Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(4): 3840–3850.
DOI:10.3934/math.2020249
Received: 31 December 2019
Accepted: 12 April 2020
Published: 22 April 2020

Research article

Uniqueness and multiplicity of positive solutions for one-dimensional prescribed mean curvature equation in Minkowski space

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Abstract: In this paper, we study the uniqueness and multiplicity of positive solutions of one-dimensional prescribed mean curvature equation

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1-u'^2}}\Big)' = \lambda f(u), \\ u(x) > 0, \quad -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where λ is a positive parameter. The main tool is the fixed point index in cones.

Keywords: mean curvature equation; positive solutions; multiplicity; uniqueness; cone
Mathematics Subject Classification: 34B18, 34C23, 35J60

1. Introduction

In this paper, we are concerned with the uniqueness and multiplicity of positive solutions for the quasilinear two-point boundary value problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1-u'^2}}\Big)' = \lambda f(u), \\ u(x) > 0, \quad -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $f : \mathbb{R} \to [0, \infty)$. This is the one-dimensional version of the Dirichlet problem associated with the prescribed mean curvature equation in Minkowski space

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The study of spacelike submanifolds of codimension one in the flat Minkowski space $\mathbb{L}^{N+1}$ with prescribed mean extrinsic curvature can lead to the type of problems (1.2), where

$$\mathbb{L}^{N+1} := \{(x,t) : x \in \mathbb{R}^N,\ t \in \mathbb{R}\}$$

endowed with the Lorentzian metric

$$\sum_{i=1}^{N}(dx_i)^2 - (dt)^2,$$

and (x, t) is the canonical coordinate in $\mathbb{R}^{N+1}$ (see [1]). Problem (1.1) is also related to classical relativity [2], the Born-Infeld model in the theory of nonlinear electrodynamics [3, 4] and cosmology [5, 6].

In 1934, in order to combine Maxwell's equation with the quantum theory, Born [7] proposed a modified energy function

$$L(a) = \frac{\sqrt{1 + a^2(H^2 - E^2)} - 1}{a^2},$$

where a is a constant of the dimensions r_0^2/e with e = charge, r_0 = radius of the electron, H is the magnetic field and E is the electric field. Then, it is easy to verify that $\lim_{a\to 0} L(a) = L = \frac{1}{2}(H^2 - E^2)$. Here L is the Lagrangian of Maxwell's equation. When $H = 0$ and $a = 1$, the Poisson equation of $L(a)$ is

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = \rho,$$

where ρ denotes the charge density and $\nabla u = E$. This is closely related to the first equation of problem (1.2). See [7, 8] for more details.

The existence/ multiplicity of positive solutions for this kind of problems have been extensively studied by using the method of lower and upper solutions, variational methods, fixed point theorem in cones as well as the bifurcation theory, see [9–14] and the references therein.

Recently, the authors of [9] obtained some existence results for positive radial solutions of problem (1.2) without parameter λ. In another paper, they [10] established some nonexistence, existence and multiplicity results for positive radial solutions of problem (1.2) with $\lambda f(|x|, s) = \lambda\mu(|x|)s^q$, where $q > 1, \mu : [0, \infty) \to \mathbb{R}$ is continuous, strictly positive on $(0, \infty)$. In [11], under some assumptions on f, the authors proved that problem (1.2) has infinitely many radial solutions. In [12], Coelho et al. discussed the existence of either one, or two, or three, or infinitely many positive solutions of the quasilinear two-point boundary value problem (1.1).

However, as far as we know, there are few works on the uniqueness of positive solutions of problem (1.1). Very recently, Zhang and Feng [13] discussed the exactness of positive solutions (1.1) under some special cases of f. For example, $f(u) = u^p (p > 0),\ u^p + u^q (0 < p < q \le 1)$.

Let us briefly recall their main results.

Theorem A. Assume that $f(u) = u^p$ with $p < 1$, then for any $\lambda > 0$ (1.1) has exactly one positive solution.
Theorem B. Assume that $f(u) = u^p + u^q$ with $0 < p < q \leq 1$, then for any $\lambda > 0$ (1.1) has exactly one positive solution.

Motivated on these studies, we are interested in investigating a general class of f, and we point out $f(u) = u^p$ with $0 < p < 1$, $f(u) = u^p + u^q$ with $0 < p < q \leq 1$ as the special case. In addition, we consider the case where f has n zeros and obtain the existences of arbitrarily many positive solutions of the problem (1.1).

Throughout this paper, we assume that

(H1): $f : \mathbb{R} \to [0, \infty)$ is continuous function, and f is not identical to zero.

If Ω is an open bounded subset in a Banach space E and $T : \bar{\Omega} \to E$ is compact, with $0 \notin (I-T)(\partial\Omega)$, then $i(T, \Omega, E)$ is the fixed point index of T on Ω with respect to E. For the definition and properties of the fixed point index we refer the reader to e.g., [15].

The rest of the paper is organized as follows. In section 2, we give some preliminary results. In section 3, under some assumptions on f, we prove the uniqueness of positive solutions for problem (1.1). In section 4, we prove the existence of arbitrarily many positive solutions for problem (1.1).

2. Preliminaries

Below, let X be the Banach space $C[0, 1]$ endowed with the norm $\|v\| = \sup_{t\in[0,1]} |v(t)|$. Let $P = \{v \in X : v(t)$ is decreasing in $t, v(1) = 0,$ and $v(t) \geq \|v\|(1 - t)$ for all $t \in [0, 1]\}$, then P is a cone in X, and the corresponding open ball of center 0 and radius $r > 0$ will be denoted by B_r.

By an argument similar to that of Lemma 2.4 of [14], we have the following

Lemma 2.1. Let (λ, u) be a positive solution of the problem (1.1) with $\|u\| = \rho_1$ and $\lambda > 0$. Let $x_0 \in (-1, 1)$ be such that $u'(x_0) = 0$. Then
(i) $x_0 = 0$;
(ii) x_0 is the unique point on which u attains its maximum;
(iii) $u'(t) > 0, t \in (-1, 0)$ and $u'(t) < 0, t \in (0, 1)$.

By a solution of problem (1.1), we understand that it is a function which belongs to $C^1[0, 1]$ with $\|u'\| < 1$, such that $u'/\sqrt{1 - u'^2}$ is differentiable and problem (1.1) is satisfied.

By Lemma 2.1, (λ, u) is a positive solution of (1.1) if and only if (λ, u) is a positive solution of the mixed boundary value problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1 - u'^2}}\Big)' = \lambda f(u), \\ u'(0) = 0, \quad u(1) = 0. \end{cases} \tag{2.1}$$

Now we treat positive classical solutions of (2.1). The following well-known result of the fixed point index is crucial in our arguments.

Lemma 2.2. [15–17] Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = K \cap B_r$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{x \in K : \|x\| = r\}$.
(i) If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K)$=0.

(ii) If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K)$=1.

Now we define a nonlinear operator T_λ on $P \cap B_1$ as follows:

$$(T_\lambda u)(t) = \int_t^1 \phi^{-1}\Big(\int_0^s \lambda f(u(\tau))d\tau\Big)ds, \quad u \in P \cap B_1,$$

where $\phi(s) = s/\sqrt{1-s^2}$.

We point out that u is a positive solution of problem (2.1) if $u \in P \cap B_1$ is a fixed point of the nonlinear operator T_λ.

Next, we give an important Lemma which will be used later.

Lemma 2.3. Let $v \in X$ with $v(t) \geq 0$ for $t \in [0, 1]$. If v concave on $[0, 1]$, then

$$v(t) \geq \min\{t, 1-t\}\|v\|, \quad t \in [0, 1].$$

In particular, for any pair $0 < \alpha < \beta < 1$ we have

$$\min_{\alpha \leq t \leq \beta} v(t) \geq \min\{\alpha, 1-\beta\}\|v\|. \tag{2.2}$$

Furthermore, if $v(0) = \|v\|$, then we have

$$v(t) \geq \|v\|(1-t) \text{ for all } t \in [0, 1]. \tag{2.3}$$

Proof. It is an immediate consequence of the fact that u is concave down in $[0, 1]$. □

Lemma 2.4. $T_\lambda(P \cap B_1) \subset P$ and the map $T_\lambda : P \cap B_\rho \to P$ for all $\rho \in (0, 1)$ is completely continuous.

Proof. From the condition (H1) one can easily see that $T_\lambda v \in X$ with $T_\lambda v(1) = 0$, and for $v \in P$, a simple computation shows that

$$(T_\lambda v)'(t) = -\lambda\phi^{-1}\Big(\int_0^s f(u(\tau))d\tau\Big)ds.$$

So $T_\lambda v(t)$ is decreasing on [0,1], which implies that

$$T_\lambda v(t) \geq T_\lambda v(1) = 0, \text{ for } t \in [0, 1].$$

Moreover, it is easy to see that $(T_\lambda v)'$ decreasing in $[0, 1]$, then it follows from Lemma 2.3 that

$$v(t) \geq \|v\|(1-t) \text{ for all } t \in [0, 1].$$

Therefore, we obtain that $T_\lambda(P \cap B_1) \subset P$. Moreover, the operator T_λ is compact on $P \cap \bar{B}_\rho$ is an immediate consequence of [9, Lemma 2]. □

Simple computations leads lead to the following lemma.

Lemma 2.5. Let $\phi(s) := s/\sqrt{1-s^2}$. Then $\phi^{-1}(s) = s/\sqrt{1+s^2}$ and

$$\phi^{-1}(s_1)\phi^{-1}(s_2) \leq \phi^{-1}(s_1 s_2) \leq s_1 s_2, \ \ \forall s_1, s_2 \in [0, \infty).$$

In particular, for $0 < s_1 \leq 1$ we have

$$\phi^{-1}(s_1 s_2) \geq s_1 \phi^{-1}(s_2).$$

3. Uniqueness results

In this section, we are going to study the positive solution of problem (1.1). Results to be proved in this section are true for any positive parameter λ. So, we may assume $\lambda = 1$ for simplicity and therefore consider

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1-u'^2}}\Big)' = f(u), \\ u'(0) = 0, \quad u(1) = 0. \end{cases} \tag{3.1}$$

and the corresponding operator

$$(T_1 u)(t) = \int_t^1 \phi^{-1}\Big(\int_0^s f(u(\tau))d\tau\Big)ds, \quad t \in [0,1], \tag{3.2}$$

defined on $P \cap B_1$.

Now, we present the main results of this section as follows.

Theorem 3.1. Assume (H1), and f is increasing, such that for any $u > 0$ and $t \in (0,1)$ there always exists some $\xi > 1$ such that

$$f(tu) \geq (1+\xi)tf(u). \tag{3.3}$$

Then problem (3.1) can have at most one positive solution.

Remark 3.1. Since f is increasing, condition (3.3) implies that $f(u) > 0$ when $u > 0$, and can be replaced by the following simpler condition:

$$f(tu) > tf(u).$$

Remark 3.2. To compare with the Theorem 3.1 and Theorem 3.4 in [13], ours in this section cover a very broad class of functions of f. For example, $f(u) = \sum_{i=1}^{i=m} u^{p_i}$ with $0 < p_i < 1$ or $f(u) = 1 + \sqrt{u}$.

Theorem 3.1 is proved via a sequence of Lemmas. But we need to have a definition first.

Definition 3.1. [16, 17] Let K be a cone in real Banach space Y and $\leq$ be the partial ordering defined by K. Let $A : K \to K$ and $u_0 > \theta$, where θ denotes the zero element Y.

(a) for any $x > \theta$, there exist α, $\beta > 0$ such that

$$\alpha u_0 \leq A(x) \leq \beta u_0;$$

(b) for any $\alpha u_0 \leq x \leq \beta u_0$ and $t \in (0,1)$, there exists some $\eta > 0$ such that

$$A(tx) \geq (1+\eta)tAx.$$

Then A is called u_0-sublinear

Lemma 3.2. Suppose that f satisfies the conditions of Theorem 3.1. Then the operator T_1, defined by 3.2, is u_0-sublinear with $u_0 = 1 - t$.

Proof. We divide the proof into two steps.

Step 1. We check condition (a) of Definition 3.1. First, we show that for any $u > 0$ from $P \cap B_1$, there exist $\alpha, \beta > 0$ such that

$$\alpha u_0 \leq T_1(u) \leq \beta u_0. \tag{3.4}$$

Let $M = \max_{t\in[0,1]}\{f(u(t))\}$. It follows from Lemma 2.5, we have

$$(T_1 u)(t) \leq \int_t^1 \phi^{-1}\Big(\int_0^s M d\tau\Big) ds \leq \frac{1}{2} M(1-t^2) \leq M(1-t).$$

So, we may take $\beta = M$.

Notice that $(T_1 u)(t)$ is strictly decreasing in t and vanishes at $t = 1$. Choose any $c \in (0,1)$ and set

$$m = \int_0^c f(u(t))dt.$$

Then for all $t \in [c, 1]$, we have

$$(T_1 u)(t) \geq \int_t^1 \phi^{-1}(m) ds = \frac{m}{\sqrt{1+m^2}}(1-t).$$

Since $(T_1 u)(t) \geq (T_1 u)(c) = \frac{m}{\sqrt{1+m^2}}(1-c)$ for all $t \in [0, c]$, we have

$$(T_1 u)(t) \geq \frac{m}{\sqrt{1+m^2}}(1-c)(1-t)$$

for all $t \in [0,1]$. So, we may choose $\alpha = \frac{m}{\sqrt{1+m^2}}(1-c)$ and (3.4) is proved.

Step 2. We check condition (b) of Definition 3.1. We need to show that for any $\alpha u_0 \leq u \leq \beta u_0$ and $\xi \in (0,1)$, there exists some $\eta > 0$ such that

$$T_1(\xi u) \geq (1+\eta)\xi T_1 u.$$

To this end, due to the conditions satisfied by f, there exists an $\eta > 0$ such that

$$f(\xi u) \geq [(1+\eta)\xi] f(u).$$

It follows from Lemma 2.5 that

$$\begin{aligned}(T_1(\xi u))(t) &= \int_t^1 \phi^{-1}\Big(\int_0^s f(\xi u) d\tau\Big) ds \\ &\geq \int_t^1 \phi^{-1}\Big[\int_0^s (1+\eta)\xi f(u) d\tau\Big] ds \\ &\geq (1+\eta)\xi \int_t^1 \phi^{-1}\Big(\int_0^s f(u) d\tau\Big) ds.\end{aligned}$$

Thus we have

$$T_1((\xi u))(t) \geq [(1+\eta)\xi](T_1 u)(t).$$

Therefore, the proof is complete. □

Proof of Theorem 3.1. Let f be from Theorem 3.1. From Lemma 3.2, we know that the operator T_1 is increasing and u_0-sublinear with $u_0 = 1 - t$.

Next, we show that T_1 can have at most one positive fixed point. Assume to the contrary that there exist two positive fixed points u and x^*. Now, we claim that there exists $c > 0$ such that $u \geq cx^*$. By (3.2), we know that

$$x^*(t) = \int_t^1 \phi^{-1}\Big(\int_0^s f(x^*(\tau))d\tau\Big)ds, \quad t \in [0, 1],$$

$$u(t) = \int_t^1 \phi^{-1}\Big(\int_0^s f(u(\tau))d\tau\Big)ds, \quad t \in [0, 1],$$

So, $x^*(t)$ is strictly decreasing in t and vanishes at $t = 1$. There exist $\sigma_1, \sigma_2 \in (0, \infty)$ such that

$$\sigma_1(1 - t) \leq x^*(t) \leq \sigma_2(1 - t), \quad t \in [0, 1].$$

Using the same method, there exist $\delta_1, \delta_2 \in (0, \infty)$ such that

$$\delta_1(1 - t) \leq u(t) \leq \delta_2(1 - t) \quad t \in [0, 1].$$

Therefore, $\forall t \in [0, 1]$

$$\begin{aligned} u(t) \geq \delta_1(1 - t) &= \frac{\delta_1}{\sigma_2} \cdot \sigma_2(1 - t) \\ &\geq \frac{\delta_1}{\sigma_2} \cdot x^*(t). \end{aligned}$$

So, we can choose $c = \frac{\delta_1}{\sigma_2}$. Set $c^* = \sup\{t | u \geq tx^*\}$. We claim that

$$c^* \geq 1.$$

If not, then there exists some $\xi > 0$ such that $T_1(c^*x^*) \geq (1 + \xi)c^*T_1x^*$. This combine the fact that T_1 is increasing imply that

$$u = T_1u \geq T_1(c^*x^*) \geq (1 + \xi)c^*T_1x^* = (1 + \xi)c^*x^*,$$

which is a contraction since $(1 + \xi)c^* > c^*$. So, $u \geq x^*$ and similarly we can show that $x^* \geq u$. Now with the above lemmas in place, Theorem 3.1 follows readily. Therefore, the proof is complete. □

Example 3.1. Let us consider the problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1 - u'^2}}\Big)' = \sqrt{u}, \\ u'(0) = 0, \ \ u(1) = 0. \end{cases} \tag{3.5}$$

By using Theorem 3.1, we know that the problem (3.5) has a unique positive solution, this is just the conclusion of [13, Theorem 3.1].

4. Multiplicity solutions

In this section, we shall investigate the positive solutions of problem (1.1). Furthermore, we are going to examine how the number of zeros of f may have a huge impact on the number of solutions of (1.1). Here is the main result of this section.

Theorem 4.1. Assume (H1), and there exist two sequences of positive numbers a_i and b_i, satisfying

$$a_i < b_i \leq a_{i+1} < b_{i+1},$$

and such that $f(a_i) = 0$ and $f(b_i) = 0$, and $f(u) > 0$ on (a_i, b_i), for all $i = 1, \cdots, n$ ($n \in \mathbb{N} = \{1, 2, \cdots\}$). Then there exists λ_0 such that for all $\lambda \geq \lambda_0$ the problem (1.1) has n distinct positive solutions, $u_1, u_2, \cdots, u_n$, such that

$$a_i < \|u_i\| \leq b_i$$

for each $i = 1, \cdots, n$.

Remark 4.1. Note that f will satisfy the conditions in the theorem if there exist numbers $a_n > a_{n-1} > \cdots > a_1 > a_0 = 0$ such that $f(a_i) = 0$ for $i = 1, \cdots, n$, and $f(u) > 0$ for $a_{i-1} < u < a_i$, $i = 1, \cdots, n$.

Now we define a nonlinear operator T_λ^i, $i = 1, \cdots n$ as follows:

$$T_\lambda^i u(t) = \int_t^1 \phi^{-1}\Big(\int_0^s \lambda f_i(u(r))dr\Big)ds, \quad u \in P \cap B_1. \tag{4.1}$$

In order to prove the main result, we define f_i by

$$f_i(u) = \begin{cases} f(u), & 0 \leq u \leq b_i, \\ 0, & b_i \leq u. \end{cases}$$

For any $0 < r < 1$, define O_r by

$$O_r = P \cap B_r.$$

Note that $\partial O_r = \{u \in P : \|u\| = r\}$. Similar to T_1, it is easy to verify that T_λ^i is compact on $P \cap B_r$ for all $r \in (0, 1)$.

In order to prove the theorem, we need the following two Lemmas.

Lemma 4.2. Suppose that f satisfies the conditions of Theorem 4.1 . If $v \in P \cap B_1$ is a positive fixed point of (4.1), i.e. $T_\lambda^i v = v$, then v is a solution of (2.1) such that

$$\sup_{t \in [0,1]} v(t) \leq b_i.$$

Proof. Notice that v satisfies

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1-u'^2}}\Big)' = \lambda f_i(u), \\ u'(0) = 0, \quad u(1) = 0. \end{cases}$$

If on the contrary that $\sup_{t\in[0,1]} v(t) = v(0) > b_i$, then there exists a $t_0 \in (0,1)$ such that $v(t) > b_i$ for $t \in [0, t_0)$ and $v(t_0) = b_i$. It follows that

$$-\Big(\frac{v'}{\sqrt{1-v'^2}}\Big)' = 0 \text{ for } t \in (0, t_0].$$

Thus, $-\Big(\frac{v'}{\sqrt{1-v'^2}}\Big)$ is constant on $[0, t_0]$. Since $v'(0) = 0$, it follows that $v'(t) = 0$ for $t \in [0, t_0]$. Consequently, $v(t)$ is a constant on $[0, t_0]$. This is a contradiction. Therefore, $\sup_{t\in[0,1]} v(t) \le b_i$. On the other hand, since $f(v) \equiv f_i(v)$ for $0 \le v \le b_i$, v is a solution of (2.1). □

Choose any ζ such that

$$\max\Big\{\frac{a_i}{b_i} : 1 \le i \le n\Big\} < \zeta < 1.$$

Lemma 4.3. Let the conditions of Theorem 4.1 be satisfied. For any $i \in \{1, \cdots, n\}$, there exists r_i such that $[\zeta r_i, r_i] \subset (a_i, b_i)$. Furthermore, for any $v \in \partial O_{r_i}$ we have

$$\|T^i_\lambda v\| \ge \frac{1}{2}(1-\zeta)^2\phi^{-1}(\lambda\omega_{r_i}),$$

where $\omega_{r_i} = \min_{\zeta r_i \le v \le r_i}\{f_i(v)\} > 0$.

Proof. Based on the choice of ζ, the existence of r_i with $[\zeta r_i, r_i] \subset (a_i, b_i)$ is obvious. Notice that for any $v \in P$ we have that $v(t) \ge v(0)(1-t)$ for all $t \in [0,1]$. In particular, we have $\zeta v(0) \le v(t) \le v(0)$ for all $t \in [0, 1-\zeta]$. Let $v \in \partial O_{r_i}$, then $f_i(v(t)) \ge \omega_{r_i}$ for $t \in [0, 1-\zeta]$. It follows that

$$\begin{aligned}
\|T^i_\lambda v\| &\ge \int_0^{1-\zeta} \phi^{-1}\Big(\int_0^t \lambda f_i(v(\tau))d\tau\Big)dt \\
&\ge \int_0^{1-\zeta} \phi^{-1}\Big(\int_0^t \lambda w_{r_i} d\tau\Big)dt \\
&\ge \int_0^{1-\zeta} t\phi^{-1}(\lambda w_{r_i})dt \\
&= \frac{1}{2}(1-\zeta)^2\phi^{-1}(\lambda\omega_{r_i}).
\end{aligned}$$

□

Proof of Theorem 4.1. Define λ_0 by

$$\lambda_0 = \max\Big\{\phi\Big(\frac{2r_i}{(1-\zeta)^2}\Big)/w_{r_i} : i = 1, 2, \cdots, n\Big\}.$$

For each $i = 1, \cdots, n$ and $\lambda > \lambda_0$, by Lemma 4.3 we infer that

$$\|T^i_\lambda v\| > \|v\| \text{ for } v \in \partial O_{r_i}.$$

On the other hand, for each fixed $\lambda > \lambda_0$, since $f_i(v)$ is bounded, this combine the fact that the rage of ϕ^{-1} is $(-1, 1)$ imply that there is an $R_i > r_i$ such that

$$\|T^i_\lambda v\| < \|v\| \text{ for } v \in \partial O_{R_i}.$$

It follows from Lemma 2.2 that

$$i(T_\lambda^i, O_{r_i}, P) = 0, \quad i(T_\lambda^i, O_{R_i}, P) = 1,$$

and hence

$$i(T_\lambda^i, O_{R_i} \backslash \bar{O}_{r_i}, P) = 1.$$

Thus, T_λ^i has a fixed point v_i^* in $O_{R_i} \backslash \bar{O}_{r_i}$. Lemma 4.2 implies that the fixed point v_i^* is a solution of (2.1) such that

$$a_i < r_i \leq \|v_i^*\| \leq b_i.$$

Consequently, (2.1) has n positive solutions, $v_1, v_2, \cdots, v_n$ for each $\lambda > \lambda_0$, such that

$$a_i < \sup_{t\in[0,1]} v_i(t) \leq b_i.$$

Therefore, the proof is complete. □

Remark 4.1. It is interesting to compare our main result(Theorem 4.1) to Candito et al. [18]. In [18], the authors showed the existence of three classical solutions of the one dimensional prescribed mean curvature equation in Euclid space by using a variational approach.

Example 4.1. Let us consider the problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1-u'^2}}\Big)' = \lambda g(u), \\ u'(0) = 0, \quad u(1) = 0, \end{cases} \tag{4.2}$$

where $g(u) = \sin 8\pi u + 1$.

Let $a_1 = \frac{3}{16}$, $a_2 = \frac{7}{16}$, $a_3 = \frac{11}{16}$, $a_4 = \frac{15}{16}$, we can easily check that $g(a_i) = 0$, and $g(u) > 0$ for $a_i < u < a_{i+1}$, $i = 1, 2, 3$.

From Theorem 4.1, the problem (4.2) has at least three positive solutions u_1, u_2, u_3 with $a_j < \sup_{r\in[0,1]} u_j(r) \leq a_{j+1}$, $j = 1, 2, 3$ provided that λ is large enough.

Acknowledgments

The authors are very grateful to the anonymous referees for their valuable suggestions.

Conflict of interest

The authors declare that they have no conflicts of interest.

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