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Research article

Multiple solutions to a quasilinear Schrödinger equation with Robin boundary condition

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Abstract: We study a quasilinear Schrödinger equation with Robin boundary condition. Using the variational methods and the truncation techniques, we prove the existence of two positive solutions when the parameter λ is large enough. We also establish the existence of infinitely many high energy solutions by using Fountain Theorem when $\lambda > 1$.

Keywords: quasilinear Schrödinger equation; Robin boundary; multiple solutions; Fountain Theorem
Mathematics Subject Classification: 35J60, 35J20

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following quasilinear Robin problem

$$\begin{cases} -\Delta u - \Delta(u^2)u + a(x)u = \lambda f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(x)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\frac{\partial u}{\partial n} = \nabla u \cdot n$ with $n(x)$ being the outward unit normed vector to $\partial\Omega$ at its point x, $a(x) \in L^\infty(\Omega)$ satisfying $\operatorname{ess}\inf_{x\in\Omega} a(x) > 0$, $\beta(x) \in C^{0,\tau}(\partial\Omega, \mathbb{R}_0^+)$ for some $\tau \in (0,1)$ and $\beta(x) \neq 0$. $\lambda > 0$ is a real parameter.

The Robin boundary problems with Laplacian operator have been widely investigated in recent years ([1–6]). For example, Papageorgiou et al. in [4] considered the Robin problems driven by negative Laplacian with a superlinear reaction and proved the existence and multiplicity theorems by variational methods. However, the Robin boundary problems with quasilinear Schrödinger operator have not been dealt with yet. Based on the above issues, this paper aims to study the existence and multiplicity of solutions to problem (1.1). In our work, one of the main difficulties is that there is no suitable space on which the energy functional is well defined. To the best of our knowledge, the first existence result for the equation of the following form

$$-\Delta u - \Delta(u^2)u + a(x)u = \lambda f(x,u), \quad x \in \mathbb{R}^N \tag{1.2}$$

is due to Poppenberg, Schmitt and Wang([7]), and their approach to the problem is the constrained minimization argument. Since then, some ideas and approaches were developed to overcome these difficulties. Liu and Wang, in [8], reduced the quasilinear equation to a semilinear one by the change of variable. In [9], using the same methods in [8], Colin and Jeanjean considered the problem (1.2) on the usual Sobolev space. In this paper, we will use the change of variable to overcome the main difficulties.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(x,0) = 0$ for a.e. $x \in \Omega$ and let

$$F(x,t) := \int_0^t f(x,\xi)d\xi, \quad (x,t) \in \Omega \times \mathbb{R}.$$

We first posit the following assumptions on f.

(f1) $|f(x,t)| \le c_0(1+|t|^{r-1})$ for a.e. $x \in \Omega$, where $c_0 > 0, 4 < r < 2\cdot 2^*$;
(f2) $\lim\limits_{t\to+\infty} \frac{F(x,t)}{t^4} = 0$ and $\lim\limits_{t\to+\infty} F(x,t) = -\infty$ uniformly for $x \in \Omega$;
(f3) There exists $\tilde{u} \in L^r(\Omega)$ such that $\int_\Omega F(x,\tilde{u})dx > 0$;
(f4) There is a constant $\delta > 0$ such that $f(x,t) \le 0$ for a.e. $x \in \Omega$ and $t \in (0,\delta)$;
(f5) For every $\rho > 0$, there exists $\mu_\rho > 0$ such that $t \mapsto \frac{f(x,t)}{\sqrt{1+2t^2}} + \mu_\rho t$ is nondecreasing on $[0,\rho]$;

Remark 1.1. The following function satisfies hypotheses (f1)-(f5), for simplicity we drop the x-dependence,

$$f(t) = \begin{cases} 0, & \text{if } t \le 0, \\ -t^{k_1-1} + 2t^{k_2-1}, & \text{if } t \in (0,1], \\ 2t^{k_3-1} - t^{k_4-1}, & \text{if } t > 1, \end{cases}$$

where $3 < k_1 < k_2 < 2k_1 < +\infty$ and $1 < k_3 < k_4 < 4$.

The theorem below is the first result of our paper.

Theorem 1.1. *If hypotheses (f1)-(f5) hold, then there exists a critical parameter value $\lambda_* > 0$ such that for all $\lambda > \lambda_*$, problem (1.1) has at least two positive solutions $v_0, v_1 \in \mathrm{int}(C_+)$ with $v_0 \le v_1$ in Ω.*

To study further the multiplicity of solutions for problem (1.1) under the assumption (f1), we give some other conditions on f as follows:

(f6) $\lim\limits_{|t|\to+\infty} \frac{F(x,t)}{t^4} = +\infty$ uniformly with respect to $x \in \Omega$, and there exist $\alpha \in (\max\{1, (r-4)\frac{N}{4}\}, 2^*)$, $\hat{c}_0 > 0$ such that

$$0 < \hat{c}_0 \le \liminf_{|t|\to+\infty} \frac{f(x,t)t - 4F(x,t)}{|t|^{2\alpha}}$$

uniformly for $x \in \Omega$.

(f7) $f(x,-t) = -f(x,t)$.

Remark 1.2. The following function satisfies hypotheses (f1), (f6) and (f7):

$$f(t) = t^3 \ln(1+|t|) \text{ for all } t \in \mathbb{R}.$$

Theorem 1.2. *If hypotheses (f1), (f6), (f7) hold and $\lambda > 1$, then problem (1.1) has infinitely many high energy solutions in $H^1(\Omega)$.*

The paper is organized as follows. In Section 2, we give some preliminary results, which will be used in this paper. We prove Theorem 1.1 and Theorem 1.2 in Section 3 and Section 4 respectively.

2. Preliminaries and variational setting

In this paper, the main working spaces are $H^1(\Omega)$, $C^1(\bar{\Omega})$ and $L^q(\partial\Omega)$, $1 \leq q \leq +\infty$. We denote the norm of $L^q(\Omega)$ and $H^1(\Omega)$ by

$$\|u\|_q = \left(\int_\Omega |u|^q dx\right)^{1/q}, \quad \|u\| = \left(\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 dx\right)^{1/2}.$$

The Banach space $C^1(\bar{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ := \{u \in C^1(\bar{\Omega}) : u(x) \geq 0, \ \forall\, x \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$\mathrm{int}(C_+) := \{u \in C_+ : u(x) > 0, \ \forall\, x \in \bar{\Omega}\}.$$

On $\partial\Omega$ we will employ the $(N-1)$-dimensional Hausdorff measure σ. By applying this measure we can define the Lebesgue spaces $L^q(\partial\Omega)$ $(1 \leq q \leq +\infty)$. The Trace Theorem says that there exists a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, known as "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega}, \quad \forall\, u \in H^1(\Omega) \cap C(\bar{\Omega}).$$

Consequently, the trace map extends the notion of boundary values to any Sobolev function, and

$$\mathrm{im}\gamma_0 = H^{\frac{1}{2},2}(\partial\Omega), \quad \ker\gamma_0 = H_0^1(\Omega).$$

Moreover, the trace map γ_0 is compact from $H^1(\Omega)$ into $L^q(\partial\Omega)$ with $q \in [1, \frac{2(N-1)}{N-2})$ if $N \geq 3$ and $q \in [1,+\infty)$ if $N = 1,\ 2$. In the sequel, for the sake of notational simplicity, the use of the trace map γ_0 will be dropped. The restrictions of all Sobolev functions on the boundary $\partial\Omega$ are understood in the sense of traces.

We know that (1.1) is the Euler-Lagrange equation associated to the energy functional

$$I(u) = \frac{1}{2}\left(\int_\Omega |\nabla u|^2(1+2u^2)dx + \int_\Omega a(x)u^2 dx + \int_{\partial\Omega} \beta(x)u^2(1+u^2)d\sigma\right) - \lambda \int_\Omega F(x,u)dx.$$

Unfortunately, the functional I could not be well defined in $H^1(\Omega)$ for $N \geq 3$. To overcome this difficulty, we use the argument developed in [9]. More precisely, we make the change of variable $u = g(v)$, which is defined by

$$\begin{aligned} g'(t) &= \frac{1}{\sqrt{1+2g^2(t)}} \quad \text{on}[0,+\infty), \\ g(t) &= -g(-t) \qquad\quad \text{on}(-\infty,0]. \end{aligned}$$

Now we present some important results about the change of variable $u = g(v)$.

Lemma 2.1. ([10]) *The function g and its derivative satisfy the following properties:*

(i) *g is uniquely defined, C^2 and invertible;*
(ii) $|g'(t)| \leq 1$ *for all* $t \in \mathbb{R}$*;*
(iii) $|g(t)| \leq t$ *for all* $t \in \mathbb{R}$*;*
(iv) $\frac{g(t)}{t} \to 1$*, as* $t \to 0$*;*
(v) $|g(t)| \leq 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$*;*
(vi) $g(t)/2 < tg'(t) < g(t)$ *for all* $t > 0$*, and the reverse inequalities hold for* $t < 0$*;*
(vii) $\frac{g(t)}{\sqrt{t}} \to a_1 > 0$*, as* $t \to +\infty$*;*
(viii) $|g(t)g'(t)| \leq 1/\sqrt{2}$ *for all* $t \in \mathbb{R}$*;*
(ix) *There exists a positive constant C_1 such that*

$$|g(t)| \geq \begin{cases} C_1|t|, & |t| \leq 1, \\ C_1|t|^{1/2}, & |t| \geq 1. \end{cases}$$

Therefore, after the change of variable, the functional $I(u)$ can be rewritten in the following way

$$\begin{aligned} J(v) =& \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \frac{1}{2}\int_\Omega a(x)g^2(v)dx + \frac{1}{2}\int_{\partial\Omega} \beta(x)g^2(v)(1+g^2(v))d\sigma \\ &- \lambda \int_\Omega F(x,g(v))dx. \end{aligned} \tag{2.1}$$

From Lemma 2.1, by a standard argument, we see that J is well defined in $H^1(\Omega)$ and $J \in C^1$. In addition,

$$\begin{aligned} J'(v)\varphi =& \int_\Omega \nabla v \nabla\varphi dx + \int_\Omega a(x)g(v)g'(v)\varphi dx + \int_{\partial\Omega} \beta(x)g(v)g'(v)(1+2g^2(v))\varphi d\sigma \\ &- \lambda\int_\Omega f(x,g(v))g'(v)\varphi dx \end{aligned} \tag{2.2}$$

for all $v, \varphi \in H^1(\Omega)$.

It is easy to see that the critical points of J correspond exactly to the weak solutions of the semilinear problem

$$\begin{cases} -\Delta v + a(x)g(v)g'(v) = \lambda f(x,g(v))g'(v) & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} + \beta(x)g(v)g'(v)(1+2g^2(v)) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Hence, to prove our main results, we shall look for solutions of problem (2.3), i.e., the critical points of the functional J.

Let X be a Banach space and X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X, X^*). We now give the definitions of the $(\mathrm{PS})_c$ condition and Cerami condition in X as follows.

Definition 2.2. Let $\psi \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. We say that ψ satisfies the $(PS)_c$ condition, if every sequence $\{u_n\} \subseteq X$ such that

$$\psi(u_n) \to c,\ \psi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty$$

admits a strongly convergent subsequence.

Definition 2.3. Let $\psi \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. We say that ψ satisfies the Cerami condition, if every sequence $\{u_n\} \subseteq X$ such that

$$\psi(u_n) \to c,\ (1 + \|u_n\|_X)\psi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty$$

admits a strongly convergent subsequence.

The following two lemmas are known as Mountain Pass Theorem and Fountain Theorem.

Lemma 2.4. (Mountain Pass Theorem, [11]) *If $\psi \in C^1(X,\mathbb{R})$ satisfies the $(PS)_c$-condition, there are $u_0,\ u_1 \in X$ with $\|u_1 - u_0\|_X > \varrho > 0$,*

$$\max\{\psi(u_0),\ \psi(u_1)\} < \inf\{\psi(u) : \|u - u_0\|_X = \varrho\} = m_\varrho$$

and $c = \inf\limits_{\gamma\in\Gamma} \max\limits_{0\le\tau\le1} \psi(\Gamma(\tau))$ where $\Gamma = \{\gamma \in C([0,1],\ X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}$, then $c \ge m_\varrho$ and c is a critical value of ψ.

If X is a reflexive and separable Banach space, then there are $e_j \in X$ and $e_j^* \in X^*$ such that

$$X = \overline{\operatorname{span}\{e_j | j = 1,\ 2,\ \cdots\}},\ X^* = \overline{\operatorname{span}\{e_j^* | j = 1,\ 2,\ \cdots\}},$$

$$\langle e_i, e_j^*\rangle = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

For convenience, we write

$$X_j := span\{e_j\},\ Y_k := \oplus_{j=0}^{k} X_j,\ Z_k := \overline{\oplus_{j=k}^{\infty} X_j}. \tag{2.4}$$

Lemma 2.5. (Fountain Theorem, [12]) *Let $\psi \in C^1(X,\mathbb{R})$ be an even functional. If, for every $k \in \mathbb{N}$, there exists $\rho_k > \gamma_k > 0$ such that*

(A1) $a_k := \sup\limits_{u\in Y_k, \|u\|_X=\rho_k} \psi(u) \le 0$,

(A2) $b_k := \inf\limits_{u\in Z_k, \|u\|_X=\gamma_k} \psi(u) \to \infty$ *as* $k \to \infty$,

(A3) ψ *satisfies the Cerami condition for every* $c > 0$.

Then ψ has an unbounded sequence of critical values.

3. Existence of positive solutions

In this section we prove the existence of positive solutions of problem (1.1). For simplicity, we may take $f(x,t) = 0$ for a.e. $x \in \Omega$, all $t \le 0$.

Proposition 1. *If (f1)–(f3) and (f5) hold, then problem (2.3) admits a solution $v_0 \in int(C_+)$.*

Proof. We consider the C^1-functional $J_+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} J_+(v) =& \frac{1}{2}\left(\|\nabla v\|_2^2 + \|v^-\|_2^2 + \int_\Omega a(x)g^2(v)dx + \int_{\partial\Omega} \beta(x)g^2(v)(1 + g^2(v))d\sigma\right) \\ &- \lambda \int_\Omega F(x, g(v^+))dx. \end{aligned} \tag{3.1}$$

We claim that J_+ is coercive. Arguing by contradiction, assume that we can find a sequence $\{v_n\}_{n\ge1} \subseteq H^1(\Omega)$ and a constant M such that

$$\|v_n\| \to +\infty \text{ as } n \to \infty \text{ and } J_+(v_n) \le M \text{ for all } n \ge 1. \tag{3.2}$$

Then from (3.2) we have

$$\begin{aligned} M \ge& \frac{1}{2}\left(\|\nabla v_n\|_2^2 + \|v_n^-\|_2^2 + \int_\Omega a(x)g^2(v_n)dx + \int_{\partial\Omega} \beta(x)g^2(v_n)(1 + g^2(v_n))d\sigma\right) \\ &- \lambda \int_\Omega F(x, g(v_n^+))dx \\ \ge& \frac{1}{2}(\|\nabla v_n\|_2^2 + \|v_n^-\|_2^2) - \lambda \int_\Omega F(x, g(v_n^+))dx \end{aligned} \tag{3.3}$$

for all $n \ge 1$. Hypotheses (f1) and (v) of Lemma 2.1 imply that there exist $c_2,\ c_3 > 0$ such that

$$|F(x, g(v_n^+))| \le c_2(1 + |g(v_n^+)|^r) \le c_3(1 + |v_n^+|^{\frac{r}{2}}).$$

Combining this with (3.2), (3.3), we get

$$\|v_n^+\| \to +\infty \text{ as } n \to \infty.$$

Let $w_n = \frac{v_n^+}{\|v_n^+\|}$, $n \ge 1$. Then $\|w_n\| = 1$ for all $n \ge 1$, and so we may assume that

$$w_n \rightharpoonup w \text{ in } H^1(\Omega),\ w_n \to w \text{ in } L^\theta(\Omega) \text{ and in } L^\theta(\partial\Omega) \text{ for each } \theta \in (1, 2^*). \tag{3.4}$$

From (3.3) we have

$$\frac{1}{2}\|\nabla w_n\|_2^2 - \lambda \int_\Omega \frac{F(x, g(v_n^+))}{\|v_n^+\|^2}dx \le \frac{M}{\|v_n^+\|^2} \quad \text{for all } n \ge 1. \tag{3.5}$$

Invoking (f1), (f2) and (v) of Lemma 2.1, we can find $c_\epsilon > 0$ such that

$$|F(x, g(t))| \le \frac{\epsilon}{2}t^2 + c_\epsilon \quad \text{for a.e. } x \in \Omega,\ \text{all } t \ge 0,$$

which yields, for $n \geq 1$ large enough,

$$\int_{\Omega} \frac{|F(x, g(v_n^+))|}{\|v_n^+\|^2} dx \leq \frac{\epsilon}{2} + \frac{c_\epsilon |\Omega|_N}{\|v_n^+\|^2} \leq \epsilon. \tag{3.6}$$

Passing to the limsup as $n \to \infty$ in (3.5) , and using (3.6), we have

$$\limsup_{n \to +\infty} \|\nabla w_n\|_2^2 \leq 0. \tag{3.7}$$

In addition, from (3.4) we have $\|\nabla w\|_2^2 \leq \liminf_{n \to +\infty} \|\nabla w_n\|_2^2$. Combining this with (3.7), we obtain that $\nabla w_n \to \nabla w = 0$ in $L^2(\Omega)$. Hence,

$$w_n \to w \text{ in } H^1(\Omega),$$

and so $\|w\| = 1,\ w \geq 0$. Since $\nabla w = 0$, we have $w = 1/\sqrt{|\Omega|_N}$, and $v_n^+(x) \to +\infty$ for a.e. $x \in \Omega$.

On the other hand, combining with (f1), (f2), (3.3), (vii) of Lemma 2.1 and Fatou's lemma, we get

$$M \geq J_+(v_n) \geq -\lambda \int_{\Omega} F(x, g(v_n^+)) dx \to +\infty \text{ as } n \to \infty,$$

which is impossible. Thus, J_+ is coercive.

Furthermore, from Lemma 2.1, Sobolev embedding theorem and trace theorem, we see that J_+ is sequentially weak lower semicontinuous. By Weierstrass-Tonelli theorem we can find $v_0 \in H^1(\Omega)$ such that

$$J_+(v_0) = \inf\{J_+(v) : v \in H^1(\Omega)\}. \tag{3.8}$$

Consider the integral functional $J_F : L^r(\Omega) \to \mathbb{R}$ defined by

$$J_F(v) = \int_{\Omega} F(x, g(v)) dx.$$

From hypothesis (f1), (i) of Lemma 2.1 and the dominated convergence theorem, we see that J_F is continuous. By hypothesis (f3) and (ix) of Lemma 2.1, there exists $\tilde{v} \in L^r(\Omega)$ such that

$$J_F(\tilde{v}) > 0.$$

Exploiting the density of $H^1(\Omega)$ in $L^r(\Omega)$, we can find $\bar{v} \in H^1(\Omega)$ such that

$$J_F(\bar{v}) > 0. \tag{3.9}$$

Therefore, from (3.1) and (3.9), there exists $\lambda_* > 0$ such that

$$J_+(v_0) < J_+(\bar{v}) < 0 = J_+(0)$$

for all $\lambda > \lambda_*$. This ensure that $v_0 \neq 0$. From (3.8), we have $J'_+(v_0) = 0$, that is,

$$\int_{\Omega} \nabla v_0 \nabla \varphi dx - \int_{\Omega} v_0^- \varphi dx + \int_{\Omega} a(x) g(v_0) g'(v_0) \varphi dx$$

$$+ \int_{\partial\Omega} \beta(x) g(v_0) g'(v_0)(1 + 2g^2(v_0)) \varphi d\sigma = \lambda \int_{\Omega} f(x, g(v_0^+)) g'(v_0^+) \varphi dx \tag{3.10}$$

for all $\varphi \in H^1(\Omega)$. In (3.10) choosing $\varphi = -v_0^-$, we obtain

$$\|\nabla v_0^-\|_2^2 + \|v_0^-\|_2^2 \leq 0,$$

which implies $v_0 \geq 0$ a.e. in Ω. Hence,

$$\begin{aligned}&\int_\Omega \nabla v_0 \nabla \varphi dx + \int_\Omega a(x) g(v_0) g'(v_0) \varphi dx \\ &+ \int_{\partial\Omega} \beta(x) g(v_0) g'(v_0)(1 + 2g^2(v_0)) \varphi d\sigma = \lambda \int_\Omega f(x, g(v_0)) g'(v_0) \varphi dx\end{aligned} \tag{3.11}$$

for all $\varphi \in H^1(\Omega)$. That is to say v_0 is a weak solution of problem (2.3). By Theorem 4.1 in [13], we know that $v_0 \in L^\infty(\Omega)$. Furthermore, applying Theorem 2 in [14], we have $v_0 \in C_+ \backslash \{0\}$. Taking $\rho = \|g(v_0)\|_\infty$, from (f5), there exists $\mu_\rho > 0$ such that

$$f(x, g(v_0)) g'(v_0) + \mu_\rho g(v_0) \geq 0 \quad \text{for a. e. } x \in \Omega. \tag{3.12}$$

From (2.3), (3.12) and (iii) of Lemma 2.1 we obtain

$$-\Delta v_0 + (\lambda \mu_\rho + \|a(x)\|_\infty) v_0 \geq 0 \quad \text{for a. e. } x \in \Omega.$$

Therefore, $v_0 \in \text{int}(C_+)$ by the strong maximum principle(see [15]). □

To look for another positive solution to problem (2.3), we consider the functional $\hat{J}_+ \in C^1(H^1(\Omega), \mathbb{R})$ defined by

$$\hat{J}_+(v) = \frac{1}{2}(\|\nabla v\|_2^2 + \|v^-\|_2^2 + \|(v - v_0)^+\|_2^2 + \int_{\partial\Omega} \hat{K}(x, v) d\sigma) - \int_\Omega \hat{F}(x, v) dx, \tag{3.13}$$

where v_0 is given in Proposition 1, $\hat{F}(x, t) = \int_0^t \hat{f}(x, s) ds$, $\hat{K}(x, t) = \int_0^t \hat{k}(x, s) ds$, and

$$\hat{f}(x, s) = \begin{cases} 0 & \text{if } s \leq 0, \\ \lambda f(x, g(s)) g'(s) - a(x) g(s) g'(s) & \text{if } 0 < s < v_0(x), \\ \lambda f(x, g(v_0)) g'(v_0) - a(x) g(v_0) g'(v_0) & \text{if } s \geq v_0(x), \end{cases} \tag{3.14}$$

for all $(x, s) \in \Omega \times \mathbb{R}$.

$$\hat{k}(x, s) = \begin{cases} 0 & \text{if } s \leq 0, \\ \beta(x) g(s) g'(s)(1 + 2g^2(s)) & \text{if } 0 < s < v_0(x), \\ \beta(x) g(v_0) g'(v_0)(1 + 2g^2(v_0)) & \text{if } s \geq v_0(x), \end{cases} \tag{3.15}$$

for all $(x, s) \in \partial\Omega \times \mathbb{R}$. It is easy to see that $\hat{f}(x, s)$ and $\hat{k}(x, s)$ are Carathéodory functions.

Proposition 2. *If (f1), (f5) hold and v is a critical point of $\hat{J}_+(v)$, then v is a solution of problem (2.3), and $v \in [0, v_0] \cap int(C_+)$.*

Proof. The assumption yields the following equality:

$$\int_\Omega \nabla v \nabla \varphi dx - \int_\Omega v^- \varphi dx + \int_\Omega (v - v_0)^+ \varphi dx + \int_{\partial\Omega} \hat{k}(x, v)\varphi d\sigma = \int_\Omega \hat{f}(x, v)\varphi dx \tag{3.16}$$

for all $\varphi \in H^1(\Omega)$. Taking $\varphi = -v^-$ in (3.16), we obtain (arguing as in Proposition 1) $v \geq 0$ a.e. in Ω.

On the one hand, letting $\varphi = (v - v_0)^+$ in (3.16) again, we have

$$\begin{aligned} &\int_\Omega \nabla v \nabla (v - v_0)^+ dx + \|(v - v_0)^+\|_2^2 + \int_\Omega a(x)g(v_0)g'(v_0)(v - v_0)^+ dx \\ &+ \int_{\partial\Omega} \beta(x)g(v_0)g'(v_0)(1 + 2g^2(v_0))(v - v_0)^+ d\sigma = \lambda \int_\Omega f(x, g(v_0))g'(v_0)(v - v_0)^+ dx. \end{aligned} \tag{3.17}$$

On the other hand, the fact that $v_0 > 0$ is a critical point of J_+ yields

$$\begin{aligned} &\int_\Omega \nabla v_0 \nabla (v - v_0)^+ dx + \int_\Omega a(x)g(v_0)g'(v_0)(v - v_0)^+ dx \\ &+ \int_{\partial\Omega} \beta(x)g(v_0)g'(v_0)(1 + 2g^2(v_0))(v - v_0)^+ d\sigma = \lambda \int_\Omega f(x, g(v_0))g'(v_0)(v - v_0)^+ dx. \end{aligned} \tag{3.18}$$

Then from (3.17) and (3.18), we obtain

$$\|\nabla(v - v_0)^+\|_2^2 + \|(v - v_0)^+\|_2^2 = 0,$$

and $(v - v_0)^+ = 0$. Therefore $v \in [0, v_0]$ a.e. in Ω. Now, (3.16) becomes

$$\begin{aligned} &\int_\Omega \nabla v \nabla \varphi dx + \int_\Omega a(x)g(v)g'(v)\varphi dx \\ &+ \int_{\partial\Omega} \beta(x)g(v)g'(v)(1 + 2g^2(v))\varphi d\sigma = \lambda \int_\Omega f(x, g(v))g'(v)\varphi dx \end{aligned}$$

for all $\varphi \in H^1(\Omega)$. So v is a weak solution of problem (2.3). Reasoning as in the proof of Proposition 1, we have $v \in \mathrm{int}(C_+)$. □

Lemma 3.1. *If hypotheses (f1)–(f5) hold, then $\hat{J}_+$ satisfies the $(PS)_c$ condition.*

Proof. We first check that $\hat{J}_+$ is coercive. Since $v_0 \in C^1(\bar{\Omega})$, using (f1), (iii) of Lemma 2.1 and (3.14), we can find a constant $c_4 > 0$ such that

$$\left| \int_\Omega \hat{F}(x, v)dx \right| \leq \lambda c_4 \|v\|_2 \quad \text{for all } v \in H^1(\Omega). \tag{3.19}$$

Because of (3.15) we have

$$\int_{\partial\Omega} \hat{K}(x, v)d\sigma \geq 0 \quad \text{for all } v \in L^2(\partial\Omega). \tag{3.20}$$

Moreover, for every $v \in H^1(\Omega)$ we see that

$$\begin{aligned}\|(v-v_0)^+\|_2^2 &= \|(v^+-v_0)^+\|_2^2 \\ &= \|v-v_0\|_2^2 - \int_{\{v^+\le v_0\}} (v_0-v)^2 dx \\ &\ge \frac{1}{2}\|v^+\|_2^2 - \frac{1}{2}\|v_0\|_2^2 - \int_{\{v^+\le v_0\}} v_0^2 dx \\ &\ge \frac{1}{2}\|v^+\|_2^2 - 2\|v_0\|_2^2.\end{aligned} \tag{3.21}$$

From (3.13), (3.19), (3.20) and (3.21) we derive

$$\hat{J}_+(v) \ge \frac{1}{4}\|v\|^2 - \|v_0\|_2^2 - \lambda c_4\|v\|_2,\ \forall\, v \in H^1(\Omega).$$

It is easy to see that $\hat{J}_+$ is coercive.

Now let $\{v_n\}_{n\ge1}$ be a sequence such that

$$\hat{J}_+(v_n) \to c,\ \hat{J}'_+(v_n) \to 0 \text{ in } H^1(\Omega) \text{ as } n \to \infty. \tag{3.22}$$

The coercivity of $\hat{J}_+$ and (3.22) imply that $\{v_n\}_{n\ge1}$ is bounded in $H^1(\Omega)$. Hence, there is $v \in H^1(\Omega)$ such that, along a relabeled subsequence $\{v_n\}_{n\ge1}$,

$$v_n \rightharpoonup v \text{ in } H^1(\Omega),\ v_n \to v \text{ in } L^\theta(\Omega) \text{ and in } L^\theta(\partial\Omega) \text{ for each } \theta \in (1, 2^*). \tag{3.23}$$

The second part of (3.22) yields

$$\begin{aligned}&\int_\Omega \nabla v_n \nabla\varphi dx - \int_\Omega v_n^-\varphi dx + \int_\Omega (v_n - v_0)^+\varphi dx \\ &+ \int_{\partial\Omega} \hat{k}(x, v_n)\varphi d\sigma - \int_\Omega \hat{f}(x, v_n)\varphi dx \to 0\end{aligned} \tag{3.24}$$

as $n \to \infty$, for all $\varphi \in H^1(\Omega)$. In (3.24) choosing $\varphi = v_n - v$, using (f1), (3.23) and Lemma 2.1, we find

$$\lim_{n\to\infty} \int_\Omega \nabla v_n \nabla(v_n - v)dx = 0.$$

Therefore $\|v_n\|^2 \to \|v\|^2$ as $n \to \infty$. Recalling that the Hilbert space $H^1(\Omega)$ is locally uniformly convex, we obtain $v_n \to v$ in $H^1(\Omega)$. This shows that $\hat{J}_+$ satisfies the $(PS)_c$ condition. □

In the next lemma, we state a consequence which depends on the regularity results in [13] and Theorem 2 in [14]. The proof is similar to Proposition 3 in [2].

Lemma 3.2. *If (f1) holds and $v \in H^1(\Omega)$ is a local $C^1(\bar{\Omega})$-minimizer of $\hat{J}_+$, then $v \in C^{1,\tau}(\bar{\Omega})$ and it is a local $H^1(\Omega)$-minimizer of $\hat{J}_+$.*

Proposition 3. *If (f1)-(f5) hold, then $\hat{J}_+$ admits a critical point $v_1 \in H^1(\Omega)$ different from 0 and v_0.*

Proof. Let $v \in C^1(\bar{\Omega})$ with $0 < \|v\|_{C^1(\bar{\Omega})} \leq \delta$. From (f4) and (iv) of Lemma 2.1, we get $\hat{F}(x, g(v)) \leq 0$ for a.e. $x \in \Omega$ which implies

$$\hat{J}_+(v) \geq -\lambda \int_\Omega \hat{F}(x, g(v))dx > 0.$$

This shows that 0 is a local $C^1(\bar{\Omega})$-minimizer of $\hat{J}_+$. Applying Lemma 3.2, we derive that 0 is a local $H^1(\Omega)$-minimizer of $\hat{J}_+$.

If 0 is not a strict local minimizer of $\hat{J}_+$, the result is obvious because any neighborhood of 0 in $H^1(\Omega)$ contains another critical point of $\hat{J}_+$.

We next only need to discuss that 0 is a strict local minimizer of $\hat{J}_+$. In such a condition, we can find a sufficiently small $\varrho \in (0, 1)$ such that

$$\hat{J}_+(0) < \inf\{\hat{J}_+(v) : \|v\| = \varrho\} = \hat{m}_\varrho. \tag{3.25}$$

Note that

$$\hat{J}_+(v_0) = J_+(v_0) < J_+(0) = \hat{J}_+(0) = 0.$$

This fact, together with (3.25) and Lemma 3.1 permit the use of the Mountain Pass Theorem, which yields a critical point v_1 of $\hat{J}_+$ different from 0 and v_0. □

Theorem 1.1 follows immediately from Proposition 1, Proposition 2 and Proposition 3.

4. Existence of infinitely many solutions

Lemma 4.1. *If (f1) and (f6) hold, then, for any $c > 0$, the functional J satisfies the Cerami condition.*

Proof. Let $\{v_n\}_{n\geq 1} \subseteq H^1(\Omega)$ be a Cerami sequence of J, that is,

$$J(v_n) \to c,\ (1 + \|v_n\|)J'(v_n) \to 0 \text{ in } H^1(\Omega) \text{ as } n \to \infty. \tag{4.1}$$

First we prove the boundedness of $\{v_n\}_{n\geq 1}$. By (4.1) we have

$$\begin{aligned} c + 1 \geq & \frac{1}{2}\Big(\int_\Omega |\nabla v_n|^2 dx + \int_\Omega a(x)g^2(v_n)dx + \int_{\partial\Omega} \beta(x)g^2(v_n)(1 + g^2(v_n))d\sigma\Big) \\ & - \lambda \int_\Omega F(x, g(v_n))dx \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \frac{\epsilon_n\|\varphi\|}{1 + \|v_n\|} \geq & J'(v_n)\varphi \\ = & \int_\Omega \nabla v_n \nabla\varphi dx + \int_\Omega a(x)g(v_n)g'(v_n)\varphi dx + \int_{\partial\Omega} \beta(x)g(v_n)g'(v_n)(1 + 2g^2(v_n))\varphi d\sigma \\ & - \lambda \int_\Omega f(x, g(v_n))g'(v_n)\varphi dx \end{aligned} \tag{4.3}$$

for all $\varphi \in H^1(\Omega)$ with $\epsilon_n \ll 1$. Choosing $\varphi = \varphi_n = -\frac{g(v_n)}{g'(v_n)}$, we obtain

$$|\nabla\varphi_n| = |\nabla v_n|(1 + 2(g(v_n)g'(v_n))^2).$$

Because of Lemma 2.1(vi, viii), $\varphi_n \in H^1(\Omega)$. Therefore,

$$\begin{aligned} 2\epsilon_n \geq J'(v_n)\varphi_n = &-\int_\Omega |\nabla v_n|^2((1+2(g(v_n)g'(v_n))^2)dx - \int_\Omega a(x)g^2(v_n)dx \\ &- \int_{\partial\Omega} \beta(x)g^2(v_n)(1+2g^2(v_n))d\sigma + \lambda \int_\Omega f(x,g(v_n))g(v_n)dx. \end{aligned} \tag{4.4}$$

Now using (f6) and (viii) of Lemma 2.1, it follows from (4.2) and (4.4) that

$$\begin{aligned} c+1+\frac{\epsilon_n}{2} \geq &\int_\Omega (\frac{1}{2}-\frac{1}{4}(1+2(g(v_n)g'(v_n))^2)|\nabla v_n|^2 dx + \frac{1}{4}\int_\Omega a(x)g^2(v_n)dx \\ &+ \frac{1}{4}\int_{\partial\Omega} \beta(x)g^2(v_n)d\sigma + \frac{\lambda}{4}\int_\Omega (f(x,g(v_n))g(v_n) - 4F(x,g(v_n)))dx \\ \geq &\frac{\lambda}{4}\int_\Omega (f(x,g(v_n))g(v_n) - 4F(x,g(v_n)))dx. \end{aligned} \tag{4.5}$$

On the other hand, from (f1) and (f6) we can find $\tilde{c}_0 \in (0, \hat{c}_0)$ and $c_5 > 0$ such that

$$f(x,t) - 4F(x,t) \geq \tilde{c}_0|t|^{2\alpha} - c_5 \quad \text{a.e. } x \in \Omega. \tag{4.6}$$

By (4.5) and (4.6), we conclude that $\{g(v_n)\}_{n\geq 1}$ is bounded in $L^{2\alpha}(\Omega)$. Furthermore, due to (ix) of Lemma 2.1,

$$\{v_n\}_{n\geq 1} \text{ is bounded in } L^\alpha(\Omega). \tag{4.7}$$

Suppose that $N \neq 2$. By (f6), without loss of generality, we may assume that $\alpha < \frac{r}{2} < 2^*$. So, we can find $z \in (0,1)$ such that

$$\frac{2}{r} = \frac{1-z}{\alpha} + \frac{z}{2^*}. \tag{4.8}$$

Invoking the interpolation inequality, Sobolev embedding theorem and (4.7) we have

$$\|v_n\|_{\frac{r}{2}} \leq \|v_n\|_\alpha^{1-z}\|v_n\|_{2^*}^z \leq c_6\|v_n\|^z \tag{4.9}$$

for some $c_6 > 0$. Taking $\varphi = -\varphi_n = \frac{g(v_n)}{g'(v_n)}$ in (4.3), we have

$$\begin{aligned} &\int_\Omega |\nabla v_n|^2((1+2(g(v_n)g'(v_n))^2)dx + \int_\Omega a(x)g^2(v_n)dx + \int_{\partial\Omega} \beta(x)g^2(v_n)(1+2g^2(v_n))d\sigma \\ &\leq 2\epsilon_n + \lambda\int_\Omega f(x,g(v_n))g(v_n)dx. \end{aligned} \tag{4.10}$$

By use of (f1) and (v) of Lemma 2.1, we can obtain that

$$f(x,g(v_n))g(v_n) \leq c_4(1+|g(v_n)|^r) \leq c_5(1+|v_n|^{\frac{r}{2}}). \tag{4.11}$$

From (f1), we assume that r is close to $2\cdot 2^*$, hence $\alpha \geq 2$. Then $\{v_n\}_{n\geq 1}$ is bounded in $L^2(\Omega)$. Combining this with (4.9), (4.10) and (4.11), we have

$$\|v_n\|^2 \leq c_6(1+\|v_n\|^{\frac{zr}{2}}). \tag{4.12}$$

Hypothesis (f6) and (4.8) lead to $zr < 4$, and consequently, $\{v_n\}_{n\geq 1}$ is bounded in $H^1(\Omega)$.

If $N = 2$, then $2^* = +\infty$ and the Sobolev embedding theorem says that $H^1(\Omega) \hookrightarrow L^\eta(\Omega)$ for all $\eta \in [1, +\infty)$. Let $\eta > \frac{r}{2} > \alpha$ and $z \in (0, 1)$ such that

$$\frac{2}{r} = \frac{1-z}{\alpha} + \frac{z}{\eta}, \tag{4.13}$$

or $zr = \frac{\eta(r-2\alpha)}{\eta-\alpha}$. Note that

$$\lim_{\eta\to+\infty} \frac{\eta(r-2\alpha)}{\eta-\alpha} = r - 2\alpha \text{ and } r - 2\alpha < 4.$$

Repeating the previous proof method, for $\eta > 1$ large enough, we again obtain that

$$\{v_n\}_{n\geq 1} \text{ is bounded in } H^1(\Omega).$$

Next, we show that $\{v_n\}_{n\geq 1}$ is strongly convergent in $H^1(\Omega)$. Since $\{v_n\}_{n\geq 1}$ is bounded, up to a subsequence(which we still denote by $\{v_n\}_{n\geq 1}$), we assume that there exists $v \in H^1(\Omega)$ such that

$$v_n \rightharpoonup v \text{ in } H^1(\Omega),\ v_n \to v \text{ in } L^\theta(\Omega) \text{ and } L^\theta(\partial\Omega) \text{ for each } \theta \in (1, 2^*). \tag{4.14}$$

Choosing $\varphi = v_n - v \in H^1(\Omega)$ in (4.3) and combining (f1) with Lemma 2.1(ii, iii, ix), we obtain $\lim\limits_{n\to\infty} \int_\Omega \nabla v_n \nabla(v_n - v)dx = 0$. Hence $\|v_n\|^2 \to \|v\|^2$ as $n \to \infty$. Recalling that the Hilbert space $H^1(\Omega)$ is locally uniformly convex, we get $v_n \to v$ in $H^1(\Omega)$. □

Lemma 4.2. ([16]) *Let $X = H^1(\Omega)$ and define Y_k, Z_k as in (2.4). If $1 \leq q < 2^*$, then*

$$d_k := \sup_{u\in Z_k, \|u\|=1} \|u\|_q \to 0 \text{ as } k \to \infty.$$

Proof of Theorem 1.2. Since $g(t)$ and $f(x, t)$ are odd respect to t, functional $J \in C^1(H^1(\Omega), \mathbb{R})$ is even obviously. Further, from Lemma 4.1, we know that J satisfies the Cerami condition. So we need only to verify J satisfying the conditions (A1) and (A2) in Lemma 2.5.

Since Y_k is finite-dimensional and all norms are equivalent on Y_k, for $v \in Y_k$, we have

$$\|v\|^2 \leq C_2\|v\|_2^2.$$

From the assumption (f6) and (ix) of Lemma 2.1, there exists $R > 0$ such that

$$F(x, g(t)) \geq 2C_2|t|^2 - C_3 \quad \text{for all } x \in \Omega,\ |t| \geq R.$$

For $v \in Y_k$, $\lambda > 1$, we have

$$J(v) \leq \|v\|^2 - \int_\Omega F(x, g(v))dx \quad \leq -C_2\|v\|_2^2 + C_3|\Omega|_N \leq -\|v\|^2 + C_3|\Omega|_N.$$

So (A1) is satisfied for sufficiently large $\|v\|$.

From (f1) and (v) of Lemma 2.1, there exists constant $C_4 > 0$ such that

$$F(x, g(t)) \leq C_4(1 + |t|^{\frac{r}{2}}), \quad \forall\, t \in \mathbb{R}. \tag{4.15}$$

Let us define

$$d_k := \sup_{v \in Z_k, \|v\|=1} \|v\|_{\frac{r}{2}}.$$

For $v \in Z_k$, using (4.15), we get

$$\begin{aligned} J(v) &\geq \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_\Omega a(x)g^2(v)dx + \frac{1}{2}\int_{\partial\Omega}\beta(x)g^2(v)(1+g^2(v))d\sigma - C_4\lambda\|v\|_{\frac{r}{2}}^{\frac{r}{2}} - C_4\lambda|\Omega|_N \\ &\geq \frac{1}{2}\|v\|^2 - C_4\lambda\|v\|_{\frac{r}{2}}^{\frac{r}{2}} - \frac{1}{2}\|v\|_2^2 - C_4\lambda|\Omega|_N \\ &\geq \frac{1}{2}\|v\|^2 - C_5 d_k^{\frac{r}{2}}\|v\|^{\frac{r}{2}} - C_6|\Omega|_N. \end{aligned}$$

Choosing $\gamma_k = (\frac{1}{2}C_5 r d_k^{\frac{r}{2}})^{\frac{2}{4-r}}$, if $\|v\| = \gamma_k$, we obtain

$$J(v) \geq (\frac{1}{2} - \frac{2}{r})(\frac{1}{2}C_5 r d_k^{\frac{r}{2}})^{\frac{4}{4-r}} - C_6|\Omega|_N.$$

By Lemma 4.2, $d_k \to 0$ and $\gamma_k \to \infty$ as $k \to \infty$, the condition (A2) is verified. In summary, all conditions of Fountain Theorem are satisfied. Thus J has an unbounded sequence of critical values. Theorem 1.2 is proved.

5. Conclusions

In this paper, we have studied a class of quasilinear Schrödinger equation in a bounded domain with Robin boundary. By giving different conditions on the reaction, we obtained two existence results of solutions to the equation. The Mountain Pass Theorem and Fountain Theorem were also employed in this study.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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