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Research article

Higher order strongly general convex functions and variational inequalities

Muhammad Aslam Noor*and Khalida Inayat Noor

Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan

* **Correspondence:** Email: noormaslam@gmail.com.

Abstract: In this paper, we define and consider some new concepts of the higher order strongly general convex functions with respect to an arbitrary function. Some properties of the higher order strongly general convex functions are investigated under suitable conditions. It is shown that the optimality conditions of the higher order strongly general convex functions are characterized by a class of variational inequalities, which is called the higher order strongly variational inequality. Auxiliary principle technique is used to suggest an implicit method for solving strongly general variational inequalities. Convergence analysis of the proposed method is investigated using the pseudo-monotonicity of the operator. It is shown that the parallelogram laws for Banach spaces can be obtained as applications of higher order strongly affine convex functions. Some special cases also discussed. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

Keywords: higher order convex functions; variational inequalities; parallelogram laws
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1. Introduction

It is well known that the convex sets and convex functions had played crucial important part in the developments of the pure and applied sciences and are continue to inspire novel and innovative applications. Convex functions have been extended and generalized in various directions in recent years. Mohsen et al [1] introduced the concept of higher order strongly convex functions and studied their properties. These results can be viewed as a significant refinement of the results of Lin and Fukushima [2]. Higher order strongly convex functions include the strongly convex functions, which were introduced and studied by Polyak [3]. Karmardian [4] used the strongly convex functions to discuss the unique existence of a solution of the nonlinear complementarity problems. Awan et al [5,6] have derived Hermite-Hadamard type inequalities for various classes of strongly convex functions, which provide upper and lower estimate for the integrand. For the applications of strongly convex functions in optimization, variational inequalities and other branches of pure and applied sciences, see [7–15] and the references therein. It is known that a set may not be convex set. However, a set can be made convex set with respect to an arbitrary function. Motivated by this fact, Noor [15] introduced the concept of general convex sets involving an arbitrary function. It has been shown that the minimum of a differentiable general convex function on the general convex set can be characterized by the general variational inequalities. Cristescu at al [16, 17] have investigated algebraic and topological properties of the g-convex sets defined by Noor [15] in order to deduce their shape. They are a subclass of star-shaped sets, which have Youness [18] type convexity. A representation theorem based on extremal points is given for the class of bounded g-convex sets. Examples showing that this convexity is a frequent property in connection with a wild range of applications are given. Noor [15] has shown that the optimality conditions of the differentiable general convex functions can be characterized a class of variational inequalities called general variational inequality, the origin of which can be traced back to Stampacchia [19]. For the formulation, applications, numerical methods, sensitivity analysis and other aspects of general variational inequalities, see [10–14, 18–22] and the references therein. These facts and observations motivated us to consider higher order strongly convex functions with respect to an arbitrary function, which is the main motivation of this paper. Several new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. We have shown that the minimum of a differentiable higher order strongly convex functions on the general convex set can be characterized by a class of variational inequality. This results inspired us to consider the higher order strongly general variational inequalities. Due to the inherent nonlinearity, the projection method and its variant form can not be used to suggest the iterative methods for solving these nonlinear variational. To overcome these draw backs, we use the technique of the auxiliary principle [12, 14, 20, 21] to suggest an implicit method for solving general variational inequalities. Convergence analysis of the proposed method is investigated under pseudo-monotonicity, which is a weaker condition than monotonicity. Parallelogram laws for Banach spaces, which are due to Bynum [23] and Chen et al [24, 25], can be obtained from these definitions, are discussed in Section 5. Some new special cases are discussed, which can be viewed itself novel and interesting applications of the higher order strongly convex functions. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. Formulations and basic facts

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively.

Definition 2.1. [10, 17, 18] *A set K in H is said to be a convex set, if*

$$u + t(v - u) \in K, \qquad \forall u, v \in K, t \in [0, 1].$$

Definition 2.2. *A function F is said to be convex function, if t*

$$F((1 - t)u + tv) \leq (1 - t)F(u) + tF(v), \quad \forall u, v \in K, \quad t \in [0, 1]. \tag{2.1}$$

It is well known that $u \in K$ of a differential convex functions F is equivalent to finding $u \in K$ such that

$$\langle F'(u), v - u \rangle \geq 0, \quad v \in K, \tag{2.2}$$

which is called the variational inequality, introduced and studied by Stampacchia [19]. Variational inequalities can be regarded as a novel and significant extension of variational principles.

We would like to mention that in many important applications the underlying the set may not be a convex set. To overcome this drawback, the underlying set can be made convex set with respect to an arbitrary function, which is called a general convex set.

Definition 2.3. [15]. *The set K_g in H is said to be a general convex set, if there exists a function g such that*

$$(1-t)u + tg(v) \in K_g, \quad \forall u, v \in H : u, g(v) \in K_g, t \in [0,1].$$

We now discuss some special cases of the general convex sets.

(I). If $g = I$, the identity operator, then general convex set reduces to the classical convex set. Clearly every convex set is a general convex set, but the converse is not true. Cristescu et al [16] discussed various applications of the general convex sets related to the necessity of adjusting investment or development projects out of environmental or social reasons. For example, the easiest manner of constructing this kind of convex sets comes from the problem of modernizing the railway transport system. Shape properties of the general convex sets with respect to a projection are investigated.

(II). If $g(v) = mv, \quad m \in (0,1)$, then the general convex set becomes the m-convex set, which is mainly due to Toader [26].

Definition 2.4. [26] *The set K_m is said to be m-convex set, if*

$$(1-t)u + tmv \in K_m, \qquad \forall u, v \in K_m, t \in [0,1].$$

For the sake of simplicity, we always assume that $\forall u, v \in H : u, g(v) \in K_g$, unless otherwise specified.

Definition 2.5. *A function F is said to be general convex(g-convex) function, if there exists an arbitrary non-negative function g, such that*

$$F((1-t)u + tg(v)) \leq (1-t)F(u) + tF(g(v)), \quad \forall u, g(v) \in K_g, \quad t \in [0,1]. \tag{2.3}$$

The general convex functions were introduced by Noor [15]. Noor [15] proved that the minimum $u \in H : g(u) \in K_g$ of the differentiable general convex functions F can be characterized by the class of variational inequalities of the type:

$$\langle F'(u), g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K_g, \tag{2.4}$$

which is known as general variational inequalities. For the applications of the general variational inequalities in various branches of pure and applied sciences, see [10, 12–15, 19–22] and the references therein.

We now introduce some new classes of higher order strongly general convex functions and higher order strongly affine general convex functions.

Definition 2.6. *A function F on the convex set K_g is said to be higher order strongly convex with respect to an arbitrary function g, if there exists a constant $\mu > 0$, such that*

$$F(u + t(g(v) - u)) \leq (1 - t)F(u) + tF(g(v)) - \mu\{t^p(1 - t) + t(1 - t)^p\}\|g(v) - u\|^p, \quad \forall u, g(v) \in K_g, t \in [0, 1]. \tag{2.5}$$

A function F is said to higher order strongly concave with respect to an arbitrary function g, if and only if, $-F$ is higher order strongly general convex with respect to an arbitrary function g.

If $t = \frac{1}{2}$, then

$$F\Big(\frac{u + g(v)}{2}\Big) \leq \frac{F(u) + F(g(v))}{2} - \mu\frac{1}{2^p}\|v - u\|^p, \forall u, g(v) \in K_g, t \in [0, 1]. \tag{2.6}$$

The function F is said to be higher order strongly general J-convex function.

We now discuss some special cases.

I. If $g = I$, the identity operator, then $K_g = K$, the convex set and Definition 2.6 reduces to:

Definition 2.7. *A function F on the convex set K is said to be higher order strongly convex function, if there exists a constant $\mu > 0$, such that*

$$F(u + t(v - u)) \leq (1 - t)F(u) + tF(v) - \mu\{t^p(1 - t) + t(1 - t)^p\}\|v - u\|^p, \quad \forall u, v \in K, t \in [0, 1], \tag{2.7}$$

which were introduced and studied by Mohsen et al [1]. It has been shown that higher order strongly convex functions can be viewed as a significant refinement of the concept introduced by Lin and Fukushima [2]. For the applications of higher order strongly convex functions, see [27–29] and the references therein.

II. If $g(v) = mv, \quad m \in (0, 1)$, the identity operator, then $K_g = K_,$ the m-convex set and Definition 2.6 reduces to:

Definition 2.8. *A function F on the convex set K is said to be higher order strongly m-convex function, if there exists a constant $\mu > 0$, such that*

$$F(u + t(mv - u)) \leq (1 - t)F(u) + tF(mv) - \mu\{t^p(1 - t) + t(1 - t)^p\}\|mv - u\|^p, \quad \forall u, v \in K_m, t \in [0, 1], \tag{2.8}$$

which appears to be new ones.

III. If $g(v) = mv, \quad m \in (0, 1)$ and $F(mv) = mF(v)$, then $K_g = K_,$ the m-convex set and Definition 2.8 reduces to:

Definition 2.9. *A function F on the convex set K is said to be higher order higher order strongly m-convex function in the sense of Toader [26], if there exists a constant $\mu > 0$, such that*

$$F(u + t(mv - u)) \leq (1 - t)F(u) + tmF(v) - \mu\{t^p(1 - t) + t(1 - t)^p\}\|mv - u\|^p, \quad \forall u, v \in K_m, t \in [0, 1], \tag{2.9}$$

IV. If $p = 2$, then the higher order strongly general convex function becomes strongly convex functions, that is,

$$F(u + t(g(v) - u)) \leq (1 - t)F(u) + tF(g(v)) - \mu t(1 - t)\|g(v) - u\|^2, \forall u, g(v) \in K_g, t \in [0, 1].$$

For the properties of the strongly convex functions in variational inequalities and equilibrium problems, see Noor [13–15].

We would like to mention that for suitable and appropriate choice of the arbitrary function g and p, we can obtain some new and previous known classes of convex functions. This show that the higher order strongly general convex functions are unifying ones.

Definition 2.10. *A function F on the convex set K_g is said to be a higher order strongly affine general convex function with respect to an arbitrary function g, if there exists a constant $\mu > 0$, such that*

$$F(u + t(g(v) - u)) = (1 - t)F(u) + tF(g(v)) - \mu\{t^p(1 - t) + t(1 - t)^p\}\|g(v) - u\|^2, \quad \forall u, g(v) \in K_g, t \in [0, 1]. \tag{2.10}$$

Note that, if a functions is both higher order strongly general convex and higher order strongly general concave, then it is higher order strongly affine general convex function.

Definition 2.11. *A function F is called higher order strongly quadratic equation with respect to an arbitrary function g, if there exists a constant $\mu > 0$, such that*

$$F\Big(\frac{u + g(v)}{2}\Big) = \frac{F(u) + F(g(v))}{2} - \mu\frac{1}{2^p}\|g(v) - u\|^p, \forall u, g(v) \in K_g, t \in [0, 1]. \tag{2.11}$$

This function F is also called higher order strongly general affine J-convex function.

Definition 2.12. *A function F on the convex set K_g is said to be higher order strongly quasi convex with respect to an arbitrary function g, if there exists a constant $\mu > 0$ such that*

$$F(u + t(g(v) - u) \leq \max\{F(u), F(g(v))\} - \mu\{t^p(1 - t) + t(1 - t)^p\}\|g(v) - u\|^p, \quad \forall u, g(v) \in K_g, t \in [0, 1].$$

Definition 2.13. *A function F on the convex set K_g is said to be higher order strongly log-convex with respect to an arbitrary function g, if there exists a constant $\mu > 0$ such that*

$$F(u + t(g(v) - u)) \leq (F(u))^{1-t}(F(g(v)))^t - \mu\{t^p(1 - t) + t(1 - t)^p\}\|g(v) - u\|^p, \quad \forall u, g(v) \in K_g, t \in [0, 1],$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} F(u+t(g(v)-u)) &\leq (F(u))^{1-t}(F(g(v))^{t}-\mu\{t^{p}(1-t)+t(1-t)^{p}\}\|g(v)-u\|^{p} \\ &\leq (1-t)F(u)+tF(g(v))-\mu\{t^{p}(1-t)+t(1-t)^{p}\}\|g(v)-u\|^{p} \\ &\leq \max\{F(u),F(g(v))\}-\mu\{t^{p}(1-t)+t(1-t)^{p}\}\|g(v)-u\|^{p}. \end{aligned}$$

This shows that every higher order strongly general log-convex function is a higher order strongly general convex function and every higher order strongly general convex function is a higher order strongly general quasi-convex function. However, the converse is not true. See also [27–32] for the applications of strongly functions convex and their variant forms.

Definition 2.14. *An operator $T : K \to H$ is said to be:*

1. *higher order strongly general monotone, if and only if, there exists a constant $\alpha > 0$ such that*

$$\langle Tu - Tg(v), u - g(v)\rangle \geq \alpha\|g(v) - u\|^{p}, \forall u, g(v) \in K_{g}.$$

2. *higher order strongly general pseudomonotone, if and only if, there exists a constant $\nu > 0$ such that*

$$\begin{aligned} &\langle Tu, g(v) - u\rangle + \nu\|g(v) - u\|^{p} \geq 0 \\ &\Rightarrow \\ &\langle Tg(v), g(v) - u\rangle - \nu\|g(v) - u\|^{p} \geq 0, \forall u, g(v) \in K_{g}. \end{aligned}$$

3. *higher order strongly general relaxed pseudomonotone, if and only if, there exists a constant $\mu > 0$ such that*

$$\begin{aligned} &\langle Tu, g(v) - u\rangle \geq 0 \\ &\Rightarrow \\ &-\langle Tg(v), u - g(v)\rangle + \mu\|g(v) - u\|^{p} \geq 0, \forall u, g(v) \in K_{g}. \end{aligned}$$

Definition 2.15. *A differentiable function F on the convex set K is said to be higher order strongly general pseudo convex function, if and only if, if there exists a constant $\mu > 0$ such that*

$$\langle F'(u), g(v) - u\rangle + \mu\|g(v) - u\|^{p} \geq 0 \Rightarrow F(g(v)) \geq F(u), \forall u, g(v) \in K_{g}.$$

3. Main results

In this section, we consider some basic properties of higher order strongly convex functions.

Theorem 3.1. *Let F be a differentiable function on the convex set K. Then the function F is higher order strongly general convex function, if and only if,*

$$F(g(v)) - F(u) \geq \langle F'(u), g(v) - u\rangle + \mu\|g(v) - u\|^{p}, \forall g(v), u \in K_{g}. \quad (3.1)$$

Proof. Let F be a higher order strongly general convex function on the general convex set K_g. Then

$$F(u + t(g(v) - u)) \leq (1 - t)F(u) + tF(g(v)) - \mu\{t^p(1 - t) + t(1 - t)^p\}\|g(v) - u\|^p, \quad \forall u, g(v) \in K_g,$$

which can be written as

$$F(g(v)) - F(u) \geq \{\frac{F(u + t(g(v) - u) - F(u)}{t}\} + \{t^{p-1}(1 - t) + (1 - t)^p\}\|g(v) - u\|^p.$$

Taking the limit in the above inequality as $t \to 0$, we have

$$F(g(v)) - F(u) \geq \langle F'(u), g(v) - u)\rangle + \mu\|g(v) - u\|^p, \forall u, g(v) \in K_g,$$

which is the required result (3.1).

Conversely, let (3.1) hold. Then, $\forall u, g(v) \in K_g, t \in [0, 1], v_t = u + t(g(v) - u) \in K_g$, we have

$$\begin{aligned} F(g(v)) - F(v_t) &\geq \langle F'(v_t), g(v) - v_t)\rangle + \mu\|g(v) - v_t\|^p \\ &= (1 - t)F'(v_t), g(v) - u\rangle + \mu(1 - t)^p\|g(v) - u\|^p, \forall u, g(v) \in K_g. \end{aligned} \tag{3.2}$$

In a similar way, we have

$$\begin{aligned} F(u) - F(v_t) &\geq \langle F'(v_t), u - v_t)\rangle + \mu\|u - v_t\|^p \\ &= -tF'(v_t), g(v) - u\rangle + \mu t^p\|g(v) - u\|^p. \end{aligned} \tag{3.3}$$

Multiplying (3.2) by t and (3.3) by $(1 - t)$ and adding the resultant, we have

$$F(u + t(g(v), u)) \leq (1 - t)F(u) + tF(g(v)) - \mu\{t^p(1 - t) + t(1 - t)^p\}\|g(v) - u\|^p, \forall u, g(v) \in K_g,$$

showing that F is a higher order strongly general convex function. □

Theorem 3.2. *Let F be a differentiable higher order strongly convex function on the general convex set K_g. Then $F'(.)$ is a higher order strongly general monotone operator.*

Proof. Let F be a higher order strongly general convex function on the general convex set K_g. Then, from Theorem 3.1. We have

$$F(g(v)) - F(u) \geq \langle F'(u), g(v) - u\rangle + \mu\|g(v) - u\|^p, \quad \forall u, g(v) \in K_g. \tag{3.4}$$

Changing the role of u and v in (3.4), we have

$$F(u) - F(g(v)) \geq \langle F'(g(v(), u - v)\rangle + \mu\|g(v) - u\|^p, \quad \forall u, g(v) \in K_g. \tag{3.5}$$

Adding (3.4) and (3.5), we have

$$\langle F'(u) - F'(g(v)), u - g(v)\rangle \geq 2\mu\|g(v) - u\|^p, \quad \forall u, g(v) \in K_g, \tag{3.6}$$

which shows that $F'(.)$ is a higher order strongly general monotone operator.

□

We remark that the converse of Theorem 3.2 is not true. However, we have the following result.

Theorem 3.3. *If the differential operator $F'(.)$ of a differentiable higher order strongly general convex function F is a higher order strongly general monotone operator, then*

$$F(g(v)) - F(u) \geq \langle F'(u), g(v) - u\rangle + 2\mu\frac{1}{p}\|g(v) - u\|^p, \forall u, g(v) \in K_g. \tag{3.7}$$

Proof. Let $F'(.)$ be a higher order strongly general monotone operator. Then, from (3.6), we have

$$\langle F'(g(v)), u - g(v)\rangle \geq \langle F'(u), u - g(v))\rangle + 2\mu\|g(v) - u\|^p. \quad \forall u, g(v) \in K_g. \tag{3.8}$$

Since K_g is a general convex set, $\forall u, g(v) \in K_g$, $t \in [0, 1]$, $v_t = u + t(g(v) - u) \in K_g$. Taking $v = v_t$ in (3.8), we have

$$\begin{aligned} \langle F'(v_t), u - v_t\rangle &\leq \langle F'(u), u - v_t\rangle - 2\mu\|g(v) - u\|^p \\ &= -t\langle F'(u), g(v) - u\rangle - 2\mu t^p\|g(v) - u\|^p, \end{aligned}$$

which implies that

$$\langle F'(v_t), g(v) - u\rangle \geq \langle F'(u), g(v) - u\rangle + 2\mu t^{p-1}\|g(v) - u\|^P. \tag{3.9}$$

Consider the auxiliary function

$$\xi(t) = F(u + t(g(v) - u), \forall u, g(v) \in K_g,$$

from which, we have

$$\xi(1) = F(g(v)), \quad \xi(0) = F(u).$$

Then, from (3.9), we have

$$\xi'(t) = \langle F'(v_t), g(v) - u\rangle \geq \langle F'(u), g(v) - u\rangle + 2\mu t^{p-1}\|g(v) - u\|^p. \tag{3.10}$$

Integrating (3.10) between 0 and 1, we have

$$\xi(1) - \xi(0) \;=\; \int_0^1 g'(t)dt \geq \langle F'(u), g(v) - u\rangle + 2\mu\frac{1}{p}\|g(v) - u\|^p.$$

Thus it follows that

$$F(g(v)) - F(u) \geq \langle F'(u), g(v) - u\rangle + 2\mu\frac{1}{p}\|g(v) - u\|^p, \forall u, g(v) \in K_g,$$

which is the required (3.7). □

We note that, if $p = 2$, then Theorem 3.3 can be viewed as the converse of Theorem 3.2.

We now give a necessary condition for higher order strongly general pseudo-convex function.

Theorem 3.4. *Let $F'(.)$ be a higher order strongly general relaxed pseudomonotone operator. Then F is a higher order strongly general pseudo-convex function.*

Proof. Let F' be a higher order strongly general relaxed pseudomonotone operator. Then, $\forall u, g(v) \in K_g$,

$$\langle F'(u), g(v) - u\rangle \geq 0.$$

implies that

$$\langle F'(g(v)), g(v) - u\rangle \geq \mu\|g(v) - u\|^p, \forall u, g(v) \in K_g. \tag{3.11}$$

Since K_g is a general convex set, $\forall u, g(v) \in K_g, \quad t \in [0, 1], v_t = u + t(g(v) - u) \in K_g.$

Taking $v = v_t$ in (3.11), we have

$$\langle F'(v_t), g(v) - u\rangle \geq \mu t^{p-1}\|g(v). - u\|^p. \tag{3.12}$$

Consider the auxiliary function

$$\xi(t) = F(u + t(g(v) - u)) = F(v_t), \quad \forall u, g(v) \in K_g, t \in [0, 1],$$

which is differentiable, since F is differentiable function. Then, using (3.12), we have

$$\xi'(t) = \langle F'(v_t), g(v) - u)\rangle \geq \mu t^{p-1}\|g(v) - u\|^p.$$

Integrating the above relation between 0 to 1, we have

$$\xi(1) - \xi(0) = \int_0^1 g'(t)dt \geq \frac{\mu}{p}\|g(v) - u\|^p,$$

that is,

$$F(g(v)) - F(u) \geq \frac{\mu}{p}\|g(v) - u\|^p), \forall u, g(v) \in K_g,$$

showing that F is a higher order strongly general pseudo-convex function. □

Definition 3.1. *A function F is said to be sharply higher order strongly general pseudo convex, if there exists a constant $\mu > 0$ such that*

$$\begin{aligned}
&\langle F'(u), g(v) - u\rangle \geq 0 \\
&\Rightarrow \\
&F(g(v)) \geq F(g(v) + t(u - g(v))) + \mu\{t^p(1 - t) + t(1 - t)^p\}\|g(v) - u\|^p, \forall u, g(v) \in K_g.
\end{aligned}$$

Theorem 3.5. *Let F be a sharply higher order strongly general pseudo convex function on K_g with a constant $\mu > 0$. Then*

$$\langle F'(g(v)), g(v) - u\rangle \geq \mu\|g(v) - u\|^p, \forall u, g(v) \in K_g.$$

Proof. Let F be a sharply higher order strongly general pseudo convex function on K_g. Then

$$F(g(v)) \geq F(g(v) + t(u - g(v))) + \mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - u\|^p, \forall u, g(v) \in K_g, t \in [0,1],$$

from which, we have

$$\{\frac{F(g(v) + t(u - g(v)) - F(g(v))}{t}\} + \mu\{t^{p-1}(1-t) + (1-t)^p\}\|g(v) - u\|^p \geq 0.$$

Taking limit in the above inequality, as $t \to 0$, we have

$$\langle F'(g(v)), g(v) - u\rangle \geq \mu\|g(v) - u\|^p, \forall u, g(v) \in K_g,$$

the required result. □

Definition 3.2. *A function F is said to be a pseudo convex function, if there exists a strictly positive bifunction $B(.,.)$, such that*

$$\begin{aligned} &F(g(v)) < F(u) \\ &\Rightarrow \\ &F(u + t(g(v) - u)) < F(u) + t(t-1)B(g(v), u), \forall u, g(v) \in K_g, t \in [0,1]. \end{aligned}$$

Theorem 3.6. *If the function F is higher order strongly general convex function such that $F(g(v)) < F(u)$, then the function F is higher order strongly general pseudo convex.*

Proof. Since $F(g(v)) < F(u)$ and F is higher order strongly general convex function, then $\forall u, g(v) \in K_g, \quad t \in [0,1]$, we have

$$\begin{aligned} F(u + t(g(v) - u)) &\leq F(u) + t(F(g(v)) - F(u)) - \mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - u\|^p \\ &< F(u) + t(a-t)(F(g(v)) - F(u)) - \mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - u\|^p \\ &= F(u) + t(t-1)(F(u) - F(g(v))) - \mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - u\|^p \\ &< F(u) + t(t-1)B(u, g(v)) \\ &\quad -\mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - u\|^p, \forall u, g(v) \in K_g, \end{aligned}$$

where $B(u, g(v) = F(u) - F(g(v)) > 0$. Consequently the function F is higher order strongly general pseudo convex. the required result. □

Theorem 3.7. *Let f be a higher order strongly general affine function. Then F is a higher order strongly general convex function, if and only if, $g = F - f$ is a general convex function.*

Proof. Let f be a higher order strongly general affine function, Then

$$f((1-t)u + tg(v)) = (1-t)f(u) + tf(g(v)) - \mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - u\|^p, \forall u, g(v) \in K_g. \quad (3.13)$$

From the higher order strongly general convexity of F, we have

$$F((1-t)u + tg(v)) \leq (1-t)F(u) + tF(g(v)) - \mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - u\|^p, \forall u, g(v) \in K_g. \quad (3.14)$$

From (3.19) and (3.14), we have

$$F((1-t)u + tg(v)) - f((1-t)f(u) + tf(g(v)) \leq (1-t)(F(u) - f(u)) + t(F(g(v)) - f(g(v))), \quad (3.15)$$

from which it follows that

$$\begin{aligned} G((1-t)u + tg(v)) &= F((1-t)u + tg(v)) - f((1-t)u + tg(v)) \\ &\leq (1-t)F(u) + tF(g(v)) - (1-t)f(u(-tf(g(v)) \\ &= (1-t)(F(u) - f(u)) + t(F(g(v) - f(v)), \end{aligned}$$

which show that $G = F - f$ is a convex function.

□

We now discuss the optimality for the differentiable generalized strongly convex functions, which is the main motivation of our next result.

Theorem 3.8. *Let F be a differentiable higher order strongly general convex function with modulus $\mu > 0$. If $u \in K_g$ is the minimum of the function F, then*

$$F'(g(v)) - F(u) \geq \mu \|g(v) - u\|^p, \quad \forall u, g(v) \in K_g. \quad (3.16)$$

Proof. Let $u \in K_g$ be a minimum of the function F. Then

$$F(u) \leq F(g(v)), \forall v \in K_g. \quad (3.17)$$

Since K_g is a general convex set, so, $\forall u, g(v) \in K_g, \quad t \in [0, 1]$,

$$v_t = (1-t)u + tg(v) \in K_g.$$

Taking $v = v_t$ in (3.17), we have

$$0 \leq \lim_{t \to 0} \{ \frac{F(u + t(g(v) - u)) - F(u)}{t} \} = \langle F'(u), g(v) - u \rangle. \quad (3.18)$$

Since F is differentiable higher order strongly convex function, so

$$F(u + t(g(v) - u)) \leq F(u) + t(F(g(v)) - F(u)) - \mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - u\|^p, \forall u, g(v) \in K_g,$$

from which, using (3.18), we have

$$\begin{aligned} F(v) - F(u) &\geq \lim_{t \to 0} \{ \frac{F(u + t(v - u)) - F(u)}{t} \} + \mu\{t^{p-1}(1-t) + (1-t)^p\}\|v - u\|^p \\ &= \langle F'(u), g(v) - u \rangle + \mu \|g(v) - u\|^p, \end{aligned}$$

the required result (3.16). □

Remark: We would like to mention that, if

$$\langle F'(u), g(v) - u \rangle + \mu \|g(v) - u\|^p \geq 0, \quad \forall u, g(v) \in K_g, \quad (3.19)$$

then $u \in K_g$ is the minimum of the function F. The inequality of the type (3.19) is called the higher order strongly general variational inequality, which appears to be new one.

4. Higher order general variational inequalities

In this section, we consider a more general variational inequality of which (3.19) is a special case. For given two operators T, g, we consider the problem of finding $u \in K$ for a constant μ such that

$$\langle Tu, g(v) - u\rangle + \mu\|g(v) - u\|^p \geq 0, \forall g(v) \in K, p > 1, \tag{4.1}$$

which is called the higher order strongly general variational inequality. It is clear that or $Tu = F'(g(u))$, problem (4.1) is exactly the general variational inequality (3.19). We also note that, if $\mu = 0$, then (3.14) is equivalent to finding $u \in K$, such that

$$\langle Tu, g(v) - u\rangle \geq 0, \forall g(v) \in K, \tag{4.2}$$

which is known as the general variational inequality, which was introduced and studied by Noor [15] in 2008. For suitable and appropriate choice of the parameter μ and p, one can obtain several new and known classes of variational inequalities, see [13, 15, 20–22]. We note that the projection method and its variant forms can be used to study the higher order strongly general variational inequalities (4.1) due to its inherent structure. This fact motivated us to consider the auxiliary principle technique, which is mainly due to Glowinski et al [21] and Lions and Stampacchia [20] as developed by Noor [13, 14]. We use this technique to suggest some iterative methods for solving the general variational inequalities (4.1).

For given $u \in K$ satisfying (4.1), consider the problem of finding $w \in K$, such that

$$\langle \rho Tw, g(v) - w\rangle + \langle w - u, v - w\rangle + \nu\|g(v) - w\|^p \geq 0, \forall g(v) \in K, p > 1, \tag{4.3}$$

where $\rho > 0$ is a parameter. The problem (4.3) is called the auxiliary higher order strongly general variational inequality. It is clear that the relation (4.3) defines a mapping connecting the problems (4.1) and (4.3). We not that, if $w(u) = u$, then w is a solution of problem (4.1). This simple observation enables to suggest an iterative method for solving (4.1).

Algorithm 4.1. *For given $u \in K$, find the approximate solution u_{n+1} by the scheme*

$$\langle \rho Tu_{n+1}, g(v) - u_{n+1}\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \nu\|g(v) - u_{n+1}\|^p \geq 0. \forall g(v) \in K, p > 1. \tag{4.4}$$

The Algorithm 4.1 is known as the implicit method. Such type of methods have been studied extensively for various classes of variational inequalities. See [13, 14] and the reference therein. If $\nu = 0$, then Algorithm 4.1 reduces to:

Algorithm 4.2. *For given $u_0 \in K$, find the approximate solution u_{n+1} by the scheme*

$$\langle \rho Tu_{n+1}, g(v) - u_{n+1}\rangle + \langle u_{n+1}u_n, v - u_{n+1}\rangle \geq 0, \forall g(v) \in K,$$

which appears to new ones even for solving the general variational inequalities (4.2).

In to study the convergence analysis of Algorithm 4.1, we need the following concept.

Definition 4.1. *The operator T is said to be pseudo g-monotone with respect to* $\mu||g(v) - u||^p, p > 1$, *if*

$$\langle \rho Tu, g(v) - u\rangle + \mu||g(v) - u||^p \geq 0, \forall g(v) \in K, p > 1,$$
$$\Longrightarrow$$
$$\langle \rho Tv, v - g(u)\rangle - \mu||g(u) - v||^p \geq 0, \forall g(v) \in K, p > 1$$

If $\mu = 0$, then Definition 4.1 reduces to:

Definition 4.2. *The operator T is said to be pseudo g-monotone, if*

$$\langle \rho Tu, g(v) - u\rangle \geq 0, \forall g(v) \in K$$
$$\Longrightarrow$$
$$\langle \rho Tv, v - g(u)\rangle \geq 0, \forall g(v) \in K,$$

which appears to be a new one.

We now study the convergence analysis of Algorithm 4.1.

Theorem 4.1. *Let* $u \in K$ *be a solution of (4.1) and* u_{n+1} *be the approximate solution obtained from Algorithm 4.1. If T is a pseudo g- monotone operator, then*

$$||u_{n+1} - u||^2 \leq ||u_n - u||^2 - ||u_{n+1} - u_n||^2. \tag{4.5}$$

Proof. Let $u \in K$ be a solution of (4.1), then

$$\langle \rho Tu, g(v) - u\rangle + \mu||g(v) - u||^p, \forall g(v) \in K,$$

implies that

$$\langle \rho Tv, g(u) - v\rangle - \mu||g(u) - v||^p, \forall g(v) \in K, \tag{4.6}$$

Now taking $v = u_{n+1}$ in (4.6), we have

$$\langle \rho Tu_{n+1}, u_{n+1} - g(u)\rangle - \mu||u_{n+1} - g(u)||^p \geq 0. \tag{4.7}$$

Taking $v = u$ in (4.4), we have

$$\langle \rho Tu_{n+1}, g(u) - u_{n+1}\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + v||g(u) - u_{n+1}||^p \geq 0. \forall g(v) \in K, p > 1. \tag{4.8}$$

Combining (4.7) and (4.8), we have

$$\langle u_{n+1} - u_n, u_{n+1} - u\rangle \geq 0.$$

Using the inequality

$$2\langle a, b\rangle = ||a + b||^2 - ||a||^2 - ||b||^2, \forall a, b \in H,$$

we obtain

$$||u_{n+1} - u||^2 \leq ||u_n - u||^2 - ||u_{n+1} - u_n||^2,$$

the required result (4.5). □

Theorem 4.2. *Let the operator T be a pseudo g-monotone. If u_{n+1} be the approximate solution obtained from Algorithm 4.1 and $u \in K$ is the exact solution (4.1), then* $\lim_{n\to\infty} u_n = u$.

Proof. Let $u \in K$ be a solution of (4.1). Then, from (4.5), it follows that the sequence $\{||u - u_n||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. From (4.5), we have

$$\sum_{n=0}^{\infty} ||u_{n+1} - u_n||^2 \leq ||u_0 - u||^2,$$

from which, it follows that

$$\lim_{n\to\infty} ||u_{n+1} - u_n|| = 0. \tag{4.9}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence u_n converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (5.5), taking the limit $n_j \to 0$ and from (4.9), we have

$$\langle T\hat{u}, g(v) - \hat{u}\rangle + \mu||g(v) - \hat{u}||^p, \quad \forall g(v) \in K, p > 1.$$

This implies that $\hat{u} \in K$ K satisfies and

$$||u_{n+1} - u_n||^2 \leq ||u_n - \hat{u}||^2.$$

Thus it follows from the above inequality that the sequence u_n has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} = \hat{u}.$$

□

In order to implement the implicit Algorithm 4.1, one uses the predictor-corrector technique. Consequently, Algorithm 4.1 is equivalent to the following iterative method for solving the general variational inequality (4.1).

Algorithm 4.3. *For a given $u_0 \in K$, find the approximate solution u_{n+1} by the schemes*

$$\begin{aligned} \langle \rho T u_n, g(v) - y_n\rangle &+ \langle y_n - u_n v - y_n\rangle + \mu||g(v) - y_n||^p \geq 0, \forall g(v) \in K, p > 1 \\ \langle \rho T y_n, g(v) - y_n\rangle &+ \langle u_n - y_n, v - y_n\rangle \mu||g(v) - u_n||^p \geq 0, \forall g(v) \in K, p > 1. \end{aligned}$$

Algorithm 4.3 is called the two-step iterative method and appears to be a new one.

Using the auxiliary principle technique, on can suggest several iterative methods for solving the higher order strongly general variational inequalities and related optimization problems. We have only given a glimpse of the higher order strongly general variational inequalities. It is an interesting problem to explore the applications of such type variational inequalities in various fields of pure and applied sciences.

5. Parallelogram laws

In this section, we discuss some characterizations of uniformly Banach spaces involving the notion of strongly general convexity, which can be viewed as novel application.

Setting $F(u) = ||u||^p$ in Definition 2.6, we have

$$||u + t(g(v) - u)||^p \leq (1 - t)||u||^p + t||v||^p - \mu\{t^p(1 - t) + t(1 - t)^p\}||g(v) - u||^p, \quad \forall u, g(v) \in K_g, t \in [0, 1]. \tag{5.1}$$

Taking $t = \frac{1}{2}$ in (5.1), we have

$$||\frac{u + g(v)}{2}||^p + \mu\frac{1}{2^p}||g(v) - u||^p \leq \frac{1}{2}||u||^p + \frac{1}{2}||g(v)||^p, \forall u, g(v) \in K_g, \tag{5.2}$$

which implies that

$$||u + g(v)||^p + \mu||g(v) - u||^p \leq 2^{p-1}\{||u||^p + ||g(v)||^p\}, \forall u, g(v) \in K_g, \tag{5.3}$$

which is known as the lower parallelogram for the Banach spaces. In a similar way, one can obtain the upper parallelogram law as

$$||u + g(v)||^p + \mu||v - u||^p \geq 2^{p-1}\{||u||^p + ||g(v)||^p\}, \forall u, g(v) \in K_g, \tag{5.4}$$

From Definition (2.10), we have

$$||u + g(v)||^p + \mu||g(v) - u||^p = 2^{p-1}\{||u||^p + ||g(v)||^p\}, \forall u, g(v) \in K_g, \tag{5.5}$$

which is known as the general parallelogram law for the Banach spaces.

We now discuss important cases.

I. If $g(v) = mv, \quad m \in (0, 1)$, then we have a new m-parallelogram law as:

$$||u + mv||^p + \mu||mv - u||^p = 2^{p-1}\{||u||^p + ||mv||^p\}, \forall u, mv \in K_m, \quad m \in (0, 1). \tag{5.6}$$

II. For $g = I$, the identity operator, we obtain the parallelogram laws obtained by Xi [33] characterising the p-uniform convexity and q-uniform smoothness of a Banach space. See also Bynum [23] and Chen et al [24, 25], who have studied the properties and applications of the parallelogram laws. For the applications of the parallelogram laws in Banach spaces in prediction theory and applied sciences, see [24, 25] and the references therein.

6. Conclusions

In this paper, we have introduced and studied a new class of convex functions, which is called higher order strongly general convex function. It is shown that several new classes of strongly convex functions can be obtained as special cases of these higher order strongly convex functions. We have studied the basic properties of these functions. We have also considered a new class of higher order strongly general variational inequalities. Using the auxiliary principle technique, an implicit iterative method is suggested for finding the approximate solution of higher order general variational inequalities. Using the pseudo-monotonicity of the operator, convergence criteria is discussed. Some special cases are considered as application of the main results. We have derived the parallelogram laws in Banach spaces, which have applications in prediction theory and stochastic analysis. The interested readers may explore the applications and other properties of the higher order strongly general convex functions in various fields of pure and applied sciences. This is an interesting direction of future research.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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