Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(4): 3634–3645.
DOI: 10.3934/math.2020235
Received: 12 February 2020
Accepted: 10 April 2020
Published: 16 April 2020

Research article

Infinitely many solutions for a class of biharmonic equations with indefinite potentials

Wen Guan[1,*], **Da-Bin Wang**[1] **and Xinan Hao**[2]

[1] Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu, 730050, P. R. China
[2] School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong, 273165, P. R. China

* **Correspondence:** Email: mathguanw@163.com; Tel: +8613919957402.

Abstract: In this paper, we consider the following sublinear biharmonic equations

$$\Delta^2 u + V(x)\, u = K(x)|u|^{p-1}u,\ \ x \in \mathbb{R}^N,$$

where $N \geq 5,\ 0 < p < 1$, and K, V both change sign in $\mathbb{R}^N$. We prove that the problem has infinitely many solutions under appropriate assumptions on K, V. To our end, we firstly infer the boundedness of PS sequence, and then prove that the PS condition was satisfied. At last, we verify that the corresponding functional satisfies the conditions of the symmetric Mountain Pass Theorem.

Keywords: biharmonic equation; indefinite potential; symmetric Mountain Pass Theorem
Mathematics Subject Classification: 35J20, 35J65

1. Introduction and main result

In this paper, we consider the following sublinear biharmonic equations

$$\Delta^2 u + V(x)\, u = K(x)|u|^{p-1}u,\ \ x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 5,\ 0 < p < 1,\ V(x), K(x) \in L^\infty(\mathbb{R}^N)$ both change sign in $\mathbb{R}^N$ and satisfies some conditions specified below. Problem (1.1) involved biharmonic operator arises in the study of traveling waves in suspension bridge. Furthermore, it is well know that biharmonic operator arises in the study of static deflection of a plate, for more details, we refer the read to [1–3].

For biharmonic equations, there have been many results [4–22] and the references therein. In these results, some authors studied biharmonic equations on the whole space $\mathbb{R}^N$ [4,6,9,10,12,14–17,19–21], in which most of them were focused on superlinear case but few results involving sublinear case [10,15, 16, 20, 21]. On the other hand, we notice that, in [4, 19], authors considered the biharmonic equations under sign-changing potential and superlinear case. However, to our best knowledge, there are no results of biharmonic equations on $\mathbb{R}^N$ in case of sign-changing potential and sublinear case. In this paper, we will investigate the nontrivial solutions for Eq 1.1 with the potential indefinite in sign and sublinear case, the tool used in our paper is the symmetric Mountain Pass Theorem.

To stated our main result, we assume that:

(H_1) $V \in L^\infty(\mathbb{R}^N)$ and there exist $\alpha, R_0 > 0$ such that

$$V(x) \geq \alpha, \text{ for any } |x| \geq R_0.$$

(H_2) $\|V^-\|_{\frac{N}{4}} < \frac{1}{S}$, where $V^\pm(x) = \max\{\pm V(x), 0\}$ and the S is the constant of Sobolev:

$$\|u\|_{2_*}^2 \leq S\|\Delta u\|_2^2, \text{ for any } u \in H^2(\mathbb{R}^N), \text{where } 2_* = \frac{2N}{N-4}.$$

(H_3) $K \in L^\infty(\mathbb{R}^N)$ and there exist $\beta > 0, R_1 > R_2 > 0, y_0 = (y_1, ..., y_N) \in \mathbb{R}^N$ such that

$$K(x) \leq -\beta, \text{ for any } |x| > R_1; \;\; K(x) > 0, \text{ for any } x \in B(y_0, R_2).$$

Our main result is as follows:

Theorem 1.1. *Assume* $(H_1) - (H_3)$ *hold. Then problem* (1.1) *possesses infinitely many nontrivial solutions.*

For biharmonic equations on the whole space $\mathbb{R}^N$, the main difficulty one may face is the Sobolev embedding $H^2(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R})$ is not compact for $s \in [2, 2_*)$. To overcome this difficulty, one can restrict the corresponding energy functional to a subspace of $H^2(\mathbb{R}^N)$, which embeds compactly into $L^s(\mathbb{R}^N)$ with certain qualifications or consisting of radially symmetric functions. For example, Yin and Wu [17] and Ye and Tang [15] considered biharmonic equations with the potential V satisfying following conditions:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V(x) \geq b > 0$ and for each $M > 0$, $meas\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty$, where b is a constant and *meas* denotes Lebesgue measure in $\mathbb{R}^N$.

In fact, due to the condition (V_1), space

$$X = \left\{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\Delta u|^2 + |\nabla u|^2 + V(x)u^2 dx < +\infty\right\}$$

can embed compactly into $L^s(\mathbb{R}^N)$ for $s \in [2, 2_*)$, which is crucial in their paper.

Subsequently, Liu, Chen and Wu [6] and Ye and Tang [16] studied biharmonic equations with λV instead of potential V under more weaker condition than (V_1), i.e.,

(V_2) $V \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V(x) \geq b > 0$ there exists $M > 0$, $meas\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty$, where b is a constant and *meas* denotes Lebesgue measure in $\mathbb{R}^N$.

Under the condition (V_2), it is obvious that $V(x)$ no longer satisfies certain coercive condition. Hence, corresponding Sobolev's embedding is not compact. Fortunately, with the aid of parameter λ ($\lambda > 0$ large enough), they obtained that the corresponding energy functional possess the property of locally compact.

In [20, 21], Zhang, Tang and Zhang considered biharmonic equations on $\mathbb{R}^N$ under more weaken conditions than (V_1) and (V_2). However, their results does not allow $V(x)$ to change sign.

Recently, Su and Chen [10] studied the following sublinear biharmonic equation

$$\begin{cases}\Delta^2 u - \Delta u + \lambda V(x)\, u = \alpha(x) f(u) + \mu K(x)|u|^{q-2}u, \ x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N),\end{cases}$$

where $N > 4, \lambda > 0, 1 < q < 2$ and $\mu \in [0, \mu_0]$. By using Ekeland's variational principle and Gigliardo-Nirenberg's inequality, they proved the existence of nontrivial solution for the above problem. It is noticed that their results require conditions $V(x) \geq 0$ and $K(x) > 0$.

However, in this paper, the condition like type (V_1) or (V_2) does not be needed. Furthermore, functions $V(x)$ and $K(x)$ both change sign in $\mathbb{R}^N$. So, the conditions in this paper more weaken than that of in [6, 10, 15–17, 20, 21]. On the other hand, our main result also supplement the results obtained by [4, 19] in which the sign-changing potential and superlinear case were considered. It is worth pointing out that there are some interesting results, for example [23–36], considered elliptic equations with an indefinite nonlinearity or sublinear condition or nonlocal terms.

2. Notations and preliminaries

In this paper, we use the following notations. Let

$$\|u\|_q = \left(\int_{\mathbb{R}^N} |u|^q dx\right)^{\frac{1}{q}}, \ 1 \leq q < +\infty.$$

Let E be a Banach space and $\varphi : E \to \mathbb{R}$ be a functional of class C^1, the Fréchet derivative of φ at u, $\varphi'(u)$, is an element of the dual space E^* and we denote $\varphi'(u)$ evaluated at $v \in E$ by $\langle \varphi'(u), v \rangle$.

The Sobolev space $E = H^2(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N), 0 < p < 1$, endowed with the norm by

$$\|u\| = \|\Delta u\|_2 + \|u\|_{p+1}.$$

Obviously, the space E is a reflexive Banach space.

The energy functional $\varphi : E \to \mathbb{R}$ corresponding to problem (1.1) is defined by

$$\varphi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x) u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K(x)|u|^{p+1} dx.$$

It is well know that, under our conditions, $\varphi \in C^1(E)$ and its critical points are solutions of problem (1.1).

Definition 2.1. *([37]) Let E be a Banach space and A a subset of E. Set A is said to be symmetric if $u \in E$ implies $-u \in E$. For a closed symmetric set A which does not contain the origin, we define a genus $\gamma(A)$ of A by the smallest integer l such that there exist an odd continuous mapping from A to $\mathbb{R}^l \setminus \{0\}$. If there does not exists such a l, we define $\gamma(A) = \infty$. We set $\gamma(\emptyset) = 0$. Let Γ_l denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq l$.*

The following result is a version of the classical symmetric Mountain Pass Theorem [37, 38]. A proof can be found in [39].

Theorem 2.1. *([39]) Let E be an infinite dimensional Banach space and let $\varphi \in C^1(E,\mathbb{R})$ satisfy:*

(1) *φ is even, bounded from below, $\varphi(0) = 0$ and φ satisfies the Palais-Smale condition.*

(2) *For each $l \in \mathbb{N}$, there exists an $A_l \in \Gamma_l$ such that*

$$\sup_{u\in A_l} \varphi(u) < 0.$$

Then either of the following two conditions holds:

(i) *There exists a sequence u_l such that $\varphi'(u_l) = 0, \varphi(u_l) < 0$ and u_l converges to zero; or*

(ii) *There exist two sequences u_l and v_l such that $\varphi'(u_l) = 0, \varphi(u_l) = 0, u_l \neq 0$, $\lim_{l\to+\infty} u_l = 0, \varphi'(v_l) = 0, \varphi(v_l) < 0, \lim_{l\to+\infty} \varphi(v_l) = 0$ and v_l converges to a non-zero limit.*

3. Proof of theorem 1.1

Lemma 3.1. *If $(H_1) - (H_3)$ hold. Then any PS sequence of φ is bounded in E.*

Proof. Let $\{u_n\} \subset E$ be such that

$$\varphi(u_n) \text{ is bounded and } \varphi'(u_n) \to 0 \text{ as } n \to \infty.$$

That is, there exists $C > 0$ such that $\varphi(u_n) \le C$. Then, according to (H_3), Hölder inequality and Sobolev embedding, one has that

$$\begin{aligned} C \ge \varphi(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u_n^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K(x)|u_n|^{p+1} dx \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u_n|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} V^-(x)u_n^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K^+(x)|u_n|^{p+1} dx \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u_n|^2 dx - \frac{1}{2}\left(\int_{\mathbb{R}^N} |V^-|^{\frac{N}{4}} dx\right)^{\frac{4}{N}} \left(\int_{\mathbb{R}^N} (|u_n|^2)^{\frac{2_*}{2}} dx\right)^{\frac{2}{2_*}} \\ &-\tfrac{1}{p+1}\int_{\mathbb{R}^N} K^+(x)|u_n|^{p+1} dx \\ &\ge \left(\frac{1}{2} - \frac{S\|V^-\|_{\frac{N}{4}}}{2}\right)\|\Delta u_n\|_2^2 - \frac{S^{\frac{p+1}{2}}}{p+1}\|K^+\|_{\frac{2_*}{2_*-(p+1)}}\|\Delta u_n\|_2^{p+1}. \end{aligned}$$

So, thanks to $0 < p < 1$, there exists $\eta > 0$ such that

$$\|\Delta u_n\|_2 \le \eta, \text{ for any } n \in \mathbb{N}. \tag{3.1}$$

On the other hand, one has that

$$\begin{aligned} C + \frac{\|u_n\|}{2} &\ge \varphi(u_n) - \tfrac{1}{2}\langle \varphi'(u_n), u_n\rangle \\ &\ge \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N} K(x)|u_n|^{p+1} dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N} K^+(x)|u_n|^{p+1} dx + \left(\frac{1}{p+1} - \frac{1}{2}\right)\int_{\mathbb{R}^N} K^-(x)|u_n|^{p+1} dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N} \left(K^+(x) + \chi_{B(0,R_1)}(x)\right)|u_n|^{p+1} dx \\ &+ \left(\tfrac{1}{p+1} - \tfrac{1}{2}\right)\int_{\mathbb{R}^N} \left(K^-(x) + \chi_{B(0,R_1)}(x)\right)|u_n|^{p+1} dx. \end{aligned}$$

By (H_3), we have that

$$K^+(x) = 0, \text{ for all } |x| > R_1.$$

Then, thanks to $K \in L^\infty(\mathbb{R}^N)$, we have that

$$\int_{\mathbb{R}^N} |K^+(x) + \chi_{B(0,R_1)}(x)|^{\frac{2_*}{2_*-(p+1)}} dx = \int_{B(0,R_1)} |K^+(x) + \chi_{B(0,R_1)}(x)|^{\frac{2_*}{2_*-(p+1)}} dx < \infty.$$

So, by Hölder inequality and Soblev inequality, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (K^+(x) + \chi_{B(0,R_1)}(x))\, |u_n|^{p+1} dx \\
&\leq \left(\int_{\mathbb{R}^N} (K^+(x) + \chi_{B(0,R_1)}(x))^{\frac{2_*}{2_*-(p+1)}}\, dx\right)^{\frac{2_*-(p+1)}{2_*}} \times \left(\int_{\mathbb{R}^N} \left(|u_n|^{p+1}\right)^{\frac{2_*}{p+1}}\, dx\right)^{\frac{p+1}{2_*}} \\
&\leq S^{\frac{p+1}{2}} \|K^+ + \chi_{B(0,R_1)}\|_{\frac{2_*}{2_*-(p+1)}} \|\Delta u_n\|_2^{p+1}. \qquad (3.2)
\end{aligned}$$

By (H_3) again, we know that $K^-(x) \geq \beta$, for all $|x| > R_1$. Then, we have that

$$\int_{\mathbb{R}^N} (K^-(x) + \chi_{B(0,R_1)}(x))\, |v_n|^{p+1} dx \geq min(\beta, 1)\|v_n\|_{p+1}^{p+1}. \qquad (3.3)$$

Thanks to (3.1), (3.2) and (3.3), there is a constant $C_1 > 0$ such that

$$\|u_n\|_{p+1}^{p+1} \leq C_1 + C_1\|u_n\|_{p+1} \text{ for all } n \in \mathbb{N}.$$

Since $0 < p < 1$, $\{u_n\}$ is bounded in $L^{p+1}(\mathbb{R}^N)$.
Then, from (3.1), we conclude that $\{u_n\}$ is bounded in E. □

Lemma 3.2. *([40]) Let x, y be for all real numbers, there exists a constant $c > 0$ such that*

$$||x + y|^{p+1} - |x|^{p+1} - |y|^{p+1}| \leq c|x|^p|y|.$$

Lemma 3.3. *If $(H_1) - (H_3)$ hold, then φ satisfies the PS condition on E.*

Proof. Let $\{u_n\} \subset E$ be such that

$$\varphi(u_n) \text{ is bounded and } \varphi'(u_n) \to 0 \text{ as } n \to \infty.$$

According to Lemma 3.1, $\{u_n\}$ is bounded in E. Passing to a subsequence in necessary, we can assume that

$$u_n \rightharpoonup u \text{ in } E,$$
$$u_n \to u \text{ in } L^q_{loc}(\mathbb{R}^N), 2 \leq q < 2_*,$$
$$u_n \to u, \text{ a.e } \mathbb{R}^N.$$

So, for any $h \in C_0^\infty(\mathbb{R}^N)$, one has that

$$\int_{\mathbb{R}^N} \Delta u_n \Delta h + V(x)u_n h dx \to \int_{\mathbb{R}^N} \Delta u \Delta h + V(x)uhdx.$$

On the other hand, by Sobolev embedding and Lebesgue's dominated convergence theorem, one has that

$$\int_{\mathbb{R}^N} K(x)|u_n|^{p-1}u_n h(x)dx \to \int_{\mathbb{R}^N} K(x)|u|^{p-1}uh(x)dx.$$

Therefore, according to above facts, one has that

$$0 = \lim_{n\to+\infty} \langle \varphi'(u_n), h\rangle = \langle \varphi'(u), h\rangle, \quad \text{for any } \; h \in C_0^\infty(\mathbb{R}^N).$$

Hence, we have that

$$\langle \varphi'(u), u\rangle = 0.$$

Let $v_n = u_n - u$, then $u_n = v_n + u$, and we have that

$$\begin{aligned}
\langle \varphi'(u_n), u_n\rangle &= \int_{\mathbb{R}^N} \left(|\Delta u_n|^2 + V(x)u_n^2\right) dx - \int_{\mathbb{R}^N} K(x)|u_n|^{p+1}dx \\
&= \int_{\mathbb{R}^N} \left(|\Delta v_n|^2 + |\Delta u|^2 + 2\Delta v_n \Delta u + V(x)v_n^2 + V(x)u^2 + 2V(x)v_n u\right) dx \\
&- \int_{\mathbb{R}^N} K(x)|u_n|^{p+1}dx + \int_{\mathbb{R}^N} K(x)|u|^{p+1}dx - \int_{\mathbb{R}^N} K(x)|u|^{p+1}dx \\
&= \langle \varphi'(u), u\rangle + \int_{\mathbb{R}^N} |\Delta v_n|^2 dx + \int_{\mathbb{R}^N} V(x)v_n^2 dx \\
&- \int_{\mathbb{R}^N} K(x)|u_n|^{p+1}dx + \int_{\mathbb{R}^N} K(x)|u|^{p+1}dx + o_n(1) \\
&\geq \int_{\mathbb{R}^N} |\Delta v_n|^2 dx - \int_{\mathbb{R}^N} V^-(x)v_n^2 dx \\
&- \int_{\mathbb{R}^N} K(x)\left(|u_n|^{p+1} - |u|^{p+1}\right) dx + o_n(1).
\end{aligned}$$

By Lemma 3.2, we have that

$$\begin{aligned}
|K(x)|\left||u_n|^{p+1} - |u|^{p+1} - |v_n|^{p+1}\right| &= |K(x)|\left||v_n + u|^{p+1} - |u|^{p+1} - |v_n|^{p+1}\right| \\
&\leq c|K(x)||u|^p|v_n|.
\end{aligned} \tag{3.4}$$

We claim that

$$\int_{\mathbb{R}^N} |K(x)||u|^p|v_n|dx \leq C\int_{\mathbb{R}^N} |u|^p|v_n|dx \to 0, \tag{3.5}$$

as $n \to +\infty$.

In fact, because $E \hookrightarrow L^{p+1}(\mathbb{R}^N)$ is continuous and $v_n \rightharpoonup 0$ in E, we obtain that $v_n \rightharpoonup 0$ in $L^{p+1}(\mathbb{R}^N)$. On the other hand, it is obvious that $|u|^p \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$. So, by $K \in L^\infty(\mathbb{R}^N)$ and definition of weakly convergence in space $L^{p+1}(\mathbb{R}^N)$, we have that

$$\int_{\mathbb{R}^N} |K(x)||u|^p|v_n|dx \leq C\int_{\mathbb{R}^N} |u|^p|v_n|dx \to 0,$$

as $n \to +\infty$.

Hence, according to (3.4) and (3.5), we have that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)[|u_n|^{p+1} - |u|^{p+1}]dx = \lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)|v_n|^{p+1}dx.$$

Then, we obtain that

$$\begin{aligned}\langle\varphi'(u_n),u_n\rangle &\geq \langle\varphi'(u),u\rangle + \int_{\mathbb{R}^N}|\Delta v_n|^2dx - \int_{\mathbb{R}^N}V^-(x)v_n^2dx\\ &- \int_{\mathbb{R}^N}K(x)|v_n|^{p+1}dx + o_n(1)\\ &= \int_{\mathbb{R}^N}|\Delta v_n|^2dx - \int_{\mathbb{R}^N}V^-(x)v_n^2dx \\ &- \int_{\mathbb{R}^N}\left(K^+(x)+\chi_{B(0,R_1)}(x)\right)|v_n|^{p+1}dx\\ &+ \int_{\mathbb{R}^N}\left(K^-(x)+\chi_{B(0,R_1)}(x)\right)|v_n|^{p+1}dx + o_n(1).\end{aligned} \tag{3.6}$$

Claim 1: $\int_{\mathbb{R}^N}V^-(x)v_n^2dx \to 0$ as $n\to+\infty$.

In fact, by (H_1), we have that $V^-(x)=0$, for all $|x|\geq R_0$. So, from $v_n\to 0$ in $L^q_{loc}(\mathbb{R}^N), 2\leq q<2_*$, and $V\in L^\infty(\mathbb{R}^N)$, we obtain $\int_{\mathbb{R}^N}V^-(x)v_n^2dx\to 0$ as $n\to+\infty$.

Claim 2: $\int_{\mathbb{R}^N}\left(K^+(x)+\chi_{B(0,R_1)}(x)\right)|v_n|^{p+1}dx\to 0$ as $n\to+\infty$.

In fact, by (H_3), we have that $K^+(x)=0$, for all $|x|>R_1$. Thanks to $K\in L^\infty(\mathbb{R}^N)$ and $v_n\to 0$ in $L^q_{loc}(\mathbb{R}^N), 2\leq q<2_*$, we get

$$\int_{\mathbb{R}^N}\left(K^+(x)+\chi_{B(0,R_1)}(x)\right)|v_n|^{p+1}dx\to 0$$

as $n\to+\infty$.

Combining claim 1, claim 2, (3.3) and (3.6), we obtain that

$$0=\lim_{n\to+\infty}\left(\|\Delta v_n\|_2^2+min(\beta,1)\|v_n\|_{p+1}^{p+1}\right).$$

That is, $v_n\to 0$ in E. The proof is complete. □

The proof of following Lemma is based on some ideas of Kajikiya [39] and very similar to the one contained in [40]. For reader convenient, we give the proof.

Lemma 3.4. *If $(H_1)-(H_3)$ hold, then for each $l\in\mathbb{N}$, there exists subset $A_l\in\Gamma_l$ such that,*

$$\sup_{u\in A_l}I(u)<0.$$

Proof. For R_2 and y_0 given by (H_3), let

$$B(R_2)=\{(x_1,\cdots,x_n)\in\mathbb{R}^N:|x_i-y_i|<R_2,\ \ 1\leq i\leq N\}.$$

Let $l\in\mathbb{N}$ be an arbitrary number and define $n=\min\{n\in\mathbb{N}:n^N\geq l\}$. By planes parallel to each face of $B(R_2)$, $B(R_2)$ be equally divided into n^N small partes B_i with $1\leq i\leq n^N$. In fact, the length a of the edge B_i is $\frac{R_2}{n}$. Let $F_i\subset B_i$ be new cubes such that F_i has the same center as that of B_i. The faces of F_i and B_i are parallel, and the length of the edge of F_i is $\frac{a}{2}$. Let $\phi_i\in C(\mathbb{R}^N)$, $1\leq i\leq l$, satisfy: $supp(\phi_i)\subset B_i$; $supp(\phi_i)\cap supp(\phi_j)=\emptyset$ $(i\neq j)$; $\phi_i(x)=1$ for $x\in F_i$; $0\leq\phi_i(x)\leq 1$, for all $x\in\mathbb{R}^N$. Let

$$S^{l-1}=\{(t_1,\cdots,t_l)\in\mathbb{R}^l:\quad \max_{1\leq i\leq l}|t_i|=1\}, \tag{3.7}$$

$$W_l=\{\textstyle\sum_{i=1}^l t_i\phi_i(x):(t_1,\cdots t_l)\in S^{l-1}\}\subset E.$$

According to the fact that the mapping $(t_1, \cdots, t_l) \to \sum_{i=1}^{l} t_i\phi_i$ from S^{l-1} to W_l is odd and homeomorphic, so $\gamma(W_l) = \gamma(S^{l-1}) = l$. Since W_l is compact in E, it follows that there exists $\alpha_l > 0$ such that

$$\|u\|^2 \le \alpha_l, \text{ for any } u \in W_l.$$

On the other hand, we claim that

$$\|u\|_2 \le C\|\Delta u\|_2^s \|u\|_{p+1}^{1-s} \le C\|u\|,$$

where $s = \frac{2_*(1-p)}{2(2_*-p-1)}$.

First, we prove that $\|u\|_{2_*} \le C\|\Delta u\|_2, \forall u \in E$.

In fact, $\forall u \in E$, $u \in H^2(\mathbb{R}^N)$, so $u, \nabla u \in H^1(\mathbb{R}^N)$. Then, by Gagliardo-Nirenberg-Sobolev inequality (since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$), we have

$$\|\nabla u\|_{2^*} \le C_1\|\Delta u\|_2$$

where $2^* = \frac{2N}{N-2}$ and C_1 depending only on N.

Since $2 < 2^* < N$, by using Gagliardo-Nirenberg-Sobolev inequality again, we have

$$\|u\|_{2_*} = \|u\|_{(2^*)^*} \le C_2\|\nabla u\|_{2^*},$$

where C_2 depending on N.

Next, since $1 < p + 1 < 2 < 2_*$, by interpolation inequality, we have

$$\|u\|_2 \le \|u\|_{2_*}^s \|u\|_{p+1}^{1-s},$$

where $\frac{1}{2} = \frac{s}{2_*} + \frac{1-s}{p+1}$ (thai is $s = \frac{2_*(1-p)}{2(2_*-p-1)}$).

So $\|u\|_2 \le C\|\Delta u\|_2^s \|u\|_{p+1}^{1-s} \le C\|u\|$.

According to above facts, there exists $c_l > 0$ such that

$$\|u\|_2^2 \le c_l \text{ for all } u \in W_l.$$

Let $t > 0$ and $v = \sum_{=1}^{l} t_i\phi_i(x) \in W_l$,

$$\begin{aligned} \varphi(tv) &= \frac{t^2}{2}\int_{\mathbb{R}^N}(|\Delta v|^2 + V(x)v^2)dx - \frac{1}{p+1}\sum_{i=1}^{l}\int_{B_i} K(x)|tt_i\phi_i|^{p+1}dx \\ &\le \frac{t^2}{2}\alpha_l + \frac{t^2}{2}\|V\|_\infty c_l - \frac{1}{p+1}\sum_{i=1}^{l}\int_{B_i} K(x)|tt_i\phi_i|^{p+1}dx. \end{aligned} \tag{3.8}$$

From (3.2), there exists $j \in [1, l]$ such that $|t_j| = 1$ and $|t_i| \le 1$ for $i \ne j$. So

$$\begin{aligned} \sum_{i=1}^{l}\int_{B_i} K(x)|tt_i\phi_i|^{p+1}dx &= \int_{F_j} K(x)|tt_j\phi_j|^{p+1}dx \\ &+ \int_{B_j\setminus F_j} K(x)|tt_j\phi_j(x)|^{p+1}dx + \sum_{i\ne j}\int_{B_i} K(x)|tt_i\phi_i|^{p+1}dx. \end{aligned} \tag{3.9}$$

According to $\phi_j(x) = 1$ for $x \in F_j$ and $|t_j| = 1$, one has that

$$\int_{F_j} K(x)|tt_j\phi_j|^{p+1}dx = |t|^{p+1}\int_{F_j} K(x)dx. \tag{3.10}$$

By (H_3), one has that

$$\int_{B_j\setminus F_j} K(x)|tt_j\phi_j(x)|^{p+1}dx + \sum_{i\neq j}\int_{B_i} K(x)|tt_i\phi_i|^{p+1}dx \geq 0. \tag{3.11}$$

So, combining (3.3), (3.4), (3.5) and (3.6), we have that

$$\frac{\varphi(tv)}{t^2} \leq \frac{1}{2}\alpha_l + \frac{1}{2}\|V\|_\infty c_l - \frac{|t|^{p+1}}{(p+1)t^2}\inf_{1\leq i\leq l}\left(\int_{F_i} K(x)dx\right).$$

Therefore, it is easy to see that

$$\lim_{t\to 0}\sup_{v\in W_l}\frac{\varphi(tv)}{t^2} = -\infty.$$

Hence, we can fixed t small enough such that $\sup\{\varphi(v), v\in A_l\} < 0$, where $A_l = tW_l \in \Gamma_l$. □

Lemma 3.5. *If $(H_1)-(H_3)$ hold. Then φ is bounded from below.*

Proof. By (H_3), Hölder inequality and Sobolev embedding, as in the proof of Lemma 3.1, we have that

$$\begin{aligned}\varphi(u) &= \tfrac{1}{2}\int_{\mathbb{R}^N}\left(|\Delta u|^2 + V(x)u^2\right)dx - \tfrac{1}{p+1}\int_{\mathbb{R}^N} K(x)|u|^{p+1}dx \\ &\geq \tfrac{1}{2}\int_{\mathbb{R}^N}\left(|\Delta u|^2 - V^-(x)u^2\right)dx - \tfrac{1}{p+1}\int_{\mathbb{R}^N} K^+(x)|u|^{p+1}dx \\ &\geq \left(\tfrac{1}{2} - \frac{S\|V^-\|_{\frac{N}{4}}}{2}\right)\|\Delta u\|_2^2 - \tfrac{S^{\frac{p+1}{2}}}{p+1}\|K^+\|_{\frac{2_*}{2_*-p-1}}\|\Delta u\|_2^{p+1}.\end{aligned}$$

Since $0 < p < 1$, we conclude the proof. □

The proof of Theorem 1.1

Proof. In fact, $\varphi(0) = 0$ and φ is an even functional. Then by Lemma 3.3, Lemma 3.4 and Lemma 3.5, the conditions (1) and (2) of Theorem 2.1 are satisfied. Therefore, by Theorem 2.1, problem (1.1) possesses infinitely many nontrivial solutions converging to 0 with negative energy. □

Remark 3.1. *By using Theorem 2.1, we obtain infinitely many nontrivial solutions to problem* (1.1). *For infinitely many nontrivial solutions converges to* 0*, we must verify that the functional φ satisfies some assumption like (A3) in [39], (see Remark 1.2 in [39]). In fact, we can verify this property by following similar inequality obtained in [[40], page 460].*

4. Conclusions

In this paper, by using the symmetric Mountain Pass Theorem, we prove a class of biharmonic equations with indefinite potentials has infinitely many solutions. Because our result mainly involves theoretical research, we don't know how to use our result to the real applications in practical problems. So, we should pay attention to both theoretical research and practical application in the follow-up work.

Acknowledgments

The paper is supported by the Natural Science Foundation of China (Grant no. 11561043,11961043) and Natural Science Foundation of China (Grant no. 11501318).

Conflict of interest

The authors declare no conflict of interest in this paper.

References

1. Y. Chen, P. J. McKenna, *Traveling waves in a nonlinearly suspension beam: Theoretical results and numerical observations*, J. Differ. Equations, **135** (1997), 325–355.
2. A. C. Lazer, P. J. McKenna, *Large-amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis*, Siam. Rev., **32** (1990), 537–578.
3. P. J. McKenna, W. Walter, *Traveling waves in a suspension bridge*, Siam J. Appl. Math., **50** (1990), 703–715.
4. C. O. Alves, J. Marcos do Ó, O. H. Miyagaki, *Nontrivial solutions for a class of semilinear biharmonic problems involving critical exponents*, Nonlinear Anal-Theor., **46** (2001), 121–133.
5. K. Kefi, K. Saoudi, *On the existence of a weak solution for some singular $p(x)$-biharmonic equation with Navier boundary conditions*, Adv. Nonlinear Anal., **8** (2018), 1171–1183.
6. J. Liu, S. X. Chen, X. Wu, *Existence and multiplicity of solutions for a class of fourth-order elliptic equations in $\mathbb{R}^N$*, J. Math. Anal. Appl., **395** (2012), 608–615.
7. A. Mao, W. Wang, *Nontrivial solutions of nonlocal fourth order elliptic equation of Kirchhoff type in $\mathbb{R}^3$*, J. Math. Anal. Appl., **459** (2018), 556–563.
8. Y. Pu, X. P. Wu, C. L. Tang, *Fourth-order Navier boundary value problem with combined nonlinearities*, J. Math. Anal. Appl., **398** (2013), 798–813.
9. M. T. O. Pimenta, S. H. M. Soares, *Singulary perturbed biharmonic problem with superlinear nonlinearities*, Adv. Differential Equ., **19** (2014), 31–50.
10. Y. Su, H. Chen, *The existence of nontrivial solution for a class of sublinear biharmonic equations with steep potential well*, Bound. Value Probl., **2018** (2018), 1–14.
11. X. Wang, A. Mao, A. Qian, *High energy solutions of modified quasilinear fourth-order elliptic equation*, Bound. Value Probl., **2018** (2018), 1–13.
12. Y. Wang, Y. Shen, *Multiple and sign-changing solutions for a class of semilinear biharmonic equation*, J. Differ. Equations, **246** (2009), 3109–3125.
13. Y. Wei, *Multiplicity results for some fourth-order elliptic equations*, J. Math. Anal. Appl., **385** (2012), 797–807.
14. M. B. Yang, Z. F. Shen, *Infinitely many solutions for a class of fourth order elliptic equations in $\mathbb{R}^N$*, Acta Math. Sin., **24** (2008), 1269–1278.
15. Y. W. Ye, C. L. Tang, *Infinitely many solutions for fourth-order elliptic equations*, J. Math. Anal. Appl., **394** (2012), 841–854.
16. Y. W. Ye, C. L. Tang, *Existence and multiplicity of solutions for fourth-order elliptic equations in $\mathbb{R}^N$*, J. Math. Anal. Appl., **406** (2013), 335–351.
17. Y. L. Yin, X. Wu, *High energy solutions and nontrivial solutions for fourth-order elliptic equations*, J. Math. Anal. Appl., **375** (2011), 699–705.

18. J. Zhang, Z. Wei, *Infinitely many nontrivial solutions for a class of biharmonic equations via variant fountain theorems*, Nonlinear Anal-Theor., **74** (2011), 7474–7485.
19. W. Zhang, X. H. Tang, J. Zhang, *Infinitely many solutions for fourth-order elliptic equations with sign-changing potential*, Taiwan. J. Math., **18** (2014), 645–659.
20. W. Zhang, X. H. Tang, J. Zhang, *Infinitely many solutions for fourth-order elliptic equations with general potentials*, J. Math. Anal. Appl., **407** (2013), 359–368.
21. W. Zhang, X. H. Tang, J. Zhang, *Existence and concentration of solutions for sublinear fourth-order elliptic equations*, Electronic J. Differ. Eq., **2015** (2015), 1–9.
22. J. W. Zhou, X. Wu, *Sign-changing solutions for some fourth-order nonlinear elliptic problems*, J. Math. Anal. Appl., **342** (2008), 542–558.
23. A. Bahrouni, H. Ounaies, V. D. Rădulescu, *Bound state solutions of sublinear Schrödinger equations with lack of compactness*, Racsam. Rev. R. Acad. A., **113** (2019), 1191–1210.
24. A. Bahrouni, V. D. Rădulescu, D. Repovs, *Double phase transonic flow problems with variable growth: nonlinear patterns and stationary waves*, Nonlinearity, **32** (2019), 2481–2495.
25. G. Bonanno, G. D'Aguì, A. Sciammetta, *Nonlinear elliptic equations involving the p-Laplacian with mixed Dirichlet-Neumann boundary conditions*, Opuscula Math., **39** (2018), 159–174.
26. Y. Li, D. B. Wang, J. Zhang, *Sign-changing solutions for a class of p-Laplacian Kirchhoff-type problem with logarithmic nonlinearity*, AIMS Math., **5** (2020), 2100–2112.
27. N. S. Papageorgiou, V. D. Radulescu, D. D. Repovs, *Nonlinear analysis-theory and methods*, Springer Monographs in Mathematics, Springer, Cham, 2019.
28. N. S. Papageorgiou, V. D. Radulescu, D. D. Repovs, *Positive solutions for nonlinear parametric singular Dirichlet problems*, Bull. Math. Sci., **9** (2019), 1950011.
29. H. R. Quoirin, K. Umezu, *An elliptic equation with an indefinite sublinear boundary condition*, Adv. Nonlinear Anal., **8** (2019), 175–192.
30. D. B. Wang, *Least energy sign-changing solutions of Kirchhoff-type equation with critical growth*, J. Math. Phys., **61** (2020), 011501. Available from: https://doi.org/10.1063/1.5074163.
31. D. B. Wang, T. Li, X. Hao, *Least-energy sign-changing solutions for Kirchhoff-Schrödinger-Poisson systems in* $\mathbb{R}^3$, Bound. Value Probl., **75** (2019). Available from: https://doi.org/10.1186/s13661-019-1183-3.
32. D. B. Wang, Y. Ma, W. Guan, *Least energy sign-changing solutions for the fractional Schrödinger-Poisson systems in* $\mathbb{R}^3$, Bound. Value Probl., **25** (2019). Available from: https://doi.org/10.1186/s13661-019-1128-x.
33. D. B. Wang, H. Zhang, W. Guan, *Existence of least-energy sign-changing solutions for Schrödinger-Poisson system with critical growth*, J. Math. Anal. Appl., **479** (2019), 2284–2301.
34. D. B. Wang, H. Zhang, Y. Ma, et al. *Ground state sign-changing solutions for a class of nonlinear fractional Schrödinger-Poisson system with potential vanishing at infinity*, J. Appl. Math. Comput., **61** (2019), 611–634.
35. D. B. Wang, J. Zhang, *Least energy sign-changing solutions of fractional Kirchhoff-Schrödinger-Poisson system with critical growth*, App. Math. Lett., **106** (2020), 106372.

36. J. Zhao, X. Liu, Z. Feng, *Quasilinear equations with indefinite nonlinearity*, Adv. Nonlinear Anal., **8** (2018), 1235–1251.

37. P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in. Math., 65, American Mathematical Society, Providence, RI, 1986.

38. A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **14** (1973), 349–381.

39. R. Kajikiya, *A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations*, J. Funct. Anal., **225** (2005), 352–370.

40. A. Bahrouni, H. Ounaies, V. D. Rădulescu, *Infinitely many solutions for a class of sublinear Schrödinger equations with indefinite potential*, P. Roy. Soc. Edinburgh, Sect. A, **145** (2015), 445–465.

AIMS Press