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Research article

Asymptotic behavior of solutions of third-order neutral differential equations with discrete and distributed delay

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Abstract: By refining the standard Riccati substitution technique, integral averaging technique and comparison principle, we obtain new oscillation and asymptotic behavior for a class of third-order neutral differential equations with discrete and distributed delay. These criteria dealing with some cases have not been covered by the existing results in the literature. We present many sufficient conditions and related examples in order to illustrate the main results.

Keywords: oscillation; third order; delay; neutral; differential equation
Mathematics Subject Classification: 34C10, 34C15, 34K11

1. Introduction

It is prudential to say that mathematical modeling with delay differential equations have drawn clear significance because of their potential applications in diverse fields, which includes biological sciences, physical sciences, gas and fluid mechanics, signal processing, robotics and traffic system, engineering, population dynamics, medicine and the like (see for example [9,16,17]). It is now realized that the oscillation and asymptotic solutions of various classes of differential equation are an important field of investigation and its theory is a lot richer than the qualitative theory of differential equations (see for example [8, 10, 22]). The problem of oscillatory and nonoscillatory of solutions of various classes of second/third order differential equations with delayed and mixed arguments has been widely investigated in the literature (see for example [2, 4–7, 11, 12, 18, 23–34]). Various types of techniques appeared for investigations of such equations.

The purpose of this work, we are concerned with third-order neutral differential equations with discrete and distributed delay

$$\left(a_2(t)\left[\left(a_1(t)z'(t)\right)'\right]^{\lambda}\right)' + q_1(t)y^{\lambda}(t-\sigma_1) + q_2(t)y^{\lambda}(t+\sigma_1) = 0, \tag{E_1}$$

and

$$\left(a_2(t)\left[\left(a_1(t)z'(t)\right)'\right]^{\lambda}\right)' + \int_c^d \tilde{q}_1(t,\xi)y^{\lambda}(t-\xi)\,d\xi + \int_c^d \tilde{q}_2(t,\xi)y^{\lambda}(t+\xi)\,d\xi = 0, \tag{E_2}$$

where $z(t) = y(t) + p_1(t)y(t-\tau_1) + p_2(t)y(t+\tau_2)$, $c < d$ and $\lambda \geq 1$. Now onwards, we assume that, $a_i(t), p_i(t) \in C([t_0, +\infty))$, $a_i(t) > 0$, $p_i(t) > 0$ for $i = 1, 2$ and $0 \leq p_i(t) \leq \mu_i$, $\mu_1 + \mu_2 < 1$ where μ_i are constants, $q_i \in C([t_0, +\infty), \mathbb{R}^+)$, $\tilde{q}_i(t,\xi) \in C([t_0, +\infty) \times [c,d], \mathbb{R}^+)$ for $i = 1, 2$, and not identically zero on $[t_*, +\infty) \times [c,d]$, $t_* \geq t$, constants $\tau_i \geq 0$, for $i = 1, 2$, and the integral of (E_2) is take in the sense of Riemann–Stieltjes.

Let us recall that, a solution $y(t) \in C([T_y, \infty), \mathbb{R})$ of (E_1) (or (E_2)) is a non-trivial or $y(t) \neq 0$ with $T_y \geq t_0$, if the functions $z \in C^1([T_y, \infty), \mathbb{R})$, $a_1 z' \in C^2([T_y, \infty), \mathbb{R})$ and $a_2[(a_1 z')']^{\lambda} \in C^1([T_y, \infty), \mathbb{R})$ for certain $T_y \geq t_0$ which satisfies (E_1) (or (E_2)). Our attention is restricted to those solutions of (E_1) (or (E_2)) which exist on half-line $[T_y, \infty)$ and the condition $\sup\{|y(t)| : t > T_*\} > 0$ satisfies for any $T_* \geq t_y$. A solution of (E_1) (or (E_2)), which is nontrivial (proper) for all large t, is called oscillatory if it has no last zero, otherwise, termed nonoscillatory.

We define the operators,

$$L^{[0]}z = z, \quad L^{[1]}z = z', \quad L^{[2]}z = (a_1 L^{[1]}z)', \quad L^{[3]}z = a_2\left[L^{[2]}z\right]^{\lambda}, \quad L^{[4]}z = (L^{[3]}z)'.$$

We shall consider the two cases,

$$\pi_1[t_0, t] = \int_{t_0}^t a_2^{-1/\lambda}(s)ds, \quad \pi_2[t_0, t] = \int_{t_0}^t a_1^{-1}(s)ds.$$

$$\pi_1[t_0, t] = \infty, \quad \pi_2[t_0, t] = \infty \text{ as } t \to \infty, \tag{1.1}$$

and

$$\pi_1[t_0, t] < \infty, \quad \pi_2[t_0, t] = \infty \text{ as } t \to \infty. \tag{1.2}$$

Recently, Candan [24] investigated the oscillatory behavior of solutions of (E_1) and (E_2) by using the Riccati substitution techniques, he presented some new oscillation criteria for (E_1) and (E_2) by the assumption of condition (1.1). We notice that in [24], no criteria were found for (E_1) (or (E_2)) to be oscillatory for the assumption of condition (1.2). It would be interesting to improve and extend them in the condition (1.2).

However, the corresponding result for (E_1) (or (E_2)) under (1.2) is still missing. In this work, we fill up this gap, also we strengthen and extend the main results of Candan [24] under the condition (1.1) and (1.2) respectively. We present several oscillatory criteria for (E_1) and (E_2), by applying three Riccati substitution techniques, integral averaging techniques and comparison principles. We present two examples in order to illustrate the main results at the end.

2. Preliminary

In this section, we present some basic Lemmas for helping to prove the main results. We use throughout this paper the following notations for convenience and for shortening the equations:

$$L_\sigma^{[0]}z(t) = z(t+\sigma), \quad L_\sigma^{[1]}z(t) = z'(t+\sigma), \quad L_\sigma^{[2]}z(t) = (a_1(t+\sigma)z'(t+\sigma))',$$
$$L_\sigma^{[3]}z(t) = a_2(t+\sigma)\left[L_\sigma^{[2]}z(t)\right]^\lambda, \quad L_\sigma^{[4]}z(t) = \left(L_\sigma^{[3]}z(t)\right)', \quad A(t) = \int_{t_0}^t \frac{\pi_1[t_0,s]}{a_1(s)}ds.$$

Lemma 2.1. *Let $\lambda \geq 1$, assume $u \geq 0$. Then*

$$(u_1+u_2+u_3)^\lambda \leq 3^{\lambda-1}\left(u_1^\lambda+u_2^\lambda+u_3^\lambda\right). \tag{2.1}$$

Lemma 2.2. *Let $\lambda \leq 1$, assume $u \geq 0$. Then*

$$(u_1+u_2+u_3)^\lambda \leq \left(u_1^\lambda+u_2^\lambda+u_3^\lambda\right). \tag{2.2}$$

Lemma 2.3. *If $\lambda > 0$ and $X, Y > 0$, then*

$$Yv - Xv^{\frac{\lambda+1}{\lambda}} \leq \frac{\lambda^\lambda}{(1+\lambda)^{1+\lambda}}\frac{Y^{1+\lambda}}{X^\lambda}. \tag{2.3}$$

Lemma 2.4. *Assume that* (1.1) *holds. Furthermore, assume that y is an eventually positive solution of* (E_1) *(or* (E_2)*). Then z for $t_1 \in [t_0,\infty)$ satisfies, eventually of the following cases:*

$$(C_1): L^{[0]}z(t) > 0, \quad L^{[1]}z(t) > 0, \quad \text{and} \quad L^{[2]}z(t) > 0;$$
$$(C_2): L^{[0]}z(t) > 0, \quad L^{[1]}z(t) < 0, \quad \text{and} \quad L^{[2]}z(t) > 0;$$

and if (1.2) *holds, then also*

$$(C_3): L^{[0]}z(t) > 0, \quad L^{[1]}z(t) > 0, \quad \text{and} \quad L^{[2]}z(t) < 0.$$

Lemma 2.5. *Assume that z satisfies* (C_1) *for $t \geq t_0$. Then*

$$z'(t) \quad \geq \quad \frac{(L^{[3]}z(t))^{1/\lambda}}{a_1(t)}\pi_1[t_0,t] \tag{2.4}$$

and

$$z(t) \quad \geq \quad (L^{[3]}z(t))^{1/\lambda}A(t). \tag{2.5}$$

Proof. Since $L^{[4]}z(t) \leq 0$, $L^{[3]}z(t)$ is nondecreasing. Then we have

$$a_1(t)z'(t) \geq a_1(t)z'(t) - a_1(t_0)z'(t_0) = \int_{t_0}^t \frac{a_2^{1/\lambda}(s)L^{[2]}z(s)}{a_2^{1/\lambda}(s)}ds \geq a_2^{1/\lambda}(t)L^{[2]}z(t)\pi_1[t_0,t].$$

Again integrate, we get

$$z(t) \geq (L^{[3]}z(t))^{1/\lambda}\int_{t_0}^t \frac{\pi_1[t_0,s]}{a_1(s)}ds = (L^{[3]}z(t))^{1/\lambda}A(t).$$

□

Lemma 2.6 (See [24]). *Assume that z is a solution of (E_1) which satisfies (C_2) in Lemma 2.4. Furthermore,*

$$\int_{t_4}^{\infty} a_1^{-1}(v) \int_{v}^{\infty} a_2^{-1/\lambda}(u) \left(\int_{u}^{\infty} (q_1(s) + q_2(s))\, ds \right)^{1/\lambda} du\, dv = \infty. \tag{2.6}$$

Then, there is $\lim_{t\to\infty} z(t) = 0$.

Lemma 2.7 (See [24]). *Assume that z is a solution of (E_2) which satisfies (C_2) in Lemma 2.4. Furthermore,*

$$\int_{t_4}^{\infty} a_1^{-1}(v) \int_{v}^{\infty} a_2^{-1/\lambda}(u) \left(\int_{u}^{\infty} \int_{a}^{b} (\tilde{q}_1(s,\xi) + \tilde{q}_2(s,\xi))\, d\xi\, ds \right)^{1/\lambda} du\, dv = \infty. \tag{2.7}$$

Then, there is $\lim_{t\to\infty} z(t) = 0$.

3. Oscillation results for (E_1)

In this section, we will establish several oscillation criteria for (E_1). The following notations for convenience and for shortening the equations:

$$\begin{aligned} P_1(t) &= \min\{q_1(t),\, q_1(t-\tau_1),\, q_1(t+\tau_2)\}, \\ P_2(t) &= \min\{q_2(t),\, q_2(t-\tau_1),\, q_2(t+\tau_2)\}, \\ P(t) &= P_1(t) + P_2(t), \quad B(t) = \int_{t_0}^{t} \frac{\int_s^\infty \frac{du}{a_2^{1/\lambda}(u)}}{a_1(s)} ds. \end{aligned}$$

Let $\mathbb{S}_0 = \{(t,s) : a \le s < t < +\infty\}$, $\mathbb{S} = \{(t,s) : a \le s \le t < +\infty\}$ the continuous function $H(t,s)$, $H : \mathbb{S} \to \mathbb{R}$ belongs to the class function $\Re$

(i) $H(t,t) = 0$ for $t \ge t_0$ and $H(t,s) > 0$ for $(t,s) \in \mathbb{S}_0$,
(ii) $\frac{\partial H(t,s)}{\partial s} \le 0$, $(t,s) \in \mathbb{S}_0$ and some locally integrable function $h(t,s)$ such that

$$-\frac{\partial}{\partial s} H(t,s) - H(t,s)\frac{m'(s)}{m(s)} = \frac{h(t,s)(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)} \quad \text{for all } (t,s) \in \mathbb{S}_0.$$

Theorem 3.1. *Let* (1.1) *hold and $\sigma_1 \ge \tau_1$. If there exists an $m(t) \in C^1([t_0,\infty), \mathbb{R}^+)$ such that* (2.6) *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_3)} \int_{t_3}^{t} \left[H(t,s)m(s)\frac{P(s)}{3^{\lambda-1}} - \frac{1+\mu_1^\lambda+\mu_2^\lambda}{(\lambda+1)^{\lambda+1}} \left(\frac{|h(t,s)|a_1(s-\sigma_1)}{m(s)\pi_1[t_0, s-\sigma_1]} \right)^{\lambda} \right] ds = \infty, \tag{3.1}$$

then every solution $y(t)$ of (E_1) is either oscillatory or tends to 0.

Proof. Suppose that (E_1) has a nonoscillatory solution y. Without loss of generality, we may take $y(t) > 0$, $y(t-\tau_1) > 0$, $y(t+\tau_2) > 0$, $y(t-\sigma_1) > 0$ and $y(t+\sigma_1) > 0$ for $t \ge t_1 \ge t_0$. Since $y(t) > 0$ for all $t \ge t_1$, in view of (E_1), we have

$$L^{[4]}z(t) = -q_1(t)y^\lambda(t-\sigma_1) - q_2(t)y^\lambda(t+\sigma_1) \le 0. \tag{3.2}$$

Assumption of (1.1), by Lemma 2.4 there exists two cases (C_1) and (C_2). If (C_2) holds, then by Lemma 2.6, $\lim_{t\to\infty} z(t) = 0$. If (C_1) holds.

$$\begin{aligned}
&L^{[4]}z(t) + q_1(t)y^{\lambda}(t-\sigma_1) + q_2(t)y^{\lambda}(t+\sigma_1)\\
&+\mu_1^{\lambda}L^{[4]}_{-\tau_1}z(t) + \mu_1^{\lambda}q_1(t-\tau_1)y^{\lambda}(t-\tau_1-\sigma_1) + \mu_1^{\lambda}q_2(t-\tau_1)y^{\lambda}(t-\tau_1+\sigma_1)\\
&+\mu_2^{\lambda}L^{[4]}_{\tau_2}z(t) + \mu_2^{\lambda}q_1(t+\tau_2)y^{\lambda}(t+\tau_2-\sigma_1) + \mu_2^{\lambda}q_2(t+\tau_2)y^{\lambda}(t+\tau_2+\sigma_1) = 0. \qquad (3.3)
\end{aligned}$$

Furthermore, from Lemma 2.1, we get

$$\begin{aligned}
&q_1(t)y^{\lambda}(t-\sigma_1) + \mu_1^{\lambda}q_1(t-\tau_1)y^{\lambda}(t-\tau_1-\sigma_1)\\
&\qquad +\mu_1^{\lambda}q_1(t+\tau_2)y^{\lambda}(t+\tau_2-\sigma_1) \geq \frac{P_1(t)}{3^{\lambda-1}}\, z^{\lambda}(t-\sigma_1). \qquad (3.4)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&q_2(t)y^{\lambda}(t+\sigma_1) + \mu_2^{\lambda}q_2(t-\tau_1)y^{\lambda}(t-\tau_1+\sigma_1)\\
&\qquad +\mu_2^{\lambda}q_2(t+\tau_2)y^{\lambda}(t+\tau_2+\sigma_1) \geq \frac{P_2(t)}{3^{\lambda-1}}\, z^{\lambda}(t+\sigma_1). \qquad (3.5)
\end{aligned}$$

Substituting (3.4), (3.5) into (3.3), we have

$$L^{[4]}z(t) + \mu_1^{\lambda}\, L^{[4]}_{-\tau_1}z(t) + \mu_2^{\lambda}\, L^{[4]}_{\tau_2}z(t) + \frac{P_1(t)}{3^{\lambda-1}}\, z^{\lambda}(t-\sigma_1) + \frac{P_2(t)}{3^{\lambda-1}}\, z^{\lambda}(t+\sigma_1) \leq 0. \qquad (3.6)$$

Using the fact of $L^{[1]}z(t) > 0$, we obtain

$$L^{[4]}z(t) + \mu_1^{\lambda}\, L^{[4]}_{-\tau_1}z(t) + \mu_2^{\lambda}\, L^{[4]}_{\tau_2}z(t) + \frac{P(t)}{3^{\lambda-1}}\, z^{\lambda}(t-\sigma_1) \leq 0. \qquad (3.7)$$

Define

$$w_1(t) = m(t)\frac{L^{[3]}z(t)}{z^{\lambda}(t-\sigma_1)}. \qquad (3.8)$$

We obtain $w_1(t) > 0$, then

$$w_1'(t) = m'(t)\frac{L^{[3]}z(t)}{z^{\lambda}(t-\sigma_1)} + m(t)\frac{L^{[4]}z(t)}{z^{\lambda}(t-\sigma_1)} - \lambda m(t)\frac{L^{[3]}z(t)z'(t-\sigma_1)}{z^{\lambda+1}(t-\sigma_1)}. \qquad (3.9)$$

By Lemma (2.5), one gets $z'(t-\sigma_1) \geq \frac{a_2^{1/\lambda}(t)}{a_1(t-\sigma_1)}\pi_1[t_0, t-\sigma_1]\, L^{[2]}z(t)$. Therefore

$$w_1'(t) \leq m'(t)\frac{L^{[3]}z(t)}{z^{\lambda}(t-\sigma_1)} + m(t)\frac{L^{[4]}z(t)}{z^{\lambda}(t-\sigma_1)} - \lambda m(t)\frac{a_2^{\frac{\lambda+1}{\lambda}}(t)\pi_1[t_0, t-\sigma_1]L^{[2]}z(t)z'(t-\sigma_1)}{z^{\lambda+1}(t-\sigma_1)a_1(t-\sigma_1)}. \qquad (3.10)$$

Using (3.8) in (3.10), we obtain

$$w_1'(t) \leq \frac{(m'(t))_+}{m(t)}w_1(t) + m(t)\frac{L^{[4]}z(t)}{z^{\lambda}(t-\sigma_1)} - \lambda\, \frac{(w_1(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-\sigma_1]}{(m(t))^{1/\lambda}a_1(t-\sigma_1)}. \qquad (3.11)$$

Next, define

$$w_2(t) = m(t)\frac{L^{[3]}_{-\tau_1}z(t)}{z^{\lambda}(t-\sigma_1)}. \tag{3.12}$$

We obtain $w_2(t) > 0$, then

$$w_2'(t) = m'(t)\frac{L^{[3]}_{-\tau_1}z(t)}{z^{\lambda}(t-\sigma_1)} + m(t)\frac{L^{[4]}_{-\tau_1}z(t)}{z^{\lambda}(t-\sigma_1)} - \lambda m(t)\frac{L^{[3]}_{-\tau_1}z(t)z'(t-\sigma_1)}{z^{\lambda+1}(t-\sigma_1)}. \tag{3.13}$$

By Lemma (2.5), one gets $z'(t-\sigma_1) \geq \frac{a_2^{1/\lambda}(t-\tau_1)}{a_1(t-\sigma_1)}\pi_1[t_0, t-\sigma_1]\, L^{[2]}_{-\tau_1}z(t)$ and using (3.12) in (3.13), we have

$$w_2'(t) \leq \frac{(m'(t))_+}{m(t)}w_2(t) + m(t)\frac{L^{[4]}_{-\tau_1}z(t)}{z^{\lambda}(t-\sigma_1)} - \lambda\,\frac{(w_2(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-\sigma_1]}{(m(t))^{1/\lambda}a_1(t-\sigma_1)}. \tag{3.14}$$

Finally, define

$$w_3(t) = m(t)\frac{L^{[3]}_{\tau_2}z(t)}{z^{\lambda}(t-\sigma_1)}. \tag{3.15}$$

We obtain $w_3(t) > 0$, then

$$w_3'(t) = m'(t)\frac{L^{[3]}_{\tau_2}z(t)}{z^{\lambda}(t-\sigma_1)} + m(t)\frac{L^{[4]}_{\tau_2}z(t)}{z^{\lambda}(t-\sigma_1)} - \lambda m(t)\frac{L^{[3]}_{\tau_2}z(t)z'(t-\sigma_1)}{z^{\lambda+1}(t-\sigma_1)}. \tag{3.16}$$

By Lemma 2.5, one gets $z'(t-\sigma_1) \geq \frac{a_2^{1/\lambda}(t+\tau_2)}{a_1(t-\sigma_1)}\pi_1[t_0, t-\sigma_1]\, L^{[2]}_{\tau_2}z(t)$ and using (3.15) in (3.16), we get

$$w_3'(t) \leq \frac{(m'(t))_+}{m(t)}w_3(t) + m(t)\frac{L^{[4]}_{\tau_2}z(t)}{z^{\lambda}(t-\sigma_1)} - \lambda\,\frac{(w_3(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-\sigma_1]}{(m(t))^{1/\lambda}a_1(t-\sigma_1)}. \tag{3.17}$$

From (3.8), (3.10) and (3.15), we have

$$\begin{aligned} w_1'(t) + \mu_1^{\lambda}w_2'(t) + \mu_2^{\lambda}w_3'(t) \;&\leq\; m(t)\left[\frac{L^{[4]}z(t) + \mu_1^{\lambda}L^{[4]}_{-\tau_1}z(t) + \mu_2^{\lambda}L^{[4]}_{\tau_2}z(t)}{z^{\lambda}(t-\sigma_1)}\right] \\ &\quad + \left[\frac{(m'(t))_+}{m(t)}w_1(t) - \lambda\,\frac{(w_1(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-\sigma_1]}{(m(t))^{1/\lambda}a_1(t-\sigma_1)}\right] \\ &\quad + \mu_1^{\lambda}\left[\frac{(m'(t))_+}{m(t)}w_2(t) - \lambda\,\frac{(w_2(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-\sigma_1]}{(m(t))^{1/\lambda}a_1(t-\sigma_1)}\right] \\ &\quad + \mu_2^{\lambda}\left[\frac{(m'(t))_+}{m(t)}w_3(t) - \lambda\,\frac{(w_3(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-\sigma_1]}{(m(t))^{1/\lambda}a_1(t-\sigma_1)}\right]. \end{aligned} \tag{3.18}$$

Using (3.7) in (3.18), we have

$$w_1'(t) + \mu_1^{\lambda}w_2'(t) + \mu_2^{\lambda}w_3'(t) \;\leq\; -m(t)\frac{P(t)}{3^{\lambda-1}} + \left[\frac{(m'(t))_+}{m(t)}w_1(t) - \lambda\,\frac{(w_1(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-\sigma_1]}{(m(t))^{1/\lambda}a_1(t-\sigma_1)}\right]$$

$$
\begin{aligned}
&+\mu_1^\lambda\left[\frac{(m'(t))_+}{m(t)}w_2(t)-\lambda\,\frac{(w_2(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-\sigma_1]}{(m(t))^{1/\lambda}a_1(t-\sigma_1)}\right]\\
&+\mu_2^\lambda\left[\frac{(m'(t))_+}{m(t)}w_3(t)-\lambda\,\frac{(w_3(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-\sigma_1]}{(m(t))^{1/\lambda}a_1(t-\sigma_1)}\right],
\end{aligned}
\tag{3.19}
$$

that is,

$$
\begin{aligned}
m(t)\frac{P(t)}{3^{\lambda-1}}\le{}& -w_1'(t)-\mu_1^\lambda w_2'(t)-\mu_2^\lambda w_3'(t)+\frac{(m'(t))_+}{m(t)}w_1(t)\\
&-\lambda\,\frac{\pi_1[t_0,t-\sigma_1]}{(m(t))^{1/\lambda}a_1(t-\sigma_1)}(w_1(t))^{\frac{\lambda+1}{\lambda}}\\
&+\mu_1^\lambda\left[\frac{(m'(t))_+}{m(t)}w_2(t)-\lambda\,\frac{\pi_1[t_0,t-\sigma_1]}{(m(t))^{1/\lambda}a_1(t-\sigma_1)}(w_2(t))^{\frac{\lambda+1}{\lambda}}\right]\\
&+\mu_2^\lambda\left[\frac{(m'(t))_+}{m(t)}w_3(t)-\lambda\,\frac{\pi_1[t_0,t-\sigma_1]}{(m(t))^{1/\lambda}a_1(t-\sigma_1)}(w_3(t))^{\frac{\lambda+1}{\lambda}}\right].
\end{aligned}
\tag{3.20}
$$

Multiply $H(t,s)$ and integrate (3.20) from t_3 to t, one can get that

$$
\begin{aligned}
\int_{t_3}^t H(t,s)m(s)\frac{P(s)}{3^{\lambda-1}}ds\le{}& -\int_{t_3}^t H(t,s)w_1'(s)ds-\mu_1^\lambda\int_{t_3}^t H(t,s)w_2'(s)ds\\
&-\mu_2^\lambda\int_{t_3}^t H(t,s)w_3'(s)ds+\int_{t_3}^t H(t,s)\frac{(m'(s))_+}{m(s)}w_1(s)\,ds\\
&-\int_{t_3}^t H(t,s)\frac{\lambda\pi_1[t_0,s-\sigma_1]}{(m(s))^{1/\lambda}a_1(s-\sigma_1)}(w_1(s))^{\frac{\lambda+1}{\lambda}}ds\\
+\mu_1^\lambda\int_{t_3}^t H(t,s)\frac{(m'(s))_+}{m(s)}w_2(s)ds&-\mu_1^\lambda\int_{t_3}^t H(t,s)\frac{\lambda\pi_1[t_0,s-\sigma_1]}{(m(s))^{1/\lambda}a_1(s-\sigma_1)}(w_2(s))^{\frac{\lambda+1}{\lambda}}ds\\
+\mu_2^\lambda\int_{t_3}^t H(t,s)\frac{(m'(s))_+}{m(s)}w_3(s)ds&-\mu_2^\lambda\int_{t_3}^t H(t,s)\frac{\lambda\pi_1[t_0,s-\sigma_1]}{(m(s))^{1/\lambda}a_1(s-\sigma_1)}(w_3(s))^{\frac{\lambda+1}{\lambda}}ds.
\end{aligned}
\tag{3.21}
$$

Thus, we obtain

$$
\begin{aligned}
\int_{t_3}^t H(t,s)m(s)\frac{P(s)}{3^{\lambda-1}}ds\le{}\quad & H(t,t_3)w_1(t_3)+\mu_1^\lambda H(t,t_3)w_2(t_3)+\mu_2^\lambda H(t,t_3)w_3(t_3)\\
&-\int_{t_3}^t\left[-\tfrac{\partial}{\partial s}H(t,s)-H(t,s)\tfrac{m'(s)}{m(s)}\right]w_1(s)\,ds\\
&-\int_{t_3}^t H(t,s)\tfrac{\lambda\pi_1[t_0,s-\sigma_1]}{(m(s))^{1/\lambda}a_1(s-\sigma_1)}(w_1(s))^{\frac{\lambda+1}{\lambda}}ds\\
&-\mu_1^\lambda\int_{t_3}^t\left[-\tfrac{\partial}{\partial s}H(t,s)-H(t,s)\tfrac{m'(s)}{m(s)}\right]w_2(s)ds\\
&-\mu_1^\lambda\int_{t_3}^t H(t,s)\tfrac{\lambda\pi_1[t_0,s-\sigma_1]}{(m(s))^{1/\lambda}a_1(s-\sigma_1)}(w_2(s))^{\frac{\lambda+1}{\lambda}}ds\\
&-\mu_2^\lambda\int_{t_3}^t\left[-\tfrac{\partial}{\partial s}H(t,s)-H(t,s)\tfrac{m'(s)}{m(s)}\right]w_3(s)ds\\
&-\mu_2^\lambda\int_{t_3}^t H(t,s)\tfrac{\lambda\pi_1[t_0,s-\sigma_1]}{(m(s))^{1/\lambda}a_1(s-\sigma_1)}(w_3(s))^{\frac{\lambda+1}{\lambda}}ds.
\end{aligned}
\tag{3.22}
$$

Then

$$\int_{t_3}^{t} H(t,s)m(s)\frac{P(s)}{3^{\lambda-1}}ds \le H(t,t_3)w_1(t_3) + \mu_1^\lambda H(t,t_3)w_2(t_3) + \mu_2^\lambda H(t,t_3)w_3(t_3)$$
$$+ \int_{t_3}^{t}\left[\frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}w_1(s) - H(t,s)\frac{\lambda\pi_1[t_0,s-\sigma_1]}{(m(s))^{1/\lambda}a_1(s-\sigma_1)}(w_1(s))^{\frac{\lambda+1}{\lambda}}\right]ds$$
$$+\mu_1^\lambda \int_{t_3}^{t}\left[\frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}w_2(s) - H(t,s)\frac{\lambda\pi_1[t_0,s-\sigma_1]}{(m(s))^{1/\lambda}a_1(s-\sigma_1)}(w_2(s))^{\frac{\lambda+1}{\lambda}}\right]ds$$
$$+\mu_2^\lambda \int_{t_3}^{t}\left[\frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}w_3(s) - H(t,s)\frac{\lambda\pi_1[t_0,s-\sigma_1]}{(m(s))^{1/\lambda}a_1(s-\sigma_1)}(w_3(s))^{\frac{\lambda+1}{\lambda}}\right]ds. \tag{3.23}$$

Setting $Y = \frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}$, $X = \frac{H(t,s)\,\lambda\pi_1[t_0,s-\sigma_1]}{(m(s))^{1/\lambda}a_1(s-\sigma_1)}$ and $u = w_i(t)$ for $i = 1,2,3$. By using the Lemma 2.3, we conclude that

$$\frac{1}{H(t,t_3)}\int_{t_3}^{t}\left[H(t,s)m(s)\frac{P(s)}{3^{\lambda-1}} - \frac{1+\mu_1^\lambda+\mu_2^\lambda}{(\lambda+1)^{\lambda+1}}\left(\frac{|h(t,s)|a_1(s-\sigma_1)}{m(s)\pi_1[t_0,s-\sigma_1]}\right)^{\lambda}\right]ds \le w_1(t_3) + \mu_1^\lambda w_2(t_3) + \mu_2^\lambda w_3(t_3) \tag{3.24}$$

which contradicts condition (3.20). □

Theorem 3.2. *Let* (1.1) *hold and* $\tau_1 \ge \sigma_1$. *If there exists an* $m(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that* (2.6) *and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_3)}\int_{t_3}^{t}\left[H(t,s)m(s)\frac{P(s)}{3^{\lambda-1}} - \frac{1+\mu_1^\lambda+\mu_2^\lambda}{(\lambda+1)^{\lambda+1}}\left(\frac{|h(t,s)|\,a_1(s-\tau_1)}{m(s)\pi_1[t_0,s-\tau_1]}\right)^{\lambda}\right]ds = \infty, \tag{3.25}$$

then every solution $y(t)$ *of* (E_1) *is either oscillatory or tends to* 0.

Proof. Suppose that (E_1) has a nonoscillatory solution y. Without loss of generality, we may take $y(t) > 0$, $y(t-\tau_1) > 0$, $y(t+\tau_2) > 0$, $y(t-\sigma_1) > 0$ and $y(t+\sigma_1) > 0$ for $t \ge t_1 \ge t_0$. Assumption of (1.1), by Lemma 2.4 there exists two cases (C_1) and (C_2). If (C_2) holds, then by Lemma 2.6, $\lim_{t\to\infty} z(t) = 0$. We only consider (C_1), by using the fact that $z'(t) > 0$ and $\tau_1 \ge \sigma_1$, we obtain that Using the fact of $L^{[1]}z(t) > 0$, we obtain

$$L^{[4]}z(t) + \mu_1^\lambda L^{[4]}_{-\tau_1}z(t) + \mu_2^\lambda L^{[4]}_{\tau_2}z(t) + \frac{P(t)}{3^{\lambda-1}}z^\lambda(t-\tau_1) \le 0. \tag{3.26}$$

Next, we categorize the functions as $w_1(t) = m(t)\frac{L^{[3]}z(t)}{z^\lambda(t-\tau_1)}$, $w_2(t) = m(t)\frac{L^{[3]}_{-\tau_1}z(t)}{z^\lambda(t-\tau_1)}$ and $w_3(t) = m(t)\frac{L^{[3]}_{\tau_2}z(t)}{z^\lambda(t-\tau_1)}$ respectively. The rest of the proof is similar to that of Theorem 3.1, therefore, it is omitted. □

Theorem 3.3. *Let* (1.2) *hold and* $\sigma_1 \ge \tau_1$. *If there exists an* $m(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that* (2.6),

$$\int_{t_3}^{\infty}\left[m(s)\frac{P(s)}{3^{\lambda-1}} - \left(1+\mu_1^\lambda+\mu_2^\lambda\right)\left(\frac{(m'(s))_+}{(\lambda+1)}\right)^{\lambda+1}\left(\frac{a_1(s-\sigma_1)}{m(s)\pi_1[t_0,s-\sigma_1]}\right)^{\lambda}\right]ds = \infty, \tag{3.27}$$

and

$$\int_{t_3}^{\infty}\left[\pi_*^\lambda(s+\tau_2)\frac{P(s)}{3^{\lambda-1}}\left(\int_{t_2}^{s+\sigma_1}\frac{du}{a_1(u)}\right)^{\lambda}\right.$$

$$-\Big(\frac{\lambda}{1+\lambda}\Big)^{1+\lambda}\frac{(1+\mu_1^\lambda)a_2(s)+\mu_2^\lambda a_2(s+\tau_2+\sigma_1)}{a_2^{1+\frac{1}{\lambda}}(s)\pi_*^\lambda(s+\tau_2)}\Bigg]ds=\infty, \tag{3.28}$$

where $(m'(t))_+ = \max\{0, m'(t)\}$, $\pi_*(t) = \int_{t+\sigma_1}^\infty a_2^{-1/\lambda}(s)ds$, *then every solution* $y(t)$ *of* (E_1) *is either oscillatory or tends to* 0*.*

Proof. Suppose that (E_1) has a nonoscillatory solution y. Without loss of generality, we may take $y(t) > 0$, $y(t-\tau_1) > 0$, $y(t+\tau_2) > 0$, $y(t-\sigma_1) > 0$ and $y(t+\sigma_1) > 0$ for $t \geq t_1 \geq t_0$. Since $y(t) > 0$ for all $t \geq t_1$. Assumption of (1.2), by Lemma 2.4 there exists three cases (C_1), (C_2) and (C_3). If case (C_1) and (C_2) holds, using the similar proof of ([24], Theorem 2.1) by using Lemma 2.1, we get the conclusion of Theorem 3.3.

If case (C_3) holds, $z'(t-\sigma_1) < 0$ for $t \geq t_1$. The facts that $z'(t) < 0$, $c+d \geq 0$ and (3.6), we obtain

$$L^{[4]}z(t)+\mu_1^\lambda L_{-\tau_1}^{[4]}z(t)+\mu_2^\lambda L_{\tau_2}^{[4]}z(t)+\frac{P(t)}{3^{\lambda-1}}z^\lambda(t+\sigma_1)\leq 0. \tag{3.29}$$

Define

$$w_*(t)=\frac{L^{[3]}z(t)}{(a_1(t+\sigma_1)z'(t+\sigma_1))^\lambda}. \tag{3.30}$$

We obtain $w_*(t) < 0$ for $t \geq t_2$. Noting that $L^{[3]}z(t)$ is decreasing, we obtain

$$a_2(s)\Big[L^{[2]}z(s)\Big]^\lambda \leq a_2(t)\Big[L^{[2]}z(t)\Big]^\lambda \tag{3.31}$$

for $s \geq t \geq t_2$. Dividing (3.31) by $a_2(s)$ and integrating from $t+\sigma_1$ to l $(l \geq t)$, we get

$$a_1(l)z'(l)\leq a_1(t+\sigma_1)z'(t+\sigma_1)+a_2^{1/\lambda}(t)\Big[L^{[2]}z(t)\Big]\int_{t+\sigma_1}^l a_2^{-1/\lambda}(s)ds.$$

letting $l \to \infty$, we get

$$-1\leq\frac{a_2^{1/\lambda}(t)\Big[L^{[2]}z(t)\Big]}{a_1(t+\sigma_1)z'(t+\sigma_1)}\pi_*(t), \tag{3.32}$$

for $t \geq t_2$. From (3.30), we have

$$-1\leq w_*(t)\pi_*^\lambda(t)\leq 0. \tag{3.33}$$

By (3.2) we have $a_1(t+\sigma_1)z'(t+\sigma_1) \leq a_1(t)z'(t)$. Differentiating (3.30) gives,

$$w_*'(t)\leq\frac{(L^{[3]}z(t))'}{(a_1(t+\sigma_1)z'(t+\sigma_1))^\lambda}-\lambda a_2(t)\Big[\frac{L^{[2]}z(t)}{a_1(t+\sigma_1)z'(t+\sigma_1)}\Big]^{\lambda+1}. \tag{3.34}$$

Using (3.30) in (3.34), we have

$$w_*'(t)\leq\frac{L^{[4]}z(t)}{(a_1(t+\sigma_1)z'(t+\sigma_1))^\lambda}-\lambda\frac{w_*^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \tag{3.35}$$

Again, we define

$$w_{**}(t) = \frac{L^{[3]}_{-\tau_1} z(t)}{(a_1(t+\sigma_1)z'(t+\sigma_1))^{\lambda}}. \tag{3.36}$$

We obtain $w_{**}(t) < 0$ and $w_{**}(t) \geq w_*(t)$ for $t \geq t_2$. By (3.33), we obtain

$$-1 \leq w_{**}(t)\pi_*^{\lambda}(t) \leq 0. \tag{3.37}$$

By (3.2) we have $a_1(t+\sigma_1)z'(t+\sigma_1) \leq a_1(t-\tau_1)z'(t-\tau_1)$. Differentiating (3.36) gives,

$$w'_{**}(t) \leq \frac{(L^{[3]}_{-\tau_1} z(t))'}{(a_1(t+\sigma_1)z'(t+\sigma_1))^{\lambda}} - \lambda a_2(t)\Big[\frac{L^{[2]}_{-\tau_1} z(t)}{a_1(t+\sigma_1)z'(t+\sigma_1)}\Big]^{\lambda+1}. \tag{3.38}$$

Using (3.36) in (3.38), we have

$$w'_{**}(t) \leq \frac{L^{[4]}_{-\tau_1} z(t)}{(a_1(t+\sigma_1)z'(t+\sigma_1))^{\lambda}} - \lambda \frac{w_{**}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \tag{3.39}$$

Finally, we define a function

$$w_{***}(t) = \frac{L^{[3]}_{\tau_2} z(t)}{(a_1(t+\tau_2+\sigma_1)z'(t+\tau_2+\sigma_1))^{\lambda}}. \tag{3.40}$$

We obtain $w_{***}(t) < 0$ and $w_{***}(t) = w_*(t+\tau_2)$ for $t \geq t_2$. By (3.33), we obtain

$$-1 \leq w_{***}(t)\pi_*^{\lambda}(t+\tau_2) \leq 0. \tag{3.41}$$

By (3.2) we have $a_1(t+\tau_2+\sigma_1)z'(t+\tau_2+\sigma_1) \leq a_1(t+\tau_2)z'(t+\tau_2)$. Differentiating (3.40) gives,

$$w'_{***}(t) \leq \frac{(L^{[3]}_{\tau_2} z(t))'}{(a_1(t+\sigma_1)z'(t+\sigma_1))^{\lambda}} - \lambda a_2(t)\Big[\frac{L^{[2]}_{\tau_2} z(t)}{a_1(t+\tau_2+\sigma_1)z'(t+\tau_2+\sigma_1)}\Big]^{\lambda+1}. \tag{3.42}$$

Using (3.40) in (3.42), we have

$$w'_{***}(t) \leq \frac{L^{[4]}_{\tau_2} z(t)}{(a_1(t+\sigma_1)z'(t+\sigma_1))^{\lambda}} - \lambda \frac{w_{***}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \tag{3.43}$$

From (3.35), (3.39), (3.43) and (3.29) which implies

$$\begin{aligned} w'_*(t) + \mu_1^{\lambda} w'_{**}(t) + \mu_2^{\lambda} w'_{***}(t) \leq & -\frac{P(t)}{3^{\lambda-1}} \frac{z^{\lambda}(t+\sigma_1)}{(a_1(t+\sigma_1)z'(t+\sigma_1))^{\lambda}} \\ & -\lambda \frac{w_*^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} - \mu_1^{\lambda}\lambda \frac{w_{**}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} - \mu_2^{\lambda}\lambda \frac{w_{***}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} \end{aligned} \tag{3.44}$$

In case (C_3), $(a_1(t)z'(t))' < 0$ we seen that

$$z(t) \geq a_1(t)z'(t) \int_{t_2}^{t} \frac{1}{a_1(s)} ds. \tag{3.45}$$

Using (3.45) in (3.44), we get

$$w'_*(t)+\mu_1^\lambda w'_{**}(t)+\mu_2^\lambda w'_{***}(t)\le -\frac{P(t)}{3^{\lambda-1}}\Big(\int_{t_2}^{t+\sigma_1}\frac{ds}{a_1(s)}\Big)^\lambda-\lambda\frac{w_*^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}-\mu_1^\lambda\lambda\frac{w_{**}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}-\mu_2^\lambda\lambda\frac{w_{***}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \quad (3.46)$$

Multiplying $\pi_*^\lambda(t+\tau_2)$ and integrating from t_3 ($t_3>t_2$) to t, yields

$$\begin{aligned}
&\pi_*^\lambda(t+\tau_2)w_*(t)-\pi_*^\lambda(t_3+\tau_2)w_*(t_3)+\pi_*^\lambda(t+\tau_2)\,\mu_1^\lambda\, w_{**}(t)\\
&-\pi_*^\lambda(t_3+\tau_2)\,\mu_1^\lambda\, w_{**}(t_3)+\pi_*^\lambda(t+\tau_2)\,\mu_2^\lambda\, w_{***}(t)-\pi_*^\lambda(t_3+\tau_2)\,\mu_2^\lambda\, w_{***}(t_3)\\
&-\lambda\int_{t_3}^{t}\Big[\frac{\pi_*^{\lambda-1}(s+\tau_2)(-w_*(s))}{a_2^{1/\lambda}(s+\tau_2)}-\frac{\pi_*^\lambda(s+\tau_2)(-w_*(s))^{1+\frac{1}{\lambda}}}{a_2^{1/\lambda}(s)}\Big]ds\\
&-\lambda\,\mu_1^\lambda\int_{t_3}^{t}\Big[\frac{\pi_*^{\lambda-1}(s+\tau_2)(-w_{**}(s))}{a_2^{1/\lambda}(s+\tau_2)}-\frac{\pi_*^\lambda(s+\tau_2)(-w_{**}(s))^{1+\frac{1}{\lambda}}}{a_2^{1/\lambda}(s)}\Big]ds\\
&-\lambda\,\mu_2^\lambda\int_{t_3}^{t}\Big[\frac{\pi_*^{\lambda-1}(s+\tau_2)(-w_{***}(s))}{a_2^{1/\lambda}(s+\tau_2)}-\frac{\pi_*^\lambda(s+\tau_2)(-w_{***}(s))^{1+\frac{1}{\lambda}}}{a_2^{1/\lambda}(s)}\Big]ds\\
&+\int_{t_3}^{t}\pi_*^\lambda(s+\tau_2)\frac{P(s)}{3^{\lambda-1}}\Big(\int_{t_2}^{s+\sigma_1}\frac{du}{a_1(u)}\Big)^\lambda ds\le 0. \qquad (3.47)
\end{aligned}$$

Applying Lemma 2.3, we conclude that

$$\begin{aligned}
&\int_{t_3}^{t}\Big[\pi_*^\lambda(s+\tau_2)\frac{P(s)}{3^{\lambda-1}}\Big(\int_{t_2}^{s+\sigma_1}\frac{du}{a_1(u)}\Big)^\lambda-\Big(\frac{\lambda}{1+\lambda}\Big)^{1+\lambda}\frac{(1+\mu_1^\lambda)a_2(s)+\mu_2^\lambda a_2(s+\tau_2+\sigma_1)}{a_2^{1+\frac{1}{\lambda}}(s)\pi_*^\lambda(s+\tau_2)}\Big]ds\\
&\le-\Big[\pi_*^\lambda(t+\tau_2)w_*(t)+\mu_1^\lambda\pi_*^\lambda(t+\tau_2)w_{**}(t)+\mu_2^\lambda\pi_*^\lambda(t+\tau_2)w_{***}(t)\Big] \qquad (3.48)
\end{aligned}$$

Using the fact of $\pi_*^\lambda(t+\tau_2)\le\pi_*^\lambda(t)$ in (3.33), (3.37), (3.41) and (3.48) imply that

$$\begin{aligned}
&\int_{t_3}^{t}\Big[\pi_*^\lambda(s+\tau_2)\frac{P(s)}{3^{\lambda-1}}\Big(\int_{t_2}^{s+\sigma_1}\frac{du}{a_1(u)}\Big)^\lambda\\
&\quad-\Big(\frac{\lambda}{1+\lambda}\Big)^{1+\lambda}\frac{(1+\mu_1^\lambda)a_2(s)+\mu_2^\lambda a_2(s+\tau_2+\sigma_1)}{a_2^{1+\frac{1}{\lambda}}(s)\pi_*^\lambda(s+\tau_2)}\Big]ds\le 1+\mu_1^\lambda+\mu_2^\lambda. \qquad (3.49)
\end{aligned}$$

a contradiction to (3.28). □

Finally, we establish new comparison theorems for (E_1) under the case when (1.2) holds.

Theorem 3.4. *Let* (1.2)*,* (2.6) *hold and* $\sigma_1>\tau_1$, $\sigma_1>\tau_2$*. If the first-order differential inequality*

$$\psi'(t)+\frac{P_1(t)}{3^{\lambda-1}}\frac{A^\lambda(t-\sigma_1)}{1+\mu_1^\lambda+\mu_2^\lambda}\psi(t-\sigma_1+\tau_1)\le 0 \qquad (3.50)$$

for $t\ge t_0$*, has no positive nonincreasing solution and the first-order differential inequality*

$$\psi'(t)-\frac{P_2(t)}{3^{\lambda-1}}\frac{B^\lambda(t+\sigma_1)}{1+\mu_1^\lambda+\mu_2^\lambda}\psi(t-\tau_2+\sigma_1)\ge 0 \qquad (3.51)$$

for $t\ge t_0$*, has no positive nondecreasing solution. Then Eq.* (E_1) *oscillatory.*

Proof. Suppose that (E_1) has a nonoscillatory solution y. Without loss of generality, we may take $y(t) > 0$, $y(t-\tau_1) > 0$, $y(t+\tau_2) > 0$, $y(t-\sigma_1) > 0$ and $y(t+\sigma_1) > 0$ for $t \geq t_1 \geq t_0$. Since $y(t) > 0$ for all $t \geq t_1$. Assumption of (1.2), by Lemma 2.4, there exists three cases (C_1), (C_2) and (C_3). If case (C_2) hold, the proof is follows from Lemma 2.6.

If case (C_1) holds, we have $L^{[2]}z(t) > 0$, from (3.6), we obtain

$$L^{[4]}z(t) + \mu_1^\lambda L^{[4]}_{-\tau_1}z(t) + \mu_2^\lambda L^{[4]}_{\tau_2}z(t) + \frac{P_1(t)}{3^{\lambda-1}}\, z^\lambda(t-\sigma_1) \leq 0. \tag{3.52}$$

By Lemma 2.5, one gets $z(t-\sigma_1) \geq (L^{[3]}_{-\sigma_1}z(t))^{1/\lambda}A(t-\sigma_1)$ and using in (3.52), we have

$$\Big(L^{[3]}z(t) + \mu_1^\lambda L^{[3]}_{-\tau_1}z(t) + \mu_2^\lambda L^{[3]}_{\tau_2}z(t)\Big)' + \frac{P_1(t)}{3^{\lambda-1}}\, L^{[3]}_{-\sigma_1}z(t)A^\lambda(t-\sigma_1) \leq 0. \tag{3.53}$$

Now, set

$$\psi(t) = L^{[3]}z(t) + \mu_1^\lambda L^{[3]}_{-\tau_1}z(t) + \mu_2^\lambda L^{[3]}_{\tau_2}z(t).$$

Then $\psi(t) > 0$ and the fact that $L^{[3]}z(t)$ is nonincreasing, we have

$$\psi(t) \leq L^{[3]}_{-\tau_1}z(t)\Big(1 + \mu_1^\lambda + \mu_2^\lambda\Big). \tag{3.54}$$

Using (3.54) and (3.53), we see that $\psi(t)$ is a nonincreasing positive solution of the first order differential inequality

$$\psi'(t) + \frac{P_1(t)}{3^{\lambda-1}}\frac{A^\lambda(t-\sigma_1)}{1+\mu_1^\lambda+\mu_2^\lambda}\psi(t-\sigma_1+\tau_1) \leq 0, \tag{3.55}$$

which is contradiction to (3.50).

If case (C_3) holds, we have $L^{[2]}z(t) < 0$, from (3.6), we obtain

$$L^{[4]}z(t) + \mu_1^\lambda L^{[4]}_{-\tau_1}z(t) + \mu_2^\lambda L^{[4]}_{\tau_2}z(t) + \frac{P_2(t)}{3^{\lambda-1}}\, z^\lambda(t+\sigma_1) \leq 0. \tag{3.56}$$

Since $L^{[3]}z(t)$ is nondecreasing . Then we get

$$L^{[3]}z(s) \leq L^{[3]}z(t) \quad \text{for all} \quad s \geq t \geq t_1 \geq t_0.$$

Integrating above inequality from t to l, we get

$$\begin{aligned} a_1(l)z'(l) &\leq a_1(t)z'(t) + \int_t^l \frac{a_2^{1/\lambda}(t)L^{[2]}z(t)}{a_2^{1/\lambda}(s)}ds \\ &\leq a_1(t)z'(t) + \Big(L^{[3]}z(s)\Big)^{1/\lambda}\int_t^l \frac{ds}{a_2^{1/\lambda}(s)}. \end{aligned}$$

Letting $l \to \infty$, we get

$$-a_1(t)z'(t) \leq \Big(L^{[3]}z(s)\Big)^{1/\lambda}\int_t^\infty \frac{ds}{a_2^{1/\lambda}(s)}.$$

Again integrating, we get

$$z(t) \geq -(L^{[3]}z(t))^{1/\lambda} \int_{t_0}^{t} \frac{\int_t^\infty \frac{du}{a_2^{1/\lambda}(u)}}{a_1(s)} ds = -(L^{[3]}z(t))^{1/\lambda} B(t). \tag{3.57}$$

From 3.57, one gets $z(t+\sigma_1) \geq -(L^{[3]}_{\sigma_1} z(t))^{1/\lambda} B(t+\sigma_1)$ and using in (3.56), we have

$$\Big(L^{[3]}z(t) + \mu_1^\lambda L^{[3]}_{-\tau_1}z(t) + \mu_2^\lambda L^{[3]}_{\tau_2}z(t)\Big)' - \frac{P_2(t)}{3^{\lambda-1}} L^{[3]}_{\sigma_1}z(t) B^\lambda(t+\sigma_1) \leq 0. \tag{3.58}$$

Now, set

$$\psi(t) = L^{[3]}z(t) + \mu_1^\lambda L^{[3]}_{-\tau_1}z(t) + \mu_2^\lambda L^{[3]}_{\tau_2}z(t).$$

Then $\psi(t) > 0$, $\psi'(t) \geq 0$ and the fact that $L^{[3]}z(t)$ is nondecreasing, we have

$$\psi(t) \leq L^{[3]}_{\tau_2}z(t)\Big(1 + \mu_1^\lambda + \mu_2^\lambda\Big). \tag{3.59}$$

Using (3.59) and (3.58), we see that $\psi(t)$ is a nonincreasing positive solution of the first order differential inequality

$$\psi'(t) - \frac{P_2(t)}{3^{\lambda-1}} \frac{B^\lambda(t+\sigma_1)}{1+\mu_1^\lambda+\mu_2^\lambda} \psi(t-\tau_2+\sigma_1) \geq 0 \tag{3.60}$$

which is contradiction to (3.51). □

Corollary 3.5. *Let* (1.2), (2.6) *hold and* $\sigma_1 > \tau_1$, $\sigma_1 > \tau_2$. *If*

$$\liminf_{t\to\infty} \int_{t-\sigma_1+\tau_1}^{t} P_1(s) A^\lambda(s-\sigma_1)\, ds > \frac{3^{\lambda-1}}{e\,(1+\mu_1^\lambda+\mu_2^\lambda)} \tag{3.61}$$

and

$$\liminf_{t\to\infty} \int_{t-\tau_2+\sigma_1}^{t} P_2(s) B^\lambda(s+\sigma_1)\, ds > \frac{3^{\lambda-1}}{e\,(1+\mu_1^\lambda+\mu_2^\lambda)} \tag{3.62}$$

hold, then Eq. (E_1) *oscillatory.*

Proof. The proof follows from Theorem 3.4 and ([10], Theorem 2.1.1), and the details are omitted. □

Example 3.6. *Consider the third order differential equation*

$$\left(\left(\left(\left(y(t) + \frac{e^{-2}}{3} y(t-2) + \frac{e}{3} y(t+1)\right)\right)'\right)'^{3/2}\right)' + \frac{3e^{-3}}{4}\Big(\frac{5}{3}\Big)^{3/2} y^{3/2}(t-2) + \frac{3e^3}{4}\Big(\frac{5}{3}\Big)^{3/2} y^{3/2}(t+2) = 0. \tag{3.63}$$

Compared with (E_1), *we can see that* $a_1(t) = a_2(t) = 1$, $p_1(t) = \frac{e^{-2}}{3}$, $p_2(t) = \frac{e^1}{3}$, $q_1(t) = \frac{3e^{-3}}{4}\left(\frac{5}{3}\right)^{3/2}$, $q_2(t) = \frac{3e^3}{4}\left(\frac{5}{3}\right)^{3/2}$, $\lambda = 3/2$, $\tau_1 = 2$, $\tau_2 = 1$ *and* $\sigma_1 = 2$. *By taking* $m(t) = 1$, $H(t,s) = (t-s)^2$, *we obtain* $h(t,s) = (3s-t)(t-s)^{-1/5}$. *It is easy to verify that all conditions of Theorem 3.1 are satisfied. Therefore, all the solutions of* (3.63) *is either oscillates or tends to* 0 *and* $y(t) = e^{-t}$ *is a such solution of* (3.63).

Example 3.7. *Consider the third order differential equation*

$$\Big[t^2(y(t) + k_1 y(t-\tau_1) + k_2 y(t+\tau_2))''\Big]' + k_3 t\, y(t-\sigma_1) + k_4 y(t+\sigma_1) = 0, \quad t \geq 1. \tag{3.64}$$

Compared with (E_1), *we can see that* $a_1(t) = 1$, $a_2(t) = t^2$, $p_1(t) = k_1$, $p_2(t) = k_2$, $q_1(t) = k_3 t$, $q_2(t) = k_4$, $\lambda = 1$ *and* k_1, k_2, k_3, k_4 *are nonnegative constants. It is easy to verify that all conditions of Corollary 3.5 are satisfied and hence all solutions of equation* (3.64) *are oscillatory.*

4. Oscillation results for (E_2)

In this section, we will establish several oscillation criteria for (E_2). For convenience, we define,

$$\begin{aligned}
Q_1(t,\xi) &= \min\{\tilde{q}_1(t,\xi),\ \tilde{q}_1(t-\tau_1,\xi),\ \tilde{q}_1(t+\tau_2,\xi)\},\\
Q_2(t,\xi) &= \min\{\tilde{q}_2(t,\xi),\ \tilde{q}_2(t-\tau_1,\xi),\ \tilde{q}_2(t+\tau_2,\xi)\},\\
Q(t,\xi) &= Q_1(t,\xi) + Q_2(t,\xi).
\end{aligned}$$

Theorem 4.1. *Let* (1.1) *holds and* $c+d \geq 0$, $b \geq \tau_1$. *If there exists an* $m(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that* (2.7) *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_3)} \int_{t_3}^{t} \left[H(t,s)m(s)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}} - \frac{1+\mu_1^\lambda+\mu_2^\lambda}{(\lambda+1)^{\lambda+1}} \left(\frac{|h(t,s)|a_1(s-d)}{m(s)\pi_1[t_0,s-d]} \right)^{\lambda} \right] ds = \infty, \tag{4.1}$$

then every solution $y(t)$ *of* (E_2) *is either oscillatory or tends to* 0.

Proof. Suppose that (E_2) has a nonoscillatory solution y. Without loss of generality, we may take $y(t) > 0$, $y(t-\tau_1) > 0$, $y(t+\tau_2) > 0$, $y(t-\xi) > 0$ and $y(t+\xi) > 0$ for $t \geq t_1 \geq t_0$ and $\xi \in [c,d]$. Since $y(t) > 0$ for all $t \geq t_1$, in view of (E_2), we have

$$L^{[4]}z(t) = -\int_c^d \tilde{q}_1(t,\xi)y^\lambda(t-\xi)\,d\xi - \int_c^d \tilde{q}_2(t,\xi)y^\lambda(t+\xi)\,d\xi \leq 0. \tag{4.2}$$

Assumption of (1.1), by Lemma 2.4 there exists two cases (C_1) and (C_2). If (C_2) holds, then by Lemma 2.7, $\lim_{t\to\infty} z(t) = 0$. If (C_1) holds.

$$\begin{aligned}
L^{[4]}z(t) &+ \int_c^d \tilde{q}_1(t,\xi)y^\lambda(t-\xi)\,d\xi + \int_c^d \tilde{q}_2(t,\xi)y^\lambda(t+\xi)\,d\xi\\
&+\mu_1^\lambda L^{[4]}_{-\tau_1}z(t) + \mu_1^\lambda \int_c^d \tilde{q}_1(t-\tau_1,\xi)y^\lambda(t-\tau_1-\xi)\,d\xi\\
&+\mu_1^\lambda \int_c^d \tilde{q}_2(t-\tau_1,\xi)y^\lambda(t-\tau_1+\xi)\,d\xi + \mu_2^\lambda L^{[4]}_{\tau_2}z(t)\\
&+\mu_2^\lambda \int_c^d \tilde{q}_1(t+\tau_2,\xi)y^\lambda(t+\tau_2-\xi)\,d\xi\\
&+\mu_2^\lambda \int_c^d \tilde{q}_2(t+\tau_2,\xi)y^\lambda(t+\tau_2+\xi)\,d\xi = 0.
\end{aligned} \tag{4.3}$$

Furthermore, from Lemma 2.1, we have

$$\begin{aligned}
&\tilde{q}_1(t,\xi)\,y^\lambda(t-\xi) + \mu_1^\lambda\,\tilde{q}_1(t-\tau_1,\xi)\,y^\lambda(t-\tau_1-\xi)\\
&\qquad +\mu_1^\lambda\,\tilde{q}_1(t+\tau_2,\xi)\,y^\lambda(t+\tau_2-\xi) \geq \frac{Q_1(t,\xi)}{3^{\lambda-1}}\,z^\lambda(t-\xi).
\end{aligned} \tag{4.4}$$

Similarly, we get

$$\tilde{q}_2(t,\xi)\,y^\lambda(t+\xi) + \mu_2^\lambda\,\tilde{q}_2(t-\tau_1,\xi)\,y^\lambda(t-\tau_1+\xi)$$

$$+\mu_2^\lambda\,\tilde{q}_2(t+\tau_2,\xi)\,y^\lambda(t+\tau_2+\xi) \geq \frac{Q_2(t,\xi)}{3^{\lambda-1}}\,z^\lambda(t+\xi). \tag{4.5}$$

Substituting (4.4), (4.5) into (4.3), we have

$$L^{[4]}z(t)+\mu_1^\lambda\,L^{[4]}_{-\tau_1}z(t)+\mu_2^\lambda\,L^{[4]}_{\tau_2}z(t)+\frac{\int_c^d Q_1(t,\xi)d\xi}{3^{\lambda-1}}\,z^\lambda(t-\xi)+\frac{\int_c^d Q_2(t,\xi)d\xi}{3^{\lambda-1}}\,z^\lambda(t+\xi)\leq 0. \tag{4.6}$$

Using the fact of $L^{[1]}z(t)>0$ and $c+d\geq 0$, we have

$$L^{[4]}z(t)+\mu_1^\lambda\,L^{[4]}_{-\tau_1}z(t)+\mu_2^\lambda\,L^{[4]}_{\tau_2}z(t)+\frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}}\,z^\lambda(t-d)\leq 0. \tag{4.7}$$

Define a function

$$w_1(t)=m(t)\frac{L^{[3]}z(t)}{z^\lambda(t-d)}. \tag{4.8}$$

We obtain $w_1(t)>0$, then

$$w_1'(t)=m'(t)\frac{L^{[3]}z(t)}{z^\lambda(t-d)}+m(t)\frac{L^{[4]}z(t)}{z^\lambda(t-d)}-\lambda m(t)\frac{L^{[3]}z(t)z'(t-d)}{z^{\lambda+1}(t-d)}. \tag{4.9}$$

By Lemma (2.5), one gets $z'(t-d)\geq \frac{a_2^{1/\lambda}(t)}{a_1(t-d)}\pi_1[t_0,t-d]\,L^{[2]}z(t)$. Therefore

$$w_1'(t)\leq m'(t)\frac{L^{[3]}z(t)}{z^\lambda(t-d)}+m(t)\frac{L^{[4]}z(t)}{z^\lambda(t-d)}-\lambda m(t)\frac{a_2^{\frac{\lambda+1}{\lambda}}(t)\pi_1[t_0,t-d]L^{[2]}z(t)z'(t-d)}{z^{\lambda+1}(t-d)a_1(t-d)}. \tag{4.10}$$

Using (4.8) in (4.10), we have

$$w_1'(t)\leq \frac{(m'(t))_+}{m(t)}w_1(t)+m(t)\frac{L^{[4]}z(t)}{z^\lambda(t-d)}-\lambda\,\frac{(w_1(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}. \tag{4.11}$$

Next, define

$$w_2(t)=m(t)\frac{L^{[3]}_{-\tau_1}z(t)}{z^\lambda(t-d)}. \tag{4.12}$$

We obtain $w_2(t)>0$, then

$$w_2'(t)=m'(t)\frac{L^{[3]}_{-\tau_1}z(t)}{z^\lambda(t-d)}+m(t)\frac{L^{[4]}_{-\tau_1}z(t)}{z^\lambda(t-d)}-\lambda m(t)\frac{L^{[3]}_{-\tau_1}z(t)z'(t-d)}{z^{\lambda+1}(t-d)}. \tag{4.13}$$

By Lemma (2.5), one gets $z'(t-d)\geq \frac{a_2^{1/\lambda}(t-\tau_1)}{a_1(t-d)}\pi_1[t_0,t-d]\,L^{[2]}_{-\tau_1}z(t)$ and using (4.12) in (4.13), we have

$$w_2'(t)\leq \frac{(m'(t))_+}{m(t)}w_2(t)+m(t)\frac{L^{[4]}_{-\tau_1}z(t)}{z^\lambda(t-d)}-\lambda\,\frac{(w_2(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}. \tag{4.14}$$

Finally, define

$$w_3(t) = m(t)\frac{L_{\tau_2}^{[3]} z(t)}{z^\lambda(t-d)}. \tag{4.15}$$

We obtain $w_3(t) > 0$, then

$$w_3'(t) = m'(t)\frac{L_{\tau_2}^{[3]} z(t)}{z^\lambda(t-d)} + m(t)\frac{L_{\tau_2}^{[4]} z(t)}{z^\lambda(t-d)} - \lambda m(t)\frac{L_{\tau_2}^{[3]} z(t) z'(t-d)}{z^{\lambda+1}(t-d)}. \tag{4.16}$$

By Lemma 2.5, one gets $z'(t-d) \geq \frac{a_2^{1/\lambda}(t+\tau_2)}{a_1(t-d)}\pi_1[t_0, t-d]\, L_{\tau_2}^{[2]} z(t)$ and using (4.15) in (4.16), we have

$$w_3'(t) \leq \frac{(m'(t))_+}{m(t)} w_3(t) + m(t)\frac{L_{\tau_2}^{[4]} z(t)}{z^\lambda(t-d)} - \lambda\,\frac{(w_3(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-d]}{(m(t))^{1/\lambda} a_1(t-d)}. \tag{4.17}$$

From (4.8), (4.10) and (4.15), we have

$$\begin{aligned} w_1'(t) + \mu_1^\lambda w_2'(t) + \mu_2^\lambda w_3'(t) \quad \leq \quad & m(t)\left[\frac{L^{[4]} z(t) + \mu_1^\lambda L_{-\tau_1}^{[4]} z(t) + \mu_2^\lambda L_{\tau_2}^{[4]} z(t)}{z^\lambda(t-d)}\right] \\ & + \left[\frac{(m'(t))_+}{m(t)} w_1(t) - \lambda\,\frac{(w_1(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-d]}{(m(t))^{1/\lambda} a_1(t-d)}\right] \\ & + \mu_1^\lambda\left[\frac{(m'(t))_+}{m(t)} w_2(t) - \lambda\,\frac{(w_2(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-d]}{(m(t))^{1/\lambda} a_1(t-d)}\right] \\ & + \mu_2^\lambda\left[\frac{(m'(t))_+}{m(t)} w_3(t) - \lambda\,\frac{(w_3(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-d]}{(m(t))^{1/\lambda} a_1(t-d)}\right]. \end{aligned} \tag{4.18}$$

Using (4.7) in (4.18), we have

$$\begin{aligned} w_1'(t) + \mu_1^\lambda w_2'(t) + \mu_2^\lambda w_3'(t) \quad \leq \quad & -m(t)\frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}} \\ & + \left[\frac{(m'(t))_+}{m(t)} w_1(t) - \lambda\,\frac{(w_1(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-d]}{(m(t))^{1/\lambda} a_1(t-d)}\right] \\ & + \mu_1^\lambda\left[\frac{(m'(t))_+}{m(t)} w_2(t) - \lambda\,\frac{(w_2(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-d]}{(m(t))^{1/\lambda} a_1(t-d)}\right] \\ & + \mu_2^\lambda\left[\frac{(m'(t))_+}{m(t)} w_3(t) - \lambda\,\frac{(w_3(t))^{\frac{\lambda+1}{\lambda}}\pi_1[t_0, t-d]}{(m(t))^{1/\lambda} a_1(t-d)}\right], \end{aligned} \tag{4.19}$$

that is,

$$\begin{aligned} m(t)\frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}} \leq & -w_1'(t) - \mu_1^\lambda w_2'(t) - \mu_2^\lambda w_3'(t) + \frac{(m'(t))_+}{m(t)} w_1(t) \\ & -\lambda\,\frac{\pi_1[t_0, t-d]}{(m(t))^{1/\lambda} a_1(t-d)}(w_1(t))^{\frac{\lambda+1}{\lambda}} \end{aligned}$$

$$+\mu_1^{\lambda}\left[\frac{(m'(t))_+}{m(t)}w_2(t)-\lambda\,\frac{\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}(w_2(t))^{\frac{\lambda+1}{\lambda}}\right]$$
$$+\mu_2^{\lambda}\left[\frac{(m'(t))_+}{m(t)}w_3(t)-\lambda\,\frac{\pi_1[t_0,t-d]}{(m(t))^{1/\lambda}a_1(t-d)}(w_3(t))^{\frac{\lambda+1}{\lambda}}\right]. \tag{4.20}$$

Multiply both sides $H(t,s)$ and integrate (4.51) from t_3 to t, one can get that

$$\begin{aligned}\int_{t_3}^{t}H(t,s)m(s)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}ds \le &-\int_{t_3}^{t}H(t,s)w_1'(s)ds-\mu_1^{\lambda}\int_{t_3}^{t}H(t,s)w_2'(s)ds\\ &-\mu_2^{\lambda}\int_{t_3}^{t}H(t,s)w_3'(s)ds+\int_{t_3}^{t}H(t,s)\frac{(m'(s))_+}{m(s)}w_1(s)\,ds\\ &-\int_{t_3}^{t}H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_1(s))^{\frac{\lambda+1}{\lambda}}ds\\ +\mu_1^{\lambda}\int_{t_3}^{t}H(t,s)\frac{(m'(s))_+}{m(s)}w_2(s)ds&-\mu_1^{\lambda}\int_{t_3}^{t}H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_2(s))^{\frac{\lambda+1}{\lambda}}ds\\ +\mu_2^{\lambda}\int_{t_3}^{t}H(t,s)\frac{(m'(s))_+}{m(s)}w_3(s)ds&-\mu_2^{\lambda}\int_{t_3}^{t}H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_3(s))^{\frac{\lambda+1}{\lambda}}ds.\end{aligned} \tag{4.21}$$

Thus, we obtain

$$\begin{aligned}\int_{t_3}^{t}H(t,s)m(s)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}ds \le &H(t,t_3)w_1(t_3)+\mu_1^{\lambda}H(t,t_3)w_2(t_3)+\mu_2^{\lambda}H(t,t_3)w_3(t_3)\\ &-\int_{t_3}^{t}\left[-\frac{\partial}{\partial s}H(t,s)-H(t,s)\frac{m'(s)}{m(s)}\right]w_1(s)\,ds\\ &-\int_{t_3}^{t}H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_1(s))^{\frac{\lambda+1}{\lambda}}ds\\ &-\mu_1^{\lambda}\int_{t_3}^{t}\left[-\frac{\partial}{\partial s}H(t,s)-H(t,s)\frac{m'(s)}{m(s)}\right]w_2(s)ds\\ &-\mu_1^{\lambda}\int_{t_3}^{t}H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_2(s))^{\frac{\lambda+1}{\lambda}}ds\\ &-\mu_2^{\lambda}\int_{t_3}^{t}\left[-\frac{\partial}{\partial s}H(t,s)-H(t,s)\frac{m'(s)}{m(s)}\right]w_3(s)ds\\ &-\mu_2^{\lambda}\int_{t_3}^{t}H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_3(s))^{\frac{\lambda+1}{\lambda}}ds.\end{aligned} \tag{4.22}$$

Then

$$\begin{aligned}\int_{t_3}^{t}H(t,s)m(s)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}ds \le &H(t,t_3)w_1(t_3)+\mu_1^{\lambda}H(t,t_3)w_2(t_3)+\mu_2^{\lambda}H(t,t_3)w_3(t_3)\\ &+\int_{t_3}^{t}\left[\frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}w_1(s)-H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_1(s))^{\frac{\lambda+1}{\lambda}}\right]ds\end{aligned}$$

$$+\mu_1^\lambda \int_{t_3}^{t}\Bigg[\frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}w_2(s)-H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_2(s))^{\frac{\lambda+1}{\lambda}}\Bigg]ds$$
$$+\mu_2^\lambda \int_{t_3}^{t}\Bigg[\frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}w_3(s)-H(t,s)\frac{\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}(w_3(s))^{\frac{\lambda+1}{\lambda}}\Bigg]ds. \tag{4.23}$$

Setting $Y = \frac{|h(t,s)|(H(t,s))^{\frac{\lambda}{\lambda+1}}}{m(s)}$, $X = \frac{H(t,s)\,\lambda\pi_1[t_0,s-d]}{(m(s))^{1/\lambda}a_1(s-d)}$ and $u = w_i(t)$ for $i = 1,2,3$. By using the Lemma 2.3, we conclude that

$$\frac{1}{H(t,t_3)}\int_{t_3}^{t}\Bigg[H(t,s)m(s)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}-\frac{1+\mu_1^\lambda+\mu_2^\lambda}{(\lambda+1)^{\lambda+1}}\bigg(\frac{|h(t,s)|a_1(s-d)}{m(s)\pi_1[t_0,s-d]}\bigg)^{\lambda}\Bigg]ds$$
$$\leq w_1(t_3)+\mu_1^\lambda w_2(t_3)+\mu_2^\lambda w_3(t_3) \tag{4.24}$$

which contradicts condition (4.51). □

Theorem 4.2. *Let* (1.1) *holds and* $c+d \geq 0$, $-c \geq \tau_1$. *If there exists an* $m(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that* (2.7) *and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_3)}\int_{t_3}^{t}\Bigg[H(t,s)m(s)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}-\frac{1+\mu_1^\lambda+\mu_2^\lambda}{(\lambda+1)^{\lambda+1}}\bigg(\frac{|h(t,s)|\,a_1(s+c)}{m(s)\,\pi_1\,[t_0,s+c]}\bigg)^{\lambda}\Bigg]ds=\infty, \tag{4.25}$$

then every solution $y(t)$ *of* (E_2) *is either oscillatory or tends to* 0.

Proof. Suppose that (E_2) has a nonoscillatory solution y. Without loss of generality, we may take $y(t) > 0$, $y(t-\tau_1) > 0$, $y(t+\tau_2) > 0$, $y(t-\xi) > 0$ and $y(t+\xi) > 0$ for $t \geq t_1 \geq t_0$ and $\xi \in [c,d]$. Assumption of (1.1), by Lemma 2.4 there exists two cases (C_1) and (C_2). If (C_2) holds, then by Lemma 2.7, $\lim_{t\to\infty} z(t) = 0$. We only consider (C_1), by using the fact that $z'(t) > 0$ and $-c \geq \tau_1$, we obtain that Using the fact of $L^{[1]}z(t) > 0$, we obtain

$$L^{[4]}z(t)+\mu_1^\lambda L_{-\tau_1}^{[4]}z(t)+\mu_2^\lambda L_{\tau_2}^{[4]}z(t)+\frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}}z^\lambda(t+c)\leq 0. \tag{4.26}$$

Next, we categorize the functions as $w_1(t) = m(t)\frac{L^{[3]}z(t)}{z^\lambda(t+c)}$, $w_2(t) = m(t)\frac{L_{-\tau_1}^{[3]}z(t)}{z^\lambda(t+c)}$ and $w_3(t) = m(t)\frac{L_{\tau_2}^{[3]}z(t)}{z^\lambda(t+c)}$ respectively. The rest of the proof is similar to that of Theorem 4.1, therefore, it is omitted. □

Theorem 4.3. *Let* (1.2) *holds and* $b \geq \tau_1$ *(or* $b \leq \tau_1$*). If there exists an* $m(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that* (2.7)*,*

$$\int_{t_3}^{\infty}\Bigg[m(s)\frac{\int_c^d Q(s,\xi)d\xi}{3^{\lambda-1}}-\Big(1+\mu_1^\lambda+\mu_2^\lambda\Big)\bigg(\frac{(m'(s))_+}{(\lambda+1)}\bigg)^{\lambda+1}\bigg(\frac{a_1(s-d)}{m(s)\pi_1[t_0,s-d]}\bigg)^{\lambda}\Bigg]ds=\infty, \tag{4.27}$$

and

$$\int_{t_3}^{\infty}\Bigg[\pi_*^\lambda(s+\tau_2)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}\Big(\int_{t_2}^{s+d}\frac{du}{a_1(u)}\Big)^{\lambda}$$

$$-\left(\frac{\lambda}{1+\lambda}\right)^{1+\lambda}\frac{(1+\mu_1^\lambda)a_2(s)+\mu_2^\lambda a_2(s+\tau_2+d)}{a_2^{1+\frac{1}{\lambda}}(s)\beta^\lambda(s+\tau_2)}\Bigg]ds=\infty, \tag{4.28}$$

where $\beta(t)=\int_{t+d}^{\infty}a_2^{-1/\lambda}(s)ds$, *then every solution* $y(t)$ *of* (E_2) *is either oscillatory or tends to* 0.

Proof. Suppose that (E_1) has a nonoscillatory solution y. Without loss of generality, we may take $y(t)>0$, $y(t-\tau_1)>0$, $y(t+\tau_2)>0$, $y(t-\xi)>0$ and $y(t+\xi)>0$ for $t\geq t_1\geq t_0$ and $\xi\in[c,d]$. Since $y(t)>0$ for all $t\geq t_1$. Assumption of (1.2), by Lemma 2.4 there exists three cases (C_1), (C_2) and (C_3). If case (C_1) and (C_2) holds, using the similar proof of ([24], Theorem 2.3) by using Lemma 2.1, we get the conclusion of Theorem 4.3

If case (C_3) holds, $z'(t-d)<0$ for $t\geq t_1$. The facts that $z'(t)<0$, $c+d\geq 0$ and (4.6), we obtain

$$L^{[4]}z(t)+\mu_1^\lambda L_{-\tau_1}^{[4]}z(t)+\mu_2^\lambda L_{\tau_2}^{[4]}z(t)+\frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}}z^\lambda(t+d)\leq 0. \tag{4.29}$$

Define

$$w_*(t)=\frac{L^{[3]}z(t)}{(a_1(t+d)z'(t+d))^\lambda}. \tag{4.30}$$

We obtain $w_*(t)<0$ for $t\geq t_2$. Noting that $L^{[3]}z(t)$ is decreasing, we obtain

$$a_2(s)\Big[L^{[2]}z(s)\Big]^\lambda\leq a_2(t)\Big[L^{[2]}z(t)\Big]^\lambda \tag{4.31}$$

for $s\geq t\geq t_2$. Dividing (4.31) by $a_2(s)$ and integrating from $t+d$ to $l(l\geq t)$, we get

$$a_1(l)z'(l)\leq a_1(t+d)z'(t+d)+a_2^{1/\lambda}(t)\Big[L^{[2]}z(t)\Big]\int_{t+d}^{l}a_2^{-1/\lambda}(s)ds.$$

letting $l\to\infty$, we get

$$-1\leq\frac{a_2^{1/\lambda}(t)\Big[L^{[2]}z(t)\Big]}{a_1(t+d)z'(t+d)}\pi_*(t),\quad t\geq t_2. \tag{4.32}$$

From (4.30), we have

$$-1\leq w_*(t)\beta^\lambda(t)\leq 0. \tag{4.33}$$

By (4.2) we have $a_1(t+d)z'(t+d)\leq a_1(t)z'(t)$. Differentiating (4.30) gives,

$$w_*'(t)\leq\frac{(L^{[3]}z(t))'}{(a_1(t+d)z'(t+d))^\lambda}-\lambda a_2(t)\Big[\frac{L^{[2]}z(t)}{a_1(t+d)z'(t+d)}\Big]^{\lambda+1}. \tag{4.34}$$

Using (4.30) in (4.34), we have

$$w_*'(t)\leq\frac{L^{[4]}z(t)}{(a_1(t+d)z'(t+d))^\lambda}-\lambda\frac{w_*^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \tag{4.35}$$

Next, we define

$$w_{**}(t) = \frac{L^{[3]}_{-\tau_1} z(t)}{(a_1(t+d)z'(t+d))^{\lambda}}. \tag{4.36}$$

We obtain $w_{**}(t) < 0$ and $w_{**}(t) \geq w_*(t)$ for $t \geq t_2$. By (4.33), we obtain

$$-1 \leq w_{**}(t)\beta^{\lambda}(t) \leq 0. \tag{4.37}$$

By (3.2) we have $a_1(t+d)z'(t+d) \leq a_1(t-\tau_1)z'(t-\tau_1)$. Differentiating (4.36) gives,

$$w'_{**}(t) \leq \frac{(L^{[3]}_{-\tau_1} z(t))'}{(a_1(t+d)z'(t+d))^{\lambda}} - \lambda a_2(t)\Big[\frac{L^{[2]}_{-\tau_1} z(t)}{a_1(t+d)z'(t+d)}\Big]^{\lambda+1}. \tag{4.38}$$

Using (4.36) in (4.38), we have

$$w'_{**}(t) \leq \frac{L^{[4]}_{-\tau_1} z(t)}{(a_1(t+d)z'(t+d))^{\lambda}} - \lambda \frac{w_{**}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \tag{4.39}$$

Finally, We define a function

$$w_{***}(t) = \frac{L^{[3]}_{\tau_2} z(t)}{(a_1(t+\tau_2+d)z'(t+\tau_2+d))^{\lambda}}. \tag{4.40}$$

We obtain $w_{***}(t) < 0$ and $w_{***}(t) = w_*(t+\tau_2)$ for $t \geq t_2$. By (4.33), we obtain

$$-1 \leq w_{***}(t)\beta^{\lambda}(t+\tau_2) \leq 0. \tag{4.41}$$

By (4.2) we have $a_1(t+\tau_2+d)z'(t+\tau_2+d) \leq a_1(t+\tau_2)z'(t+\tau_2)$. Differentiating (4.40) gives,

$$w'_{***}(t) \leq \frac{(L^{[3]}_{\tau_2} z(t))'}{(a_1(t+d)z'(t+d))^{\lambda}} - \lambda a_2(t)\Big[\frac{L^{[2]}_{\tau_2} z(t)}{a_1(t+\tau_2+d)z'(t+\tau_2+d)}\Big]^{\lambda+1}. \tag{4.42}$$

Using (4.40) in (4.42), we have

$$w'_{***}(t) \leq \frac{L^{[4]}_{\tau_2} z(t)}{(a_1(t+d)z'(t+d))^{\lambda}} - \lambda \frac{w_{***}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)}. \tag{4.43}$$

From (4.35), (4.39), (4.43) and (4.29) which implies

$$\begin{aligned} w'_*(t) + \mu_1^{\lambda} w'_{**}(t) + \mu_2^{\lambda} w'_{***}(t) \leq & -\frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}} \frac{z^{\lambda}(t+d)}{(a_1(t+d)z'(t+d))^{\lambda}} \\ & -\lambda \frac{w_*^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} - \mu_1^{\lambda}\lambda \frac{w_{**}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} - \mu_2^{\lambda}\lambda \frac{w_{***}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} \end{aligned} \tag{4.44}$$

In case (C_3), $(a_1(t)z'(t))' < 0$ we seen that

$$z(t) \geq a_1(t)z'(t) \int_{t_2}^{t} \frac{ds}{a_1(s)}. \tag{4.45}$$

Using (4.45) in (4.44), we get

$$w_*'(t) + \mu_1^\lambda w_{**}'(t) + \mu_2^\lambda w_{***}'(t) \leq -\frac{\int_c^d Q(t,\xi)\,d\xi}{3^{\lambda-1}}\Big(\int_{t_2}^{t+d} \frac{ds}{a_1(s)}\Big)^\lambda - \lambda\frac{w_*^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} -\mu_1^\lambda\lambda\frac{w_{**}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} - \mu_2^\lambda\lambda\frac{w_{***}^{1+\frac{1}{\lambda}}(t)}{a_2^{1/\lambda}(t)} \tag{4.46}$$

Multiplying $\beta^\lambda(t+\tau_2)$ and integrating from t_3 ($t_3 > t_2$) to t, yields

$$\begin{aligned}
&\beta^\lambda(t+\tau_2)w_*(t) - \beta^\lambda(t_3+\tau_2)w_*(t_3) + \beta^\lambda(t+\tau_2)\,\mu_1^\lambda\, w_{**}(t) \\
&-\beta^\lambda(t_3+\tau_2)\,\mu_1^\lambda\, w_{**}(t_3) + \beta^\lambda(t+\tau_2)\,\mu_2^\lambda\, w_{***}(t) - \beta^\lambda(t_3+\tau_2)\,\mu_2^\lambda\, w_{***}(t_3) \\
&-\lambda\int_{t_3}^t\Big[\frac{\beta^{\lambda-1}(s+\tau_2)(-w_*(s))}{a_2^{1/\lambda}(s+\tau_2)} - \frac{\beta^\lambda(s+\tau_2)(-w_*(s))^{1+\frac{1}{\lambda}}}{a_2^{1/\lambda}(s)}\Big]ds \\
&-\lambda\,\mu_1^\lambda\int_{t_3}^t\Big[\frac{\beta^{\lambda-1}(s+\tau_2)(-w_{**}(s))}{a_2^{1/\lambda}(s+\tau_2)} - \frac{\beta^\lambda(s+\tau_2)(-w_{**}(s))^{1+\frac{1}{\lambda}}}{a_2^{1/\lambda}(s)}\Big]ds \\
&-\lambda\,\mu_2^\lambda\int_{t_3}^t\Big[\frac{\beta^{\lambda-1}(s+\tau_2)(-w_{***}(s))}{a_2^{1/\lambda}(s+\tau_2)} - \frac{\beta^\lambda(s+\tau_2)(-w_{***}(s))^{1+\frac{1}{\lambda}}}{a_2^{1/\lambda}(s)}\Big]ds \\
&+\int_{t_3}^t\beta^\lambda(s+\tau_2)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}\Big(\int_{t_2}^{s+d}\frac{du}{a_1(u)}\Big)^\lambda ds \leq 0.
\end{aligned} \tag{4.47}$$

Applying Lemma 2.3, we conclude that

$$\begin{aligned}
&\int_{t_3}^t\Big[\beta^\lambda(s+\tau_2)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}\Big(\int_{t_2}^{s+d}\frac{du}{a_1(u)}\Big)^\lambda \\
&\qquad-\Big(\frac{\lambda}{1+\lambda}\Big)^{1+\lambda}\frac{(1+\mu_1^\lambda)a_2(s)+\mu_2^\lambda a_2(s+\tau_2+d)}{a_2^{1+\frac{1}{\lambda}}(s)\,\beta^\lambda(s+\tau_2)}\Big]ds \\
&\leq -\Big[\beta^\lambda(t+\tau_2)w_*(t) + \mu_1^\lambda\beta^\lambda(t+\tau_2)w_{**}(t) + \mu_2^\lambda\beta^\lambda(t+\tau_2)w_{***}(t)\Big]
\end{aligned} \tag{4.48}$$

Using the fact of $\beta^\lambda(t+\tau_2) \leq \beta^\lambda(t)$ in (4.33), (4.37), (4.41) and (4.48) imply that

$$\begin{aligned}
&\int_{t_3}^t\Big[\beta^\lambda(s+\tau_2)\frac{\int_c^d Q(s,\xi)\,d\xi}{3^{\lambda-1}}\Big(\int_{t_2}^{s+d}\frac{du}{a_1(u)}\Big)^\lambda \\
&\qquad-\Big(\frac{\lambda}{1+\lambda}\Big)^{1+\lambda}\frac{(1+\mu_1^\lambda)a_2(s)+\mu_2^\lambda a_2(s+\tau_2+d)}{a_2^{1+\frac{1}{\lambda}}(s)\beta^\lambda(s+\tau_2)}\Big]ds \leq 1+\mu_1^\lambda+\mu_2^\lambda.
\end{aligned} \tag{4.49}$$

a contradiction to (4.28). □

Example 4.4. *Consider a third-order differential equation*

$$\Big(\frac{1}{2}\Big(y(t) + (1/3)y(t-\pi/4) + (2/3)y(t+\pi/2)\Big)''\Big)' + \int_0^\pi y(t-\xi)d\xi + \frac{3}{2}\int_0^\pi y(t+\xi)d\xi = 0, \tag{4.50}$$

Compared with (E_2)*, we can see that* $c = 0$*,* $d = \pi$*,* $a_1(t) = 1/2$*,* $a_2(t) = 1$*,* $p_1(t) = \frac{1}{3}$*,* $p_2(t) = \frac{2}{3}$*,* $\tilde{q}_1(t,\xi) = \tilde{q}_2(t,\xi) = 1$*,* $\lambda = 1$*,* $\tau_1 = \pi/4$ *and* $\tau_2 = \pi/2$*. By taking* $m(t) = 1$*, we obtain*

$$\frac{1}{2}\int_{t_4}^{\infty}\int_{v}^{\infty}\int_{u}^{\infty} 2\pi\, ds\, du\, dv = \infty$$

and we take $H(t,s) = (t-s)^2$ *then* $h(t,s) = (3s-t)(t-s)^{-1/5}$ *and* $0 < \mu_1 + \mu_2 < 1$*, we see that*

$$\limsup_{t\to\infty} \frac{1}{(t-t_3)^2}\int_{t_3}^{t}\left[2\pi\,(t-s)^2 - \frac{1+\mu_1+\mu_2}{8}\left(\frac{(3s-t)(t-s)^{-1/5}}{s-\pi-t_0}\right)^{\lambda}\right]ds = \infty. \tag{4.51}$$

Since all the conditions of Theorem 4.1 hold, (4.50) *is either oscillates or tends to* 0*.*

5. Conclusion

In this paper, we have used Riccati substitution techniques, integral averaging technique and some new oscillation and asymptotic theorems for (E_1) and (E_2) under the conditions (1.1) and (1.2) have been established. Additionally, we established new comparison theorem that permit to study properties of (E_1) regardless under the conditions (1.2). The results obtained indicated that it improved theorems reported by Candan [24]. Similar results can be presented under the assumption that $\lambda \leq 1$. In this case, using Lemma 2.2, one has to simply replace $3^{\lambda-1}$ by 1 and proceed as above. In literature, very few works has been paid in the research activities related to qualitative behavior of solutions of various types of stochastic differential equations, see the recent works [1, 3, 13–15, 19–21]. The results of this paper could be extended to the stochastic differential equations with time delay in further research.

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Conflict of interest

The authors declare there are no conflicts of interest.

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