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Research article

The generalized conjugate direction method for solving quadratic inverse eigenvalue problems over generalized skew Hamiltonian matrices with a submatrix constraint

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Abstract: In this paper, we consider a class of constrained quadratic inverse eigenvalue Problem 1.1. Then, a generalized conjugate direction method is proposed to obtain the generalized skew Hamiltonian matrix solutions with a submatrix constraint. In addition, by choosing a special kind of initial matrices, it is shown that the unique least Frobenius norm solutions can be obtained consequently. Some numerical results are reported to demonstrate the efficiency of our algorithm.

Keywords: generalized conjugate direction method; constrained matrix; quadratic inverse eigenvalue problem; generalized skew Hamiltonian matrix; least Frobenius norm solution group
Mathematics Subject Classification: 15A57, 15A24

1. Introduction

The transmission eigenvalue problem in inverse scattering theory is to find $k \in C, w, \sigma \in L^2(D), w - \sigma \in H_0^2(D)$ such that

$$\begin{cases} \Delta^2 w - k^2 n(x) w = 0, & in \;\; D, \\ \Delta^2 \sigma - k^2 \sigma = 0, & in \;\; D, \end{cases} \tag{1.1}$$

where D is a bounded inhomogeneous media. $n(x)$ is the index of refraction to be determined(see, e.g. [1, 2]). The quadratic eigenvalue problem

$$Q(\lambda)x := (\lambda^2 A + \lambda B + C)x = 0, \tag{1.2}$$

can be derived from the finite element discretization of the transmission eigenvalue problem (1.1). The quadratic inverse eigenvalue problem is to construct the matrices A, B and C such that the quadratic eigenvalue problem (1.2) holds. That is, for given $X = (x_1, x_2, \cdots, x_n) \in \mathbb{C}^{n\times m}$ and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m) \in \mathbb{C}^{m\times m}$, finding $A, B, C \in \mathbb{C}^{n\times n}$ such that

$$AX\Lambda^2 + BX\Lambda + CX = 0. \tag{1.3}$$

If $A = 0$, then quadratic inverse eigenvalue problem (1.3) reduces to generalized inverse eigenvalue problem

$$BX\Lambda + CX = 0. \tag{1.4}$$

If $A = 0$, $B = I$, then quadratic inverse eigenvalue problem (1.3) reduces to inverse eigenvalue problem

$$X\Lambda + CX = 0. \tag{1.5}$$

Due to its wide application in engineering and scientific computation (see, e.g., [3–7]), inverse eigenvalue problem has raised much attention. So far, many researches are devoted to the solvability conditions and numerical solutions. Bai et al. [8] considered the sufficient and necessary conditions for the inverse eigenvalue problem with Hermitian generalized skew-Hamiltonian matrices. Dai and Liang [9] considered solving the generalized inverse eigenvalue problem for the (P, Q)-conjugate matrices and the associated approximation problem by using generalized singular value decomposition and canonical correlation decomposition. Gao et al. [10] considered that solving generalized inverse eigenvalue problem with the coefficient matrices is reflexive or anti-reflexive matrices by the direct method. Wei et al. [11] discussed the generalized inverse eigenvalue problem for Hermitian generalized Hamiltonian matrices and derived general expressions for the solution and the optimal approximation solution by the matrix decomposition theory and Hilbert space approximation theory. Mo and Li [12] discussed the inverse eigenvalue problem $AX = XB$ for Hermitian and generalized skew-Hamiltonian matrices with a leading principle submatrix constraint using singular value decomposition and Moore-Penrose generalized inverse. Cai et al. [13, 14] considered the optimal approximation problem of the generalized inverse eigenvalue problem and proposed an iterative method to obtain the least-squares solutions of generalized inverse eigenvalue problem over Hermitian-Hamiltonian matrices with a submatrix constraint.

Inspired by previous work, we consider the following constrained quadratic inverse eigenvalue problems:

Problem 1.1. *Given* $X \in \mathbb{C}^{n\times m}$, $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_m) \in \mathbb{C}^{m\times m}$, $s = (s_1, s_2, \cdots, s_p, n+1-s_p, \cdots, n+1-s_2, n+1-s_1) \in D_{2p,n}$, $t = (t_1, t_2, \cdots, t_q, n+1-t_q, \cdots, n+1-t_2, n+1-t_1) \in D_{2q,n}$, $u = (u_1, u_2, \cdots, u_r, n+1-u_r, \cdots, n+1-u_2, n+1-u_1) \in D_{2r,n}$, $A_p \in \mathbb{C}^{2p\times 2p}$, $B_q \in \mathbb{C}^{2q\times 2q}$ *and* $C_r \in \mathbb{C}^{2r\times 2r}$. *Let* $S_1 = \{X|X[s|s] = A_p, X[\overline{s}|\overline{s}] \in GSH^{(n-2p)\times(n-2p)}\}$, $S_2 = \{X|X[t|t] = B_q, X[\overline{t}|\overline{t}] \in GSH^{(n-2q)\times(n-2q)}\}$, $S_3 = \{X|X[u|u] = C_r, X[\overline{u}|\overline{u}] \in GSH^{(n-2r)\times(n-2r)}\}$, *find* $A^* \in S_1$, $B^* \in S_2$ *and* $C^* \in S_3$ *such that*

$$A^*X\Lambda^2 + B^*X\Lambda + C^*X = 0. \tag{1.6}$$

The rest of this paper is organized as follows: In Section 2, by reformulating Problem 1.1 as its equivalent Problem 2.1, we present a generalized conjugate direction method to solve the constrained quadratic inverse eigenvalue Problem. The convergence properties of the proposed algorithm are reported later; In Section 3, by choosing a special kind of initial matrices, we discuss the unique least Frobenius norm solution of Problem 1.1; Some numerical results are reported in section 4; The conclusions are given in Section 5 at last.

In our notation, let $R^{m\times n}$ and $C^{m\times n}$ be the sets of all real and complex $m\times n$ matrices, respectively. Let $A \in C^{m\times n}$, we write $Re(A), Im(A), \overline{A}, A^T, A^H, \|A\|, A^{-1}$, and $\mathcal{R}(A)$ to denote the real part, imaginary part, conjugation, transpose, conjugate transpose, Frobenius norm, inverse, and the column spaces of matrix A, respectively. For any matrix $A = (a_{ij})$, $B = (b_{ij})$, matrix $A \bigotimes B$ denotes the Kronecker product defined as $A \bigotimes B = (a_{ij}B)$. For the matrix $X = (x_1, x_2, \cdots, x_n) \in \mathbb{C}^{n\times n}$, vec$(X)$ denotes the vec operator defined as $\text{vec}(X) = (x_1^T, x_2^T, \cdots, x_n^T)^T \in C^{mn}$. Let $I_n = (e_1, e_2, \cdots, e_n)$ and $S_n = (e_n, e_{n-1}, \cdots, e_1)$ be the $n \times n$ unit matrix and reverse unit matrix, respectively, where e_i denotes its ith column of unit matrix. Let $D_{p,n} = \{d = (d_1, d_2, \cdots, d_p) : 1 \le d_1 < d_2 < \cdots < d_p \le n\}$ denote the strictly increasing sequences of p elements from $1, 2, \cdots, n$. For $s = (s_1, s_2, \cdots, s_p) \in D_{p,n}$, $t = (t_1, t_2, \cdots, t_q) \in D_{q,n}$, $u = (u_1, u_2, \cdots, u_r) \in D_{r,n}$, we assume that $E_s = (e_{s_1}, e_{s_2}, \cdots, e_{s_p}) \in \mathbb{C}^{n\times p}$, $E_t = (e_{t_1}, e_{t_2}, \cdots, e_{t_q}) \in \mathbb{C}^{n\times q}$, $E_u = (e_{u_1}, e_{u_2}, \cdots, e_{u_r}) \in \mathbb{C}^{n\times r}$. Also, $A[s|t]$ stands the submatrix of A determined by rows indexed by s and columns indexed by t. Moreover, the notation $A[\overline{s}, \overline{t}]$ represents the submatrix of A determined by deleting rows indexed by s and columns indexed by t.

Let $ASOR^{m\times m}$ stand for the set of all $m \times m$ antisymmetric orthogonal matrices, i.e.,

$$ASOR^{m\times m} = \{J | J^T J = J J^T = I_m, J = -J^T, J \in R^{m\times m}\}.$$

In the space $C^{m\times n}$, the inner product can be defined as

$$\langle A, B\rangle = Re[tr(A^H B)]. \tag{1.7}$$

Definition 1.1. *Let $J \in ASOR^{n\times n}$ be given.*

(1) *Matrix $A \in C^{n\times n}$ is called a generalized Hamiltonian matrix if $JAJ = A^H$. The set of all $n \times n$ generalized Hamiltonian matrices are denoted by $GH^{n\times n}$.*

(2) *Matrix $A \in C^{n\times n}$ is called a generalized skew Hamiltonian matrix if $JAJ = -A^H$. The set of all $n \times n$ generalized skew Hamiltonian matrices are denoted by $GSH^{n\times n}$.*

The generalized skew Hamiltonian matrices have practical applications in information theory, linear system theory, linear estimate theory and numerical analysis, see, for instance, [15]. More literatures on generalized skew Hamiltonian matrices can be found in [16–21].

2. An iterative methods for solving Problem 1.1

Notice that the solution set of Problem 1.1 is not a linear subspace. Some effective iterative algorithms cannot be extended directly to solve Problem 1.1. Therefore, we shall first transform the solution set of Problem 1.1 into the following equivalent form:

$$S_1 = \widehat{S}_1 \oplus \widetilde{S}_1 = \{X | X[s|s] = A_p, X[\overline{s}|\overline{s}] = 0\} \oplus \{X | X \in GSH^{n\times n}, X[s|s] = 0\}, \tag{2.1a}$$
$$S_2 = \widehat{S}_2 \oplus \widetilde{S}_2 = \{X | X[t|t] = B_q, X[\overline{t}|\overline{t}] = 0\} \oplus \{X | X \in GSH^{n\times n}, X[t|t] = 0\}, \tag{2.1b}$$
$$S_3 = \widehat{S}_3 \oplus \widetilde{S}_3 = \{X | X[u|u] = C_r, X[\overline{u}|\overline{u}] = 0\} \oplus \{X | X \in GSH^{n\times n}, X[u|u] = 0\}. \tag{2.1c}$$

Let

$$\begin{aligned} A &= \widetilde{A}_p + \widetilde{A}, \\ B &= \widetilde{B}_p + \widetilde{B}, \\ C &= \widetilde{C}_p + \widetilde{C}, \end{aligned}$$

then we can reformulate Problem 1.1 into the following Problem 2.1

Problem 2.1. *Given $X \in \mathbb{C}^{n\times m}$, $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_m) \in \mathbb{C}^{m\times m}$, $s = (s_1, s_2, \cdots, s_p, n+1-s_p, \cdots, n+1-s_2, n+1-s_1) \in D_{2p,n}$, $t = (t_1, t_2, \cdots, t_q, n+1-t_q, \cdots, n+1-t_2, n+1-t_1) \in D_{2q,n}$, $u = (u_1, u_2, \cdots, u_r, n+1-u_r, \cdots, n+1-u_2, n+1-u_1) \in D_{2r,n}$, $A_p \in \mathbb{C}^{2p\times 2p}$, $B_q \in \mathbb{C}^{2q\times 2q}$ and $C_r \in \mathbb{C}^{2r\times 2r}$, find $\widetilde{A}^* \in \tilde{S}_1$, $\widetilde{B}^* \in \tilde{S}_2$ and $\widetilde{C}^* \in \tilde{S}_3$ such that*

$$\widetilde{A}^* X\Lambda^2 + \widetilde{B}^* X\Lambda + \widetilde{C}^* X = \widetilde{Z}, \tag{2.2}$$

where $\widetilde{Z} = -\widetilde{A}_p X\Lambda^2 - \widetilde{B}_q X\Lambda - \widetilde{C}_r X$, $\widetilde{A}_p$, $\widetilde{B}_q$, $\widetilde{C}_r$ satisfied: $\widetilde{A}_p[s|s] = A_p$, $\widetilde{B}_q[t|t] = B_q$, $\widetilde{C}_r[u|u] = C_r$ and $\widetilde{A}_p[\bar{s}|\bar{s}] = 0$, $\widetilde{B}_q[\bar{t}|\bar{t}] = 0$, $\widetilde{C}_r[\bar{u}|\bar{u}] = 0$.

It is not difficult to find that $(\widetilde{A}^*, \widetilde{B}^*, \widetilde{C}^*)$ is a solution of Problem 2.1 if and only if $(A, B, C) = (\widetilde{A}^* + \widetilde{A}_p, \widetilde{B}^* + \widetilde{B}_q, \widetilde{C}^* + \widetilde{C}_r)$ is a solution of Problem 1.1.

Lemma 2.1. *A matrix $X \in GSH^{n\times n}$ if and only if $X^H = -JXJ$.*

Lemma 2.2. *Let $A \in \mathbb{C}^{n\times n}$ and $X \in GSH^{n\times n}$. Then*

$$\langle X, \frac{A - JA^H J}{2} \rangle = \langle A, X \rangle.$$

Proof. For any $A \in C^{n\times n}$, $X \in GSH^{n\times n}$, we have

$$\begin{aligned} \langle X, \frac{A - JA^H J}{2} \rangle &= \frac{1}{2}\langle X, A\rangle - \frac{1}{2}\langle X, JA^H J\rangle \\ &= \frac{1}{2}\langle X, A\rangle - \frac{1}{2}\langle JXJ, A^H\rangle \\ &= \frac{1}{2}\langle X, A\rangle + \frac{1}{2}\langle X^H, A^H\rangle = \langle X, A\rangle. \end{aligned}$$

Lemma 2.3. *Suppose that $X \in \mathbb{C}^{n\times n}$, $W_1 \in \widetilde{S}_1$, $W_2 \in \widetilde{S}_2$ and $W_3 \in \widetilde{S}_3$. Then $\langle E_s E_s^T X E_s E_s^T, W_1\rangle = 0$, $\langle E_t E_t^T X E_t E_t^T, W_2\rangle = 0$ and $\langle E_u E_u^T X E_u E_u^T, W_3\rangle = 0$.*

Proof. Since $W_1 \in \widetilde{S}_1$, $W_2 \in \widetilde{S}_2$ and $W_3 \in \widetilde{S}_3$, we immediately have $E_s^T W_1 E_s = 0$, $E_t^T W_2 E_t = 0$ and $E_u^T W_3 E_u = 0$. Hence

$$\begin{aligned} \langle E_s E_s^T X E_s E_s^T, W_1\rangle &= \mathrm{Re}\{\mathrm{tr}(W_1^H E_s E_s^T X E_s E_s^T)\} = \mathrm{Re}\{\mathrm{tr}(E_s E_s^T X^H E_s E_s^T W_1)\} \\ &= \mathrm{Re}\{\mathrm{tr}(E_s^T X^H E_s E_s^T W_1 E_s)\} = 0. \end{aligned}$$

Similarly, we have

$$\langle E_t E_t^T X E_t E_t^T, W_2\rangle = 0 \;\text{ and }\; \langle E_u E_u^T X E_u E_u^T, W_3\rangle = 0.$$

The proof is completed. □

In this section, we propose a generalized conjugate direction method (GCD) with the matrix form to solve Problem 2.1.

Algorithm 2.1. (GCD *method for solving Problem 2.1*)

Step 0 *Input* $X \in \mathbb{C}^{n\times m}$, $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_m) \in \mathbb{C}^{m\times m}$, $s = (s_1, s_2, \cdots, s_p, n+1-s_p, \cdots, n+1-s_2, n+1-s_1) \in D_{2p,n}$, $t = (t_1, t_2, \cdots, t_q, n+1-t_q, \cdots, n+1-t_2, n+1-t_1) \in D_{2q,n}$, $u = (u_1, u_2, \cdots, u_r, n+1-u_r, \cdots, n+1-u_2, n+1-u_1) \in D_{2r,n}$, $A_p \in \mathbb{C}^{2p\times 2p}$, $B_q \in \mathbb{C}^{2q\times 2q}$, $C_r \in \mathbb{C}^{2r\times 2r}$, $\widetilde{A}_p$, $\widetilde{B}_q$, $\widetilde{C}_r$ *satisfied:* $\widetilde{A}_p[s|s] = A_p$, $\widetilde{B}_q[t|t] = B_q$, $\widetilde{C}_r[u|u] = C_r$ *and* $\widetilde{A}_p[\bar{s}|\bar{s}] = 0$, $\widetilde{B}_q[\bar{t}|\bar{t}] = 0$, $\widetilde{C}_r[\bar{u}|\bar{u}] = 0$, *and* $\widetilde{Z} = -\widetilde{A}_p X\Lambda^2 - \widetilde{B}_q X\Lambda - \widetilde{C}_r X$. *Choose arbitrary initial matrices* $\widetilde{A}(1), P_1(1) \in \widetilde{S}_1$, $\widetilde{B}(1), P_2(1) \in \widetilde{S}_2$ *and* $\widetilde{C}(1), P_3(1) \in \widetilde{S}_3$, *respectively.Set* $k := 1$.

Step 1 *Compute*

$$R(1) = \widetilde{A}(1)X\Lambda^2 + \widetilde{B}(1)X\Lambda + \widetilde{C}(1)X - \widetilde{Z},$$

set $S(1) = R(1)$ *and* $T_j(1) = P_j(1)$ *for* $i = 1, 2, 3$*; and*

$$T(1) = P(1) = \begin{pmatrix} P_1(1) & 0 & 0 \\ 0 & P_2(1) & 0 \\ 0 & 0 & P_3(1) \end{pmatrix}.$$

Step 2 *If* $\|R(k)\| = 0$, *stop; otherwise, go to step 3.*

Step 3 *Compute*

$$W(k) = T_1(k)X\Lambda^2 + T_2(k)X\Lambda + T_3(k)X.$$

$$\begin{aligned} Z_1(k) &= S(k)(X\Lambda^2)^H - JX\Lambda^2 S^H(k)J - E_s E_s^T[S(k)(X\Lambda^2)^H - JX\Lambda^2 S^H(k)J]E_s E_s^T, \\ Z_2(k) &= S(k)(X\Lambda)^H - JX\Lambda S^H(k)J - E_t E_t^T[S(k)(X\Lambda)^H - JX\Lambda S^H(k)J]E_t E_t^T, \\ Z_3(k) &= S(k)X^H - JXS^H(k)J - E_u E_u^T[S(k)X^H - JXS^H(k)J]E_u E_u^T. \end{aligned}$$

$$Z(k) = \begin{pmatrix} Z_1(k) & 0 & 0 \\ 0 & Z_2(k) & 0 \\ 0 & 0 & Z_3(k) \end{pmatrix}.$$

Update the sequences

$$\widetilde{A}(k+1) = \widetilde{A}(k) - \alpha_k T_1(k),$$

$$\widetilde{B}(k+1) = \widetilde{B}(k) - \alpha_k T_2(k),$$

$$\widetilde{C}(k+1) = \widetilde{C}(k) - \alpha_k T_3(k),$$

$$R(k+1) = \widetilde{A}(k+1)X\Lambda^2 + \widetilde{B}(k+1)X\Lambda + \widetilde{C}(k+1)X - \widetilde{Z} = R(k) - \alpha_k W(k).$$

$$P_j(k+1) = P_j(k) - \beta_k Z_j(k),\ j = 1, 2, 3;$$

$$T_j(k+1) = P_j(k+1) + \gamma_{k+1} T_j(k),\ j = 1, 2, 3;$$

$$S(k+1) = R(k+1) + \delta_{k+1} S(k);$$

where

$$\alpha_k = \frac{\|R(k)\|^2}{\langle S(k), W(k)\rangle}, \beta_k = \frac{\|P(k)\|^2}{2\langle S(k), W(k)\rangle},$$

$$\gamma_{k+1} = \frac{\|P(k+1)\|^2}{\|P(k)\|^2},\ \ \delta_{k+1} = \frac{\|R(k+1)\|^2}{\|R(k)\|^2},$$

Denote

$$P(k+1) := \begin{pmatrix} P_1(k+1) & 0 & 0 \\ 0 & P_2(k+1) & 0 \\ 0 & 0 & P_3(k+1) \end{pmatrix},$$

$$T(k+1) := \begin{pmatrix} T_1(k+1) & 0 & 0 \\ 0 & T_2(k+1) & 0 \\ 0 & 0 & T_3(k+1) \end{pmatrix}.$$

Step 4 *Set* $k := k+1$*, go to Step 2.*

To establish the convergence property of GCD method, we first give some basic properties of Algorithm 2.1 as follows.

Lemma 2.4. *Let the sequences* $\{\widetilde{A}(k)\}$, $\{\widetilde{B}(k)\}$,$\{\widetilde{C}(k)\}$,$\{P_j(k)\}$, $\{T_j(k)\}$, $\{Z_j(k)\}$ *be generated by Algorithm 2.1. Then, we have*
(1) $\{\widetilde{A}(k)\}, \{P_1(k)\}, \{Z_1(k)\}, \{T_1(k)\} \subset \widetilde{S}_1$;
(2) $\{\widetilde{B}(k)\}, \{P_2(k)\}, \{Z_2(k)\}, \{T_2(k)\} \subset \widetilde{S}_2$;
(3) $\{\widetilde{C}(k)\}, \{P_3(k)\}, \{Z_3(k)\}, \{T_3(k)\} \subset \widetilde{S}_3$;

The proof is obvious and thus omitted here.

Lemma 2.5. *Let the sequences* $\{W(k)\}$, $\{S(k)\}$, $\{T(k)\}$ *and* $\{Z(k)\}$ *be generated by Algorithm 2.1. Then we have*

$$\langle T(v), Z(k)\rangle = 2\langle W(v), S(k)\rangle.$$

Proof. By Algorithm 2.1 and Lemmas 2.2, 2.3, 2.4, we have

$$\begin{aligned} & \langle T_1(v), Z_1(k)\rangle \\ = \; & \langle T_1(v), S(k)(X\Lambda^2)^H - JX\Lambda^2 S^H(k)J - E_sE_s^T[S(k)(X\Lambda^2)^H - JX\Lambda^2 S^H(k)J]E_sE_s^T\rangle \\ = \; & \langle T_1(v), S(k)(X\Lambda^2)^H - JX\Lambda^2 S^H(k)J\rangle \\ = \; & 2\langle T_1(v), S(k)(X\Lambda^2)^H\rangle \\ = \; & 2\langle T_1(v)X\Lambda^2, S(k)\rangle. \end{aligned} \tag{2.3}$$

Similarly, we have

$$\langle T_2(v), Z_2(k)\rangle = 2\langle T_2(v)X\Lambda, S(k)\rangle,$$

and

$$\langle T_3(v), Z_3(k)\rangle = 2\langle T_3(v)X, S(k)\rangle.$$

Hence,

$$\langle T(v), Z(k)\rangle = \sum_{j=1}^{3}\langle T_j(v), Z_j(k)\rangle = 2\langle W(v), S(k)\rangle.$$

We complete the proof. □

Lemma 2.6. *Let the sequence* $\{R(k)\}, \{P(k)\}, \{W(k)\}, \{S(k)\}$ *and* $\{T(k)\}$ *be generated by Algorithm 2.1, then we have*

$$\langle R(u), R(v)\rangle = 0, \quad for \ \ u, v = 1, 2, \cdots, k, u \neq v; \tag{2.4}$$

$$\langle P(u), P(v)\rangle = 0, \quad for \ \ u, v = 1, 2, \cdots, k, u \neq v; \tag{2.5}$$

$$\langle W(u), S(v)\rangle = \langle S(u), W(v)\rangle = 0, \quad for \ \ u > v; \tag{2.6}$$

$$\langle P(u), T(v)\rangle = 0, \quad for \ \ u > v; \tag{2.7}$$

and

$$\langle R(u), S(v)\rangle = 0, \quad for \ \ u > v; \tag{2.8}$$

Proof. By mathematical induction, for $k = 2$, by lemma 2.4 we have

$$\begin{aligned}
\langle R(2), R(1)\rangle &= \langle R(1) - \alpha_1 W(1), R(1)\rangle \\
&= \| R(1) \|^2 - \alpha_1 \langle W(1), S(1)\rangle = 0,
\end{aligned}$$

$$\begin{aligned}
\langle P(2), P(1)\rangle &= \langle P(1) - \beta_1 Z(1), P(1)\rangle \\
&= \| P(1) \|^2 - \beta_1 \langle Z(1), P(1)\rangle \\
&= \| P(1) \|^2 - \beta_1 \langle Z(1), T(1)\rangle \\
&= \| P(1) \|^2 - 2\beta_1 \langle S(1), W(1)\rangle \\
&= \| P(1) \|^2 - \frac{\|P(1)\|^2}{\langle S(1), W(1)\rangle}\langle S(1), W(1)\rangle \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\langle W(2), S(1)\rangle &= \langle T_1(2)X\Lambda^2 + T_2(2)X\Lambda + T_3(2)X, S(1)\rangle \\
&= \langle T_1(2)X\Lambda^2, S(1)\rangle + \langle T_2(2)X\Lambda, S(1)\rangle + \langle T_3(2)X, S(1)\rangle \\
&= \langle T_1(2), S(1)(X\Lambda^2)^H\rangle + \langle T_2(2), S(1)(X\Lambda)^H\rangle + \langle T_3(2), S(1)X^H\rangle \\
&= \frac{1}{2}\langle T_1(2), S(1)(X\Lambda^2)^H - JX\Lambda^2 S^H(1)J \\
&\quad - E_s E_s^T[S(1)(X\Lambda^2)^H - JX\Lambda^2 S^H(1)J]E_s E_s^T\rangle \\
&\quad + \frac{1}{2}\langle T_2(2), S(1)(X\Lambda)^H - JX\Lambda S^H(1)J \\
&\quad - E_t E_t^T[S(1)(X\Lambda)^H - JX\Lambda S^H(1)J]E_t E_t^T\rangle \\
&\quad + \frac{1}{2}\langle T_3(2), S(1)X^H - JXS^H(1)J \\
&\quad - E_u E_u^T[S(1)X^H - JXS^H(1)J]E_u E_u^T\rangle \\
&= \frac{1}{2}\langle T_1(2), Z_1(1)\rangle + \frac{1}{2}\langle T_2(2), Z_2(1)\rangle + \frac{1}{2}\langle T_3(2), Z_3(1)\rangle \\
&= \frac{1}{2}\langle P_1(2) + \gamma_2 T_1(1), Z_1(1)\rangle + \frac{1}{2}\langle P_2(2) + \gamma_2 T_2(1), Z_2(1)\rangle + \frac{1}{2}\langle P_3(2) + \gamma_2 T_3(1), Z_3(1)\rangle \\
&= \frac{1}{2}\langle P(2), Z(1)\rangle + \frac{1}{2}\gamma_2\langle T(1), Z(1)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\beta_1}\langle P(2), P(1) - P(2)\rangle + \gamma_2\langle W(1), S(1)\rangle \\
&= -\frac{1}{2\beta_1}\langle P(2), P(2)\rangle + \gamma_2\langle W(1), S(1)\rangle \\
&= -\frac{1}{2}\frac{2\langle S(1), W(1)\rangle}{\|P(1)\|^2}\langle P(2), P(2)\rangle + \frac{\|P(2)\|^2}{\|P(1)\|^2}\langle W(1), S(1)\rangle \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\langle S(2), W(1)\rangle &= \langle R(2) + \delta_2 S(1), W(1)\rangle \\
&= \langle R(2), W(1)\rangle + \delta_2\langle S(1), W(1)\rangle \\
&= \frac{1}{\alpha_1}\langle R(2), R(1) - R(2)\rangle + \delta_2\langle S(1), W(1)\rangle \\
&= -\frac{\langle S(1), W(1)\rangle}{\|R(1)\|^2}\langle R(2), R(2)\rangle + \frac{\|R(2)\|^2}{\|R(1)\|^2}\langle S(1), W(1)\rangle \\
&= 0.
\end{aligned}
$$

$$
\langle P(2), T(1)\rangle = \langle P(2), P(1)\rangle = 0,
$$

and

$$
\langle R(2), S(1)\rangle = \langle R(2), R(1)\rangle = 0.
$$

Suppose that (2.4)–(2.8) hold for $k = w (w \geq 2)$. For $k = w + 1, t = w$, it follows from Lemma 2.4, 2.5 and Algorithm 2.1 that

$$
\begin{aligned}
&\langle P(w+1), P(w)\rangle \\
&= \langle P(w) - \beta_w Z(w), P(w)\rangle \\
&= \|P(w)\|^2 - \beta_w\langle Z(w), P(w)\rangle \\
&= \|P(w)\|^2 - \beta_w\langle Z(w), T(w) - \gamma_w T(w-1)\rangle \\
&= \|P(w)\|^2 - \beta_w\langle Z(w), T(w)\rangle + \beta_w\gamma_w\langle Z(w), T(w-1)\rangle \\
&= \|P(w)\|^2 - 2\beta_w\langle W(w), S(w)\rangle + 2\beta_w\gamma_w\langle W(w-1), S(w)\rangle \\
&= \|P(w)\|^2 - 2\beta_w\langle W(w), S(w)\rangle \\
&= \|P(w)\|^2 - 2\frac{\|P(w)\|^2}{2\langle S(w), W(w)\rangle}\langle W(w), S(w)\rangle \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
&\langle W(w+1), S(w)\rangle \\
&= \frac{1}{2}\langle T(w+1), Z(w)\rangle \\
&= \frac{1}{2}\langle P(w+1) + \gamma_{w+1}T(w), Z(w)\rangle \\
&= \frac{1}{2}\langle P(w+1), Z(w)\rangle + \tfrac{1}{2}\gamma_{w+1}\langle T(w), Z(w)\rangle \\
&= \frac{1}{2\beta_w}\langle P(w+1), P(w) - P(w+1)\rangle + \gamma_{w+1}\langle W(w), S(w)\rangle \\
&= -\frac{1}{2\beta_w}\langle P(w+1), P(w+1)\rangle + \gamma_{w+1}\langle W(w), S(w)\rangle \\
&= -\frac{\langle S(w), W(w)\rangle}{\|P(w)\|^2}\langle P(w+1), P(w+1)\rangle + \frac{\|P(w+1)\|^2}{\|P(w)\|^2}\langle W(w), S(w)\rangle \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\langle R(w+1), R(w)\rangle &= \langle R(w) - \alpha_w W(w), R(w)\rangle \\
&= \|R(w)\|^2 - \alpha_w\langle W(w), R(w)\rangle \\
&= \|R(w)\|^2 - \alpha_w\langle W(w), S(w) - \delta_w S(w-1)\rangle \\
&= \|R(w)\|^2 - \alpha_w\langle W(w), S(w)\rangle \\
&= \|R(w)\|^2 - \frac{\|R(w)\|^2}{\langle S(w), W(w)\rangle}\langle W(w), S(w)\rangle \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\langle S(w+1), W(w)\rangle &= \langle R(w+1) + \delta_{w+1}S(w), W(w)\rangle \\
&= \frac{1}{\alpha_w}\langle R(w+1), R(w) - R(w+1)\rangle + \delta_{w+1}\langle S(w), W(w)\rangle \\
&= -\frac{\langle S(w), W(w)\rangle}{\|R(w)\|^2}\|R(w+1)\|^2 + \frac{\|R(w+1)\|^2}{\|R(w)\|^2}\langle S(w), W(w)\rangle \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\langle P(w+1), T(w)\rangle &= \langle P(w) - \beta_w Z(w), T(w)\rangle \\
&= \langle P(w), T(w)\rangle - \beta_w\langle Z(w), T(w)\rangle \\
&= \|P(w)\|^2 - 2\beta_w\langle S(w), W(w)\rangle = 0.
\end{aligned}
$$

and

$$
\begin{aligned}
\langle R(w+1), S(w)\rangle &= \langle R(w) - \alpha_w W(w), S(w)\rangle \\
&= \langle R(w), S(w)\rangle - \alpha_w\langle W(w), S(w)\rangle \\
&= \|R(w)\|^2 - \alpha_w\langle W(w), S(w)\rangle = 0.
\end{aligned}
$$

For $k = w + 1, t = 1$, by Lemma 2.4, 2.5, Algorithm 2.1 and the induction, we get

$$\begin{aligned}
&\langle P(w+1), P(1)\rangle \\
&= \langle P(w) - \beta_w Z(w), P(1)\rangle \\
&= -\beta_w \langle Z(w), P(1)\rangle \\
&= -\beta_w \langle Z(w), T(1)\rangle \\
&= -2\beta_w \langle W(w), S(1)\rangle \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\langle R(w+1), R(1)\rangle &= \langle R(w) - \alpha_w W(w), R(1)\rangle \\
&= -\alpha_w \langle W(w), R(1)\rangle \\
&= -\alpha_w \langle W(w), S(1)\rangle \\
&= 0.
\end{aligned}$$

Similarly, for $k = w + 1, t = 2, 3, \cdots, w - 1$, we have

$$\begin{aligned}
\langle P(w+1), P(t)\rangle &= \langle P(w) - \beta_w Z(w), P(t)\rangle \\
&= -\beta_w \langle Z(w), P(t)\rangle \\
&= -\beta_w \langle Z(w), T(t) - \gamma_t T(t-1)\rangle \\
&= -\beta_w \langle Z(w), T(t)\rangle + \beta_w \gamma_t \langle Z(w), T(t-1)\rangle \\
&= -2\beta_w \langle S(w), W(t)\rangle + 2\beta_w \gamma_t \langle S(w), W(t-1)\rangle \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\langle R(w+1), R(t)\rangle &= \langle R(w) - \alpha_w W(w), R(t)\rangle \\
&= -\alpha_w \langle W(w), R(t)\rangle \\
&= -\alpha_w \langle W(w), S(t) - \delta_t S(t-1)\rangle = 0.
\end{aligned}$$

So relation (2.4) and (2.5) hold for $k = w + 1$ and $1 \le v < u \le k$. For $u < v$, we obtain

$$\langle P(v), P(u)\rangle = \langle P(u), P(v)\rangle = 0.$$

$$\langle R(v), R(u)\rangle = \langle R(u), R(v)\rangle = 0.$$

Furthermore, for $k = w + 1, t = 1, 2, \cdots, w - 1$, by Lemma 2.5 and Algorithm 2.1, we have

$$\begin{aligned}
\langle W(w+1), S(t)\rangle &= \frac{1}{2}\langle T(w+1), Z(t)\rangle \\
&= \frac{1}{2}\langle P(w+1) + \gamma_{k+1} T(w), Z(t)\rangle \\
&= \frac{1}{2}\langle P(w+1), Z(t)\rangle + \frac{1}{2}\gamma_{k+1}\langle T(w), Z(t)\rangle \\
&= \frac{1}{2\beta_t}\langle P(w+1), P(t) - P(t+1)\rangle + \gamma_{k+1}\langle W(w), S(t)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2\beta_t}\langle P(w+1), P(t+1)\rangle \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\langle S(w+1), W(t)\rangle &= \langle R(w+1) + \delta_{w+1} S(w), W(t)\rangle \\
&= \frac{1}{\alpha_t}\langle R(w+1), R(t) - R(t+1)\rangle + \delta_{w+1}\langle S(w), W(t)\rangle \\
&= -\frac{1}{\alpha_t}\langle R(w+1), R(t+1)\rangle = 0,
\end{aligned}
$$

$$
\begin{aligned}
\langle P(w+1), T(t)\rangle &= \langle P(w) - \beta_w Z(w), T(t)\rangle \\
&= -\beta_w\langle Z(w), T(t)\rangle \\
&= -2\beta_w\langle S(w), W(t)\rangle = 0.
\end{aligned}
$$

and

$$
\begin{aligned}
\langle R(w+1), S(t)\rangle &= \langle R(w) - \alpha_w W(w), S(t)\rangle \\
&= -\alpha_w\langle W(w), S(t)\rangle = 0.
\end{aligned}
$$

So the relations (2.4)–(2.8) hold for $k = w + 1$. The lemma is proved using the induction principle. □

Theorem 2.1. *The exact solution of Problem 2.1 can be derived with the generalized GCD method 2.1 in at most nm iterative steps in the absence of round-off errors.*

Proof. Assume that $R(k) \neq 0$ for $k = 1, 2, \cdots, nm$, then $R(nm + 1)$ can be derived by Algorithm 2.1. According to Lemma 2.6, we know $\langle R(u), R(v)\rangle = 0$ for all $u, v = 1, 2, \cdots, nm, u \neq v$. So, the matrix sequence of $R(1), R(2), \cdots, R(nm)$ is an orthogonal basis of the linear space F, where $F \in C^{n\times m}$. Since $R(nm + 1) \in F$ and $\langle R(nm + 1), R(k)\rangle = 0$ for $k = 1, 2, \cdots, nm$, which indicates that $R(nm + 1) = 0$. We complete the proof. □

3. The least Frobenius norm solution

Before considering the least norm solution of Problem 2.1, we give the following lemma.

Lemma 3.1. *Suppose that $(\widetilde{A}^*, \widetilde{B}^*, \widetilde{C}^*)$ is a solution group of Problem 2.1. Then, arbitrary solution group $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ of Problem 2.1 can be expressed as*

$$(\widetilde{A}, \widetilde{B}, \widetilde{C}) = (\widetilde{A}^* + \mathcal{W}_1, \widetilde{B}^* + \mathcal{W}_2, \widetilde{C}^* + \mathcal{W}_3), \tag{3.1}$$

where $(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3) \in \widetilde{S}_1 \times \widetilde{S}_2 \times \widetilde{S}_3$ satisfies

$$\mathcal{W}_1 X\Lambda^2 + \mathcal{W}_2 X\Lambda + \mathcal{W}_3 X = 0. \tag{3.2}$$

Proof. For arbitrary solution group $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ of Problem 2.1, first we define the following matrices

$$\mathcal{W}_1 = \widetilde{A} - \widetilde{A}^*, \ \ \mathcal{W}_2 = \widetilde{B} - \widetilde{B}^*, \ \ \mathcal{W}_3 = \widetilde{C} - \widetilde{C}^*.$$

It is obvious that $(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3) \in \widetilde{S}_1 \times \widetilde{S}_2 \times \widetilde{S}_3$. Since $(\widetilde{A}^*, \widetilde{B}^*, \widetilde{C}^*)$ is a solution group of Problem 2.1, then we have

$$\widetilde{A}^* X\Lambda^2 + \widetilde{B}^* X\Lambda + \widetilde{C}^* X - \widetilde{Z} = 0.$$

It then follows that

$$\begin{aligned}
0 &= \|\widetilde{A}X\Lambda^2 + \widetilde{B}X\Lambda + \widetilde{C}X - \widetilde{Z}\|^2 \\
&= \|(\widetilde{A}^* + \mathcal{W}_1)X\Lambda^2 + (\widetilde{B}^* + \mathcal{W}_2)X\Lambda + (\widetilde{C}^* + \mathcal{W}_3)X - \widetilde{Z}\|^2 \\
&= \|\widetilde{A}^* X\Lambda^2 + \widetilde{B}^* X\Lambda + \widetilde{C}^* X - \widetilde{Z}\|^2 + 2\langle \widetilde{A}^* X\Lambda^2 + \widetilde{B}^* X\Lambda + \widetilde{C}^* X - \widetilde{Z}, \mathcal{W}_1 X\Lambda^2 + \mathcal{W}_2 X\Lambda + \mathcal{W}_3 X\rangle \\
&\quad + \|\mathcal{W}_1 X\Lambda^2 + \mathcal{W}_2 X\Lambda + \mathcal{W}_3 X\|^2 \\
&= \|\widetilde{A}^* X\Lambda^2 + \widetilde{B}^* X\Lambda + \widetilde{C}^* X - \widetilde{Z}\|^2 + \|\mathcal{W}_1 X\Lambda^2 + \mathcal{W}_2 X\Lambda + \mathcal{W}_3 X\|^2.
\end{aligned}$$

Hence

$$\mathcal{W}_1 X\Lambda^2 + \mathcal{W}_2 X\Lambda + \mathcal{W}_2 X = 0.$$

The proof is completed. □

In the following, it will be show that the unique least Frobenius norm solution of Problem 2.1 can be derived by Algorithm 2.1 with special initial matrices.

Theorem 3.1. *If we choose the initial matrices* $\widetilde{A}(1)$, $\widetilde{B}(1)$, $\widetilde{C}(1)$ *and* $P_1(1),\ P_2(1),\ P_3(1)$ *as follows:*

$$\begin{cases}
\widetilde{A}(1) = M(1)(X\Lambda^2)^H - JX\Lambda^2 M^H(1)J - E_s E_s^T[M(1)(X\Lambda^2)^H - JX\Lambda^2 M^H(1)J]E_s E_s^T, \\
\widetilde{B}(1) = M(1)(X\Lambda)^H - JX\Lambda M^H(1)J - E_t E_t^T[M(1)(X\Lambda)^H - JX\Lambda M^H(1)J]E_t E_t^T, \\
\widetilde{C}(1) = M(1)X^H - JXM^H(1)J - E_u E_u^T[M(1)X^H - JXM^H(1)J]E_u E_u^T, \\
P_1(1) = N(1)(X\Lambda^2)^H - JX\Lambda^2 N^H(1)J - E_s E_s^T[N(1)(X\Lambda^2)^H - JX\Lambda^2 N^H(1)J]E_s E_s^T, \\
P_2(1) = N(1)(X\Lambda)^H - JX\Lambda N^H(1)J - E_t E_t^T[N(1)(X\Lambda)^H - JX\Lambda N^H(1)J]E_t E_t^T, \\
P_3(1) = N(1)X^H - JXN^H(1)J - E_u E_u^T[N(1)X^H - JXN^H(1)J]E_u E_u^T,
\end{cases} \tag{3.3}$$

where $M(1) \in \mathbb{C}^{n\times m},\ N(1) \neq 0 \in \mathbb{C}^{n\times m}$, *are arbitrary matrices, then, the solution group* $(\widetilde{A}^*, \widetilde{B}^*, \widetilde{C}^*)$ *given by Algorithm 2.1 is the unique least Frobenius norm solution of Problem 2.1 .*

Proof. Let $\widetilde{A}(1), \widetilde{B}(1), \widetilde{C}(1), P_1(1), P_2(1), P_3(1)$ have the form of (3.3). According to Algorithm 2.1, we have

$$\begin{aligned}
\widetilde{A}(2) &= \widetilde{A}(1) - \alpha_1 T_1(1) \\
&= M(1)(X\Lambda^2)^H - JX\Lambda^2 M^H(1)J - E_s E_s^T[M(1)(X\Lambda^2)^H - JX\Lambda^2 M^H(1)J]E_s E_s^T \\
&- \alpha_1\{N(1)(X\Lambda^2)^H - JX\Lambda^2 N^H(1)J - E_s E_s^T[N(1)(X\Lambda^2)^H - JX\Lambda^2 N^H(1)J]E_s E_s^T\} \\
&= (M(1) - \alpha_1 N(1))(X\Lambda^2)^H - JX\Lambda^2(M(1) - \alpha_1 N(1))^H J \\
&- E_s E_s^T[(M(1) - \alpha_1 N(1))(X\Lambda^2)^H - JX\Lambda^2(M(1) - \alpha_1 N(1))^H J]E_s E_s^T \\
&= M(2)(X\Lambda^2)^H - JX\Lambda^2 M^H(2)J - E_s E_s^T[M(2)(X\Lambda^2)^H - JX\Lambda^2 M^H(2)J]E_s E_s^T,
\end{aligned}$$

where $M(2) = M(1) - \alpha_1 N(1)$. Similarly, we have

$$\begin{aligned}\widetilde{B}(2) &= M(2)(X\Lambda)^H - JX\Lambda M^H(2)J - E_tE_t^T[M(2)(X\Lambda)^H - JX\Lambda M^H(2)J]E_tE_t^T,\\ \widetilde{C}(2) &= M(2)X^H - JXM^H(2)J - E_uE_u^T[M(2)X^H - JXM^H(2)J]E_uE_u^T.\end{aligned}$$

Combining with Theorem 2.1, we can conclude that the solution $(\widetilde{A}^*, \widetilde{B}^*, \widetilde{C}^*)$ can be obtained within finite iteration steps, which can be represented as

$$\begin{cases}\widetilde{A}^* = M(X\Lambda^2)^H - JX\Lambda^2M^HJ - E_sE_s^T[M(X\Lambda^2)^H - JX\Lambda^2M^HJ]E_sE_s^T,\\ \widetilde{B}^* = M(X\Lambda)^H - JX\Lambda M^HJ - E_tE_t^T[M(X\Lambda)^H - JX\Lambda M^HJ]E_tE_t^T,\\ \widetilde{C}^* = MX^H - JXM^HJ - E_uE_u^T[MX^H - JXM^HJ]E_uE_u^T.\end{cases} \tag{3.4}$$

Now suppose that $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is arbitrary solution group of Problem 2.1, it then follows from Lemma 3.1 that there exists $(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3) \in \widetilde{S}_1 \times \widetilde{S}_2 \times \widetilde{S}_3$ satisfy

$$(\widetilde{A}, \widetilde{B}, \widetilde{C}) = (\widetilde{A}^* + \mathcal{W}_1, \widetilde{B}^* + \mathcal{W}_2, \widetilde{C}^* + \mathcal{W}_3) \tag{3.5}$$

and

$$\mathcal{W}_1X\Lambda^2 + \mathcal{W}_2X\Lambda + \mathcal{W}_3X = 0. \tag{3.6}$$

Then, by using (3.4) we have

$$\begin{aligned}&\langle\widetilde{A}^*, \mathcal{W}_1\rangle + \langle\widetilde{B}^*, \mathcal{W}_2\rangle + \langle\widetilde{C}^*, \mathcal{W}_3\rangle\\ =\ & 2\langle Y(X\Lambda^2)^H, \mathcal{W}_1\rangle + 2\langle Y(X\Lambda)^H, \mathcal{W}_2\rangle + 2\langle YX^H, \mathcal{W}_3\rangle\\ =\ & 2\langle Y, \mathcal{W}_1X\Lambda^2 + \mathcal{W}_2X\Lambda + \mathcal{W}_3X\rangle\\ =\ & 0.\end{aligned} \tag{3.7}$$

The combination of (3.5) and (3.7) yields

$$\begin{aligned}&\|\widetilde{A}\|^2 + \|\widetilde{B}\|^2 + \|\widetilde{C}\|^2\\ =\ & \|\widetilde{A}^* + \mathcal{W}_1\|^2 + \|\widetilde{B}^* + \mathcal{W}_2\|^2 + \|\widetilde{C}^* + \mathcal{W}_3\|^2\\ =\ & \|\widetilde{A}^*\|^2 + \|\widetilde{B}^*\|^2 + \|\widetilde{C}^*\|^2 + \|\mathcal{W}_1\|^2 + \|\mathcal{W}_2\|^2 + \|\mathcal{W}_3\|^2\\ & +2\Big[\langle\widetilde{A}^*, \mathcal{W}_1\rangle + \langle\widetilde{B}^*, \mathcal{W}_2\rangle + \langle\widetilde{C}^*, \mathcal{W}_3\rangle\Big]\\ =\ & \|\widetilde{A}^*\|^2 + \|\widetilde{B}^*\|^2 + \|\widetilde{C}^*\|^2 + \|\mathcal{W}_1\|^2 + \|\mathcal{W}_2\|^2 + \|\mathcal{W}_3\|^2\\ \geq\ & \|\widetilde{A}^*\|^2 + \|\widetilde{B}^*\|^2 + \|\widetilde{C}^*\|^2.\end{aligned}$$

This implies that the solution group $(\widetilde{A}^*, \widetilde{B}^*, \widetilde{C}^*)$ is the least Frobenius norm solution of Problem 2.1. The proof is completed. □

4. Numerical experiments

In this section, we report some numerical results to illustrate the effectiveness of the proposed algorithm. The experiments has been carried out by MATLAB R2011b, Intel(R)Core(TM)i5-2430M, CPU2. 40GHz, RAM 8 GB PC Environment. In view of the influence of round-off errors, we regard a matrix T as the zero matrix if $\langle T, T\rangle < 10^{-10}$, where $\langle\cdot, \cdot\rangle$ denotes the inner product defined by (1.7).

Example 4.1. *In this example, we consider the following quadratic inverse eigenvalue problem*

$$AX\Lambda^2 + BX\Lambda + CX = 0,$$

where

$$X=\begin{pmatrix} 3.1472+4.5717i & 4.1338-3.5811i & -2.2150+2.9221i & 4.6489-4.6429i & 6.723+1.2123i \\ 4.0579-0.1462i & 1.3236-0.7824i & 0.4688+4.5949i & -3.4239+3.4913i & 2.000+1.6755i \\ -3.7301+3.0028i & -4.0246+4.1574i & 4.5751+1.5574i & 4.7059+4.3399i & 3.000-2.4321i \\ 3.5303+4.0272i & 0.1325-0.1075i & -2.6008-1.3075i & -2.6005-1.1026i & 2.4325+1.000i \\ 1.2206+4.4479i & -0.9819-1.6228i & -3.7668-3.8880i & -0.8273-2.5831i & 3.2317-2.1267i \\ -1.4905-0.0914i & -4.2403+4.0005i & -3.1609+2.8025i & -4.5035-0.9609i & 2.3495-1.0421i \end{pmatrix}$$

$$\Lambda=\begin{pmatrix} 2.4321+0.2341i & 0 & 0 & 0 & 0 \\ 0 & 0.9812+1.0441i & 0 & 0 & 0 \\ 0 & 0 & 1.8751+0.2132i & 0 & 0 \\ 0 & 0 & 0 & -1.8732+1.0121i & 0 \\ 0 & 0 & 0 & 0 & -0.4391+2.9012i \end{pmatrix}.$$

Let $s = \{2,5\}$, $t = \{2,5\}$, $u = \{3,4\}$, $A_p=\begin{pmatrix} 2+i & -5+i \\ -2-i & 1+i \end{pmatrix}$, $B_q=\begin{pmatrix} 5 & -2i \\ -5-i & 2+i \end{pmatrix}$, $C_r=\begin{pmatrix} 5+i & 1+2i \\ 2+i & 2-i \end{pmatrix}$, *then*

$$\widetilde{A}_p=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2+i & 0 & 0 & -5+i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2-i & 0 & 0 & 1+i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ \widetilde{B}_q=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5-i & 0 & 0 & 2+i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{C}_r=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5+i & 1+2i & 0 & 0 \\ 0 & 0 & 2+i & 2-i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{Z} = -\widetilde{A}_pX\Lambda^2 - \widetilde{B}_qX\Lambda - \widetilde{C}_rX =$$

$$\begin{pmatrix} -10.2414+2.8131i & -6.5680-5.6994i & 12.4720+0.5147i & 11.2590-1.2639i & -2.6222+24.3481i \\ 3.4700+3.6332i & -0.4012-0.1856i & -2.8751-2.7625i & 2.1603+1.0206i & 0.7737+2.8362i \\ 11.2432+5.9463i & 4.0224+4.7689i & -8.8280-2.1618i & -8.9101-10.9506i & -4.2535-11.7322i \\ 1.5677-2.2806i & 0.4102+0.3327i & -0.0478+3.3802i & 0.1671-2.6704i & -0.2873-0.5648i \\ 1.6693-0.4214i & 1.5723+0.2339i & -0.6157-0.4610i & -0.2488-1.4681i & -1.6223-0.2241i \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If we choose $\widetilde{A}(1) = \widetilde{B}(1) = \widetilde{C}(1) = zeros(6,6), N(1) = 0.5 * ones(6,5) + 3 * i * ones(6,5)$, *and* $P_1(1), P_2(1), P_3(1)$, *have the form of (3.3). Then by using Algorithm 2.1, after* 84 *iteration steps, we obtain the unique least Frobenius norm solution of Problem 2.1:*

$\widetilde{A}(84) =$

$$\begin{pmatrix} 0.8441-1.1815i & 0.8999-0.7114i & -2.5005-1.5452i & 3.8047+1.5856i & -2.3664+4.0612i & 1.7260+0.0000i \\ -0.9659+2.7543i & 0 & 0.3295-4.9417i & -2.3941+5.3146i & 0 & 2.3664+4.0612i \\ -2.1664+1.1079i & -8.2386-2.7482i & -2.9876+2.5889i & -1.1264+0.0000i & 2.3941+5.3146i & 3.8047-1.5856i \\ 0.6653-1.2347i & -0.2723+0.7023i & -1.2946+0.0000i & 2.9876+2.5889i & 0.3295+4.9417i & 2.5005-1.5452i \\ 1.8897+1.3489i & 0 & 0.2723+0.7023i & -8.2386+2.7482i & 0 & 0.8999+0.7114i \\ 2.9656+0.0000i & -1.8897+1.3489i & 0.6653+1.2347i & 2.1664+1.1079i & -0.9659-2.7543i & -0.8441-1.1815i \end{pmatrix},$$

$$\widetilde{B}(84) =$$

$$\begin{pmatrix} 1.4883-4.7200i & -0.3546-1.2510i & -2.6436+1.8236i & -0.3317-3.1903i & -0.0363+5.6767i & 7.0430+0.0000i \\ 11.7698+3.7515i & 0 & -2.0284+3.3924i & 9.3574-1.9698i & 0 & 0.0363+5.6767i \\ 5.8103-2.3157i & 12.5363+4.6009i & 0.6514-8.3676i & 10.9132+0.0000i & -9.3574-1.9698i & -0.3317+3.1903i \\ 1.8741-4.9631i & 1.9491-3.1209i & -1.2799+0.0000i & -0.6514-8.3676i & -2.0284-3.3924i & 2.6436+1.8236i \\ 6.5289+0.5833i & 0 & -1.9491-3.1209i & 12.5363-4.6009i & 0 & -0.3546+1.2510i \\ -0.4786+0.0000i & -6.5289+0.5833i & 1.8741+4.9631i & -5.8103-2.3157i & 11.7698-3.7515i & -1.4883-4.7200i \end{pmatrix},$$

$$\widetilde{C}(84) =$$

$$\begin{pmatrix} 0.6030+5.3560i & -2.1553+0.3928i & -0.1398+3.6903i & -1.5865-0.0585i & 1.4352+2.5339i & 3.0260+0.0000i \\ -13.7993-4.7247i & 3.9183-10.6060i & -2.5907-5.1673i & -2.2921-7.6398i & -9.6510+0.0000i & -1.4352+2.5339i \\ 1.2187-9.2566i & 4.7554+3.3998i & 0 & 0 & 2.2921-7.6398i & -1.5865+0.0585i \\ -3.6265+9.8061i & -4.5678-2.0375i & 0 & 0 & -2.5907+5.1673i & 0.1398+3.6903i \\ -5.1867-14.5513i & 8.7330+0.0000i & 4.5678-2.0375i & 4.7554-3.3998i & -3.9183-10.6060i & -2.1553-0.3928i \\ -8.0598+0.0000i & 5.1867-14.5513i & -3.6265-9.8061i & -1.2187-9.2566i & -13.7993+4.7247i & -0.6030+5.3560i \end{pmatrix}.$$

Hence, the least Frobenius norm solution of Problem 1.1 can be obtained as follows:

$$\widetilde{A} = \widetilde{A}(84) + \widetilde{A}_p =$$

$$\begin{pmatrix} 0.8441-1.1815i & 0.8999-0.7114i & -2.5005-1.5452i & 3.8047+1.5856i & -2.3664+4.0612i & 1.7260+0.0000i \\ -0.9659+2.7543i & 2+i & 0.3295-4.9417i & -2.3941+5.3146i & -5+i & 2.3664+4.0612i \\ -2.1664+1.1079i & -8.2386-2.7482i & -2.9876+2.5889i & -1.1264+0.0000i & 2.3941+5.3146i & 3.8047-1.5856i \\ 0.6653-1.2347i & -0.2723+0.7023i & -1.2946+0.0000i & 2.9876+2.5889i & 0.3295+4.9417i & 2.5005-1.5452i \\ 1.8897+1.3489i & -2-i & 0.2723+0.7023i & -8.2386+2.7482i & 1+i & 0.8999+0.7114i \\ 2.9656+0.0000i & -1.8897+1.3489i & 0.6653+1.2347i & 2.1664+1.1079i & -0.9659-2.7543i & -0.8441-1.1815i \end{pmatrix},$$

$$\widetilde{B} = \widetilde{B}(84) + \widetilde{B}_q =$$

$$\begin{pmatrix} 1.4883-4.7200i & -0.3546-1.2510i & -2.6436+1.8236i & -0.3317-3.1903i & -0.0363+5.6767i & 7.0430+0.0000i \\ 11.7698+3.7515i & 5 & -2.0284+3.3924i & 9.3574-1.9698i & -2i & 0.0363+5.6767i \\ 5.8103-2.3157i & 12.5363+4.6009i & 0.6514-8.3676i & 10.9132+0.0000i & -9.3574-1.9698i & -0.3317+3.1903i \\ 1.8741-4.9631i & 1.9491-3.1209i & -1.2799+0.0000i & -0.6514-8.3676i & -2.0284-3.3924i & 2.6436+1.8236i \\ 6.5289+0.5833i & -5-i & -1.9491-3.1209i & 12.5363-4.6009i & 2+i & -0.3546+1.2510i \\ -0.4786+0.0000i & -6.5289+0.5833i & 1.8741+4.9631i & -5.8103-2.3157i & 11.7698-3.7515i & -1.4883-4.7200i \end{pmatrix},$$

$$\widetilde{C} = \widetilde{C}(84) + \widetilde{C}_r =$$

$$\begin{pmatrix} 0.6030+5.3560i & -2.1553+0.3928i & -0.1398+3.6903i & -1.5865-0.0585i & 1.4352+2.5339i & 3.0260+0.0000i \\ -13.7993-4.7247i & 3.9183-10.6060i & -2.5907-5.1673i & -2.2921-7.6398i & -9.6510+0.0000i & -1.4352+2.5339i \\ 1.2187-9.2566i & 4.7554+3.3998i & 5+i & 1+2i & 2.2921-7.6398i & -1.5865+0.0585i \\ -3.6265+9.8061i & -4.5678-2.0375i & 2+i & 2-i & -2.5907+5.1673i & 0.1398+3.6903i \\ -5.1867-14.5513i & 8.7330+0.0000i & 4.5678-2.0375i & 4.7554-3.3998i & -3.9183-10.6060i & -2.1553-0.3928i \\ -8.0598+0.0000i & 5.1867-14.5513i & -3.6265-9.8061i & -1.2187-9.2566i & -13.7993+4.7247i & -0.6030+5.3560i \end{pmatrix}.$$

At this moment, the norm of $R(k)$ defined as Algorithm 2.1 is

$$||R(k)|| = 6.6747e - 11.$$

The relationship between the number of iterations and the norm of $R(k)$ is shown in Figure 1.

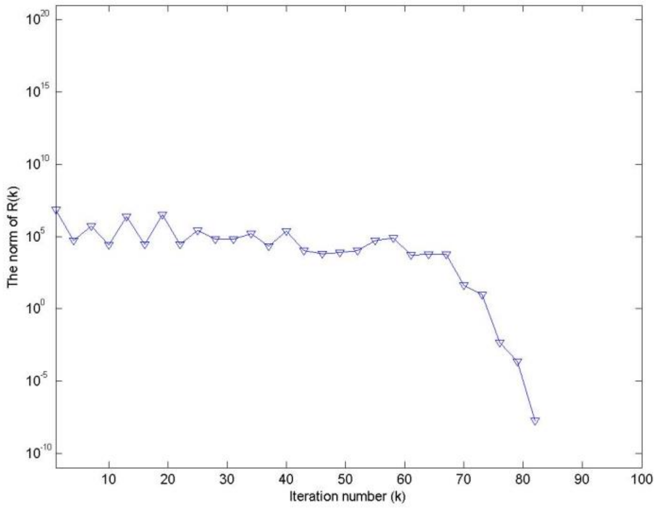

Figure 1. The relationship between the number of iterations and the norm of $R(k)$ for Example 4.1.

5. Conclusions

In this paper, we consider a class of constrained quadratic inverse eigenvalue Problem 1.1: for given $X \in \mathbb{C}^{n\times m}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m) \in \mathbb{C}^{m\times m}$, find $A^*, B^*, C^* \in \mathbb{C}^{n\times n}$, such that $AX\Lambda^2 + BX\Lambda + CX = 0$, where A^*, B^*, C^* are generalized skew Hamiltonian matrices with a constrained submatrix. By reformulating Problem 1.1 as its equivalent Problem 2.1, a generalized conjugate direction method is proposed later. It is shown that the proposed algorithm always converge to the generalized skew Hamiltonian solutions with constrained submatrix of Problem 1.1 within finite iterative steps in the absence of roundoff error. In addition, by choosing a special kind of initial matrices, the unique least Frobenius norm solution of Problem 1.1 can be obtained consequently. Some numerical results are reported to demonstrate the efficiency of our algorithm.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

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