsilon_h^{n+1}\|_0^2, \\
2\mu\tau|(B^n \cdot \nabla e_h^{n+1}, \eta^{n+1})| &\leq C\tau\|B^n\|_2\|\nabla e_h^{n+1}\|_0\|\eta^{n+1}\|_0 \\
&\leq \frac{\nu\tau}{20}\|\nabla e_h^{n+1}\|_0^2 + C\tau\|\eta^{n+1}\|_0^2, \\
2\mu\tau|(e_h^{n+1} \cdot \nabla \varepsilon_h^n, B_h^n)| &\leq C\tau\|\nabla e_h^{n+1}\|_0\|\varepsilon_h^n\|_0\|B_h^n\|_{L^\infty} \\
&\leq \frac{\nu\tau}{20}\|\nabla e_h^{n+1}\|_0^2 + C\tau\|\varepsilon_h^n\|_0^2, \\
2\mu\tau|(e_h^{n+1} \cdot \nabla B^n, \varepsilon_h^{n+1})| &\leq C\tau\|\nabla e_h^{n+1}\|_0\|B^n\|_2\|\varepsilon_h^n\|_0 \\
&\leq \frac{\nu\tau}{20}\|\nabla e_h^{n+1}\|_0^2 + C\tau\|\varepsilon_h^n\|_0^2, \\
2\mu\tau|(e_h^{n+1} \cdot \nabla B^n, \eta^n)| &\leq C\tau\|\nabla e_h^{n+1}\|_0\|B^n\|_2\|\eta^n\|_0 \\
&\leq \frac{\nu\tau}{20}\|\nabla e_h^{n+1}\|_0^2 + C\tau\|\eta^n\|_0^2, \\
2\mu\tau|(e_h^{n+1} \cdot \nabla \eta^n, B_h^n)| &\leq C\tau\|\nabla e_h^{n+1}\|_0\|\eta^n\|_0\|B_h^n\|_{L^\infty} \\
&\leq \frac{\nu\tau}{20}\|\nabla e_h^{n+1}\|_0^2 + C\tau\|\eta^n\|_0^2, \\
2\tau|(D_\tau B^{n+1} - R_{0h}D_\tau B^{n+1}, \varepsilon_h^{n+1})| &\leq C\tau h^2\|\varepsilon_h^{n+1}\|_0\|D_\tau B^{n+1}\|_2\|\eta^n\|_0 \\
&\leq \frac{\mu\tau}{16}\|\mathrm{curl}\varepsilon_h^{n+1}\|_0^2 + C\tau h^4\|D_\tau B^{n+1}\|_2^2, \\
2\mu|(\varepsilon_h^n - \hat{\varepsilon}_h^n, \varepsilon_h^{n+1})| &\leq C\|\varepsilon_h^n - \hat{\varepsilon}_h^n\|_0\|\varepsilon_h^{n+1}\|_0 \\
&\leq C\tau\|\mathrm{curl}\varepsilon_h^n\|_0\|B_h^n\|_{L^\infty}\|\varepsilon_h^{n+1}\|_0 \\
&\leq \frac{\mu\tau}{16}\|\mathrm{curl}\varepsilon_h^{n+1}\|_0^2 + C\tau\|\varepsilon_h^{n+1}\|_0^2, \\
2\mu|(\eta^n - \hat{\eta}^n, \varepsilon_h^{n+1})| &\leq C\|\eta^n - \hat{\eta}^n\|_{-1}\|\nabla\varepsilon_h^{n+1}\|_0 \\
&\leq C\tau\|\eta^n\|_0\|B_h^n\|_{W^{1,\infty}}\|\mathrm{curl}\varepsilon_h^{n+1}\|_0 \\
&\leq \frac{\mu\tau}{16}\|\mathrm{curl}\varepsilon_h^{n+1}\|_0^2 + C\tau\|\eta^n\|_0^2, \\
2\mu\tau|(\varepsilon_h^n \cdot \nabla u_h^n, \varepsilon_h^{n+1})| &\leq C\tau\|\varepsilon_h^n\|_0\|\nabla u_h^n\|_{L^3}\|\varepsilon_h^{n+1}\|_{L^6} \\
&\leq \frac{\mu\tau}{16}\|\mathrm{curl}\varepsilon_h^{n+1}\|_0^2 + C\tau\|\varepsilon_h^{n+1}\|_0^2, \\
2\mu\tau|(\eta^n \cdot \nabla u_h^n, \varepsilon_h^{n+1})| &\leq C\tau\|\eta^n\|_0\|\nabla u_h^n\|_{L^3}\|\varepsilon_h^{n+1}\|_{L^6} \\
&\leq \frac{\mu\tau}{16}\|\mathrm{curl}\varepsilon_h^{n+1}\|_0^2 + C\tau\|\eta^n\|_0^2, \\
2\mu\tau|(B^n \cdot \nabla(R_h^n - U^n), \varepsilon_h^{n+1})| &\leq C\tau\|B^n\|_2\|\theta^n\|_0\|\nabla\varepsilon_h^{n+1}\|_0 \\
&\leq \frac{\mu\tau}{16}\|\mathrm{curl}\varepsilon_h^{n+1}\|_0^2 + C\tau\|\theta^n\|_0^2, \\
2\mu\tau|(B^n \cdot \nabla e_h^n, \varepsilon_h^{n+1})| &\leq C\tau\|B^n\|_{L^\infty}\|e_h^n\|_0\|\mathrm{curl}\varepsilon_h^{n+1}\|_0
\end{aligned}
$$

$$\leq \frac{\mu\tau}{16}\|\text{curl}\varepsilon_h^{n+1}\|_0^2 + C\tau\|e_h^n\|_0^2.$$

Substituting these above estimates into (3.83), we obtain

$$\begin{aligned}
&\|e_h^{n+1}\|_0^2 - \|e_h^n\|_0^2 + \mu\|\varepsilon_h^{n+1}\|_0^2 - \mu\|\varepsilon_h^n\|_0^2 + \|e_h^{n+1} - e_h^n\|_0^2 + \mu\|\varepsilon_h^{n+1} - \varepsilon_h^n\|_0^2 \\
&+ \nu\tau\|\nabla e_h^{n+1}\|_0^2 + \mu\tau\|\text{curl}\varepsilon_h^{n+1}\|_0^2 \\
\leq& C\tau(\|e_h^n\|_0^2 + \|\varepsilon_h^n\|_0^2 + \|\varepsilon_h^{n+1}\|_0^2) + C\tau h^4. \qquad (3.84)
\end{aligned}$$

Taking the sum of the (3.76) for n from 0 to $m \leq N$ and using the discrete Gronwall's inequality Lemma 2.1, we get

$$\begin{aligned}
&\|e_h^{m+1}\|_0^2 + \mu\|\varepsilon_h^{m+1}\|_0^2 + \sum_{n=0}^{m}(\|e_h^{n+1} - e_h^n\|_0^2 + \mu\|\varepsilon_h^{n+1} - \varepsilon_h^n\|_0^2) \\
&+ \tau\sum_{n=0}^{m}\left(\nu\|\nabla e_h^{n+1}\|_0^2 + \mu\|\text{curl}\varepsilon_h^{n+1}\|_0^2\right) \leq Ch^4.
\end{aligned}$$

Then, we can deduce

$$\begin{aligned}
\max_{0\leq n\leq m}\|u_h^{n+1}\|_{L^\infty} &\leq \max_{0\leq n\leq m}(\|R_h^{n+1}\|_{L^\infty} + \|e_h^{n+1}\|_{L^\infty}) \\
&\leq C\max_{0\leq n\leq m}\|U^{n+1}\|_2 + C\max_{0\leq n\leq m}\|P^{n+1}\|_1 + Ch^{-\frac{1}{2}}\max_{0\leq n\leq m}\|e_h^{n+1}\|_0 \leq C, \\
\max_{0\leq n\leq m}\|B_h^{n+1}\|_{L^\infty} &\leq \max_{0\leq n\leq m}(\|R_{0h}^{n+1}\|_{L^\infty} + \|\varepsilon_h^{n+1}\|_{L^\infty}) \\
&\leq C\max_{0\leq n\leq m}\|B^{n+1}\|_2 + Ch^{-\frac{1}{2}}\max_{0\leq n\leq m}\|\varepsilon_h^{n+1}\|_0 \leq C, \\
\tau\sum_{n=0}^{m}\|u_h^{n+1}\|_{W^{1,\infty}}^2 &\leq 2\tau\sum_{n=0}^{m}(\|R_h^{n+1}\|_{W^{1,\infty}}^2 + \|e_h^{n+1}\|_{W^{1,\infty}}^2) \\
&\leq C\tau\sum_{n=0}^{m}(\|U^{n+1}\|_{W^{1,\infty}}^2 + \|P^{n+1}\|_{L^\infty}^2) + C\tau h^{-2}\sum_{n=0}^{m}\|\nabla e_h^{n+1}\|_0^2 \leq C, \\
\tau\sum_{n=0}^{m}\|B_h^{n+1}\|_{W^{1,\infty}}^2 &\leq 2\tau\sum_{n=0}^{m}(\|R_{0h}^{n+1}\|_{W^{1,\infty}}^2 + \|\varepsilon_h^{n+1}\|_{W^{1,\infty}}^2) \\
&\leq C\tau\sum_{n=0}^{m}\|B^{n+1}\|_{W^{1,\infty}}^2 + C\tau h^{-2}\sum_{n=0}^{m}\|\nabla\varepsilon_h^{n+1}\|_0^2 \leq C.
\end{aligned}$$

Thus, all the results in Theorem 3.2 have been proved. □

4. Numerical results

In order to show the effect of our method, we present some numerical results. Here, we consider the

$$\begin{aligned}
u_1 &= (y + y^4) * cos(t), \\
u_2 &= (x + x^2) * cos(t),
\end{aligned}$$

$$p = (2.0 * x - 1.0) * (2.0 * y - 1.0) * cos(t),$$
$$B_1 = (sin(y) + y) * cos(t),$$
$$B_2 = (sin(x) + x^2) * cos(t).$$

The boundary conditions and the forcing terms are given by the exact solution. The finite element spaces are chosen as P1b-P1-P1b finite element spaces. Here, we choose $\tau = h^2$ and $h = 1/n$, $n = 8, 16, 24, 32, 40, 48$. Different Reynolds number Re and magnetic Reynolds number Rm are chosen to show the effect of our method, the numerical results were shown in Tables 1–6. Here, we use the software package FreeFEM++ [32] for our program.

Tables 1 and 2 give the numerical results for $Re = 1.0$ and $Rm = 1.0$. In order to show the effect of our method for high Reynolds number, we give the numerical results for $Re = 100.0,\ 1000.0$ and $Rm = 100.0,\ 1000.0$. Tables 3 and 4 give the numerical results for $Re = 100.0$ and $Rm = 100.0$. Tables 5 and 6 give the numerical results for $Re = 1000.0$ and $Rm = 1000.0$. It shows that our method is effect for high Reynolds numbers. It shows that the errors are goes small as the space step goes small and the convergence orders are optimal. We can see that $|\int_\Omega \nabla \cdot Bdx|$ and $|\int_\Omega \nabla \cdot \mathbf{u}dx|$ are small, which means that our method can conserve the Gauss's law very well.

Table 1. The numerical results for $Re = 1.0$ and $Rm = 1.0$ for different h at $T = 1.0$.

$1/h$	$\|\|\mathbf{u}_h^1 - \mathbf{u}\|\|_0$	$\|\|\nabla(\mathbf{u}_h^1 - \mathbf{u}_1)\|\|_0$	$\|\|p_h^1 - p\|\|_0$	$\|\|\mathbf{B}_h^1 - \mathbf{B}\|\|_0$	$\|\|\nabla(\mathbf{B}_h^1 - \mathbf{B})\|\|_0$	$\|\|E_h^1 - E\|\|_0$	$\|\int_\Omega \nabla \cdot Bdx\|$	$\|\int_\Omega \nabla \cdot \mathbf{u}dx\|$
8	0.00395939	0.104518	0.0633311	0.00146071	0.0443105	0.0402426	4.95209e-17	1.99024e-14
16	0.00109704	0.0527177	0.0212408	0.000384435	0.0221793	0.0208612	1.41995e-16	4.15508e-15
24	0.000473611	0.0347495	0.0118957	0.000175071	0.0148256	0.0141051	5.03032e-17	5.95751e-15
32	0.000256132	0.0258999	0.010131	9.82405e-05	0.0111161	0.010656	3.35182e-16	7.38832e-15
40	0.000160786	0.0206654	0.00831346	6.2778e-05	0.00889124	0.00856254	7.97451e-16	5.55468e-15
48	0.000110796	0.0171977	0.00648101	4.35568e-05	0.00740848	0.00715659	1.00015e-16	4.15077e-15

Table 2. Convergence order for $Re = 1.0$ and $Rm = 1.0$ for different h at $T = 1.0$.

$1/h$	$\mathbf{u} - L^2$	$\mathbf{u} - H_1$	$P - L^2$	$\mathbf{B} - L^2$	$\mathbf{B} - H_1$
16	1.85166	0.987389	1.57608	1.92586	0.998435
24	2.07166	1.02792	1.42982	1.93995	0.99343
32	2.13671	1.0217	0.5582	2.00837	1.00097
40	2.08665	1.01181	0.88607	2.00685	1.00084
48	2.04246	1.00748	1.36571	2.00491	1.00066

Table 3. The numerical results for $Re = 100.0$ and $Rm = 100.0$ for different h at $T = 1.0$.

$1/h$	$\|\|\mathbf{u}_h^1 - \mathbf{u}\|\|_0$	$\|\|\nabla(\mathbf{u}_h^1 - \mathbf{u}_1)\|\|_0$	$\|\|p_h^1 - p\|\|_0$	$\|\|\mathbf{B}_h^1 - \mathbf{B}\|\|_0$	$\|\|\nabla(\mathbf{B}_h^1 - \mathbf{B})\|\|_0$	$\|\|E_h^1 - E\|\|_0$	$\|\int_\Omega \nabla \cdot B dx\|$	$\|\int_\Omega \nabla \cdot \mathbf{u} dx\|$
8	0.0172667	0.279161	0.0354359	0.0127351	0.19731	0.00779509	3.41253e-17	3.17954e-14
16	0.00233046	0.0697115	0.01409	0.00240034	0.0479792	0.00159673	9.63754e-17	1.3337e-14
24	0.000831767	0.0398145	0.00892872	0.000977778	0.0269431	0.000711847	8.03834e-18	8.42217e-15
32	0.000422379	0.0280313	0.00649219	0.000527044	0.0184095	0.000385869	2.36579e-16	5.31439e-15
40	0.000256947	0.0217553	0.00511173	0.000330314	0.013926	0.000239563	7.44033e-16	2.85026e-15
48	0.000173343	0.0178297	0.00421705	0.000226765	0.0112143	0.000162916	7.14376e-17	2.41418e-15

Table 4. Convergence order for $Re = 100.0$ and $Rm = 100.0$ for different h at $T = 1.0$.

$1/h$	$\mathbf{u} - L^2$	$\mathbf{u} - H_1$	$P - L^2$	$\mathbf{B} - L^2$	$\mathbf{B} - H_1$
16	2.88931	2.00163	1.33054	2.4075	2.03998
24	2.54095	1.38147	1.12512	2.21495	1.42315
32	2.35555	1.21978	1.10773	2.1482	1.32391
40	2.22742	1.13589	1.07134	2.09391	1.2508
48	2.1588	1.09144	1.05529	2.063	1.1878

Table 5. The numerical results for $Re = 1000.0$ and $Rm = 1000.0$ for different h at $T = 1.0$.

$1/h$	$\|\|\mathbf{u}_h^1 - \mathbf{u}\|\|_0$	$\|\|\nabla(\mathbf{u}_h^1 - \mathbf{u}_1)\|\|_0$	$\|\|p_h^1 - p\|\|_0$	$\|\|\mathbf{B}_h^1 - \mathbf{B}\|\|_0$	$\|\|\nabla(\mathbf{B}_h^1 - \mathbf{B})\|\|_0$	$\|\|E_h^1 - E\|\|_0$	$\|\int_\Omega \nabla \cdot B dx\|$	$\|\int_\Omega \nabla \cdot \mathbf{u} dx\|$
8	0.0397337	1.68458	0.0330041	0.029124	1.17999	0.0213862	1.41814e-16	1.99053e-14
16	0.00470743	0.368592	0.0137702	0.00403407	0.219747	0.00228304	7.44847e-17	1.26213e-14
24	0.00158913	0.177215	0.00901776	0.00143967	0.106852	0.000843484	8.59434e-17	8.328e-15
32	0.000750714	0.102493	0.00644056	0.000756826	0.0713168	0.000456685	2.75943e-16	5.37049e-15
40	0.000430867	0.0675245	0.00509083	0.000460115	0.0510193	0.000286991	8.76007e-16	2.73806e-15
48	0.000276193	0.0481754	0.00421193	0.000306103	0.038122	0.000195792	1.25266e-16	2.4751e-15

Table 6. Convergence order for $Re = 1000.0$ and $Rm = 1000.0$ for different h at $T = 1.0$.

$1/h$	$\mathbf{u} - L^2$	$\mathbf{u} - H_1$	$P - L^2$	$\mathbf{B} - L^2$	$\mathbf{B} - H_1$
16	3.07735	2.19229	1.26109	2.8519	2.42487
24	2.6783	1.80614	1.04402	2.54119	1.77829
32	2.60675	1.90339	1.16997	2.23523	1.40541
40	2.48819	1.8701	1.05392	2.23021	1.50095
48	2.43909	1.85191	1.03949	2.23536	1.59834

5. Conclusions

In this paper, we have given a modified characteristics projection finite element method for the unsteady incompressible magnetohydrodynamics(MHD) equations. In this method, the modified characteristics finite element method and the projection method were combined for solving the incompressible MHD equations. Both the stability and the optimal error estimates in L^2 and H^1 norms for the modified characteristics projection finite element method have been given. In order to demonstrate the effectiveness of our method, some numerical results were given at the end of the manuscript.

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Conflict of interest

The authors declare there is no conflict of interest in this paper.

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