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Research article

On nonlocal fractional symmetric Hanh integral boundary value problems for fractional symmetric Hahn integrodifference equation

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Abstract: In this paper, we propose a boundary value problems for fractional symmetric Hahn integrodifference equation. The problem contains two fractional symmetric Hahn difference operators and three fractional symmetric Hahn integral with different numbers of order. The existence and uniqueness result of problem is studied by using the Banach fixed point theorem. The existence of at least one solution is also studied, by using Schauder's fixed point theorem.

Keywords: fractional symmetric Hahn integral; Riemann-Liouville fractional symmetric Hahn difference; boundary value problems; existence
Mathematics Subject Classification: 39A10, 39A13, 39A70

1. Introduction

Concurrent with the development of classic calculus theory, quantum calculus (calculus without limit) have received a great deal of attention in the last three decades. Quantum calculus have been found in many problems such as particle physics, quantum mechanics, and calculus of variations. In this paper, we study on the development of Hahn calculus, which is a type of quantum calculus. Hahn difference operator was first introduced by Hahn [1] in 1949 in the form of

$$D_{q,\omega}f(t) := \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega}, \quad t \neq \omega_0 := \frac{\omega}{1-q}.$$

This operator has been further employed in many research works such as the studies of the right inverse and its properties of Hahn difference operator [2, 3], Hahn quantum variational calculus [4, 5, 6], the initial value problems [7, 8, 9], and the boundary value problems [10, 11]. The approximation problems and constructing families of orthogonal polynomials [12, 13, 14], Hahn difference operator is an important tool used to study in these areas.

Based on the iadea of Hahn, in 2017, Brikshavana and Sitthiwirattham [15] introduced a general case of order of Hahn's operator, the so-called fractional Hahn difference operators. This operator has been used in the study of existencne and uniqueness of solution of boundary value problems for fractional Hahn difference equations (see [16, 17, 18, 19]).

The symmetric Hahn difference operator $\tilde{D}_{q,\omega}$ is another opertor related to Hahn's operator. It was introduced by Artur *et* al. in 2013 [20] where

$$\tilde{D}_{q,\omega}f(t) := \frac{f(qt+\omega) - f(q^{-1}(t-\omega))}{(q-q^{-1})t + (1+q^{-1})\omega} \quad \text{for } \ t \neq \omega_0.$$

Recently, Patanarapeelert and Sitthiwirattham [21] introduced the fractional symmetric Hahn integral, Riemann-Liouville and Caputo fractional symmetric Hahn difference operators and their properties. To present the advantage of this newest knowledge, in this paper, we devote our attention to study the solutions of boundary value problem for fractional symmetric Hahn difference equation.

Our problem is a nonlocal fractional symmetric Hanh integral boundary value problem for fractional symmetric Hahn integrodifference equation of the form

$$\begin{aligned} \tilde{D}^{\alpha}_{q,\omega}u(t) &= F\left(t, u(t), \tilde{D}^{\beta}_{q,\omega}u(t), \tilde{\Psi}^{\gamma}_{q,\omega}u(t),\right), \quad t \in I^T_{q,\omega}, \\ u(\omega_0) &= \lambda_1 \tilde{\mathcal{I}}^{\theta_1}_{q,\omega} g(\eta_1)u(\eta_1), \\ u(T) &= \lambda_2 \tilde{\mathcal{I}}^{\theta_2}_{q,\omega} g(\eta_2)u(\eta_2), \quad \eta_1, \eta_2 \in I^T_{q,\omega} - \{\omega_0, T\}, \end{aligned} \tag{1.1}$$

where $I^T_{q,\omega} := \{q^kT + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{\omega_0\}$; $\alpha \in (1,2]$; $\beta, \gamma, \theta_1, \theta_2 \in (0,1]$; $\omega > 0$; $q \in (0,1)$; $\lambda_1, \lambda_2 \in \mathbb{R}^+$; $F \in C(I^T_{q,\omega} \times \mathbb{R}^3, \mathbb{R})$ and $g_1, g_2 \in C(I^T_{q,\omega}, \mathbb{R}^+)$ are given functions; and for $\varphi \in C(I^T_{q,\omega} \times I^T_{q,\omega}, [0,\infty))$, we define

$$\tilde{\Psi}^{\gamma}_{q,\omega}u(t) := \left(\tilde{\mathcal{I}}^{\gamma}_{q,\omega}\varphi\, u\right)(t) = \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_{q,\omega}(\gamma)} \int_{\omega_0}^{t} \widetilde{(t-s)}^{\,\underline{\gamma-1}}_{q,\omega} \varphi(t, \sigma^{\alpha-1}_{q,\omega}(s))\, u(\sigma^{\alpha-1}_{q,\omega}(s))\, \tilde{d}_{q,\omega}s.$$

In the next section, we give some definitions and lemmas related to fractional symmetric Hahn calculus. In section 3, we analyze the existence and uniqueness of a solution of problem (1.1) by using the Banach fixed point theorem. Moreover, we show the existence of at least one solution of problem (1.1) by using the Schuader's fixed point theorem. Finally, we present an example to illustrate our results in the last section.

2. Preliminaries

2.1. Basic knowledge

In this section, we provide some notations, definitions, and lemmas related to the fractional symmetric Hahn difference calculus as follows [20, 21, 22, 23].
For $0 < q < 1$, $\omega > 0$, $\omega_0 = \frac{\omega}{1-q}$ and $[k]_q = \frac{1-q^k}{1-q}$, we define

$$\widetilde{[k]}_q := \begin{cases} \dfrac{1-q^{2k}}{1-q^2} = [k]_{q^2}, & k \in \mathbb{N} \\ 1, & k = 0, \end{cases}$$

$$\widetilde{[k]}_q! \; := \begin{cases} \widetilde{[k]}_q\widetilde{[k-1]}_q \cdots \widetilde{[1]}_q = \prod_{i=1}^{k} \dfrac{1-q^{2i}}{1-q^2}, & k \in \mathbb{N} \\ 1, & k = 0. \end{cases}$$

The q,ω-forward jump operator is defined by

$$\sigma^k_{q,\omega}(t) := q^k t + \omega[k]_q,$$

and the q,ω-backward jump operator is defined by

$$\rho^k_{q,\omega}(t) := \frac{t-\omega[k]_q}{q^k},$$

where $k \in \mathbb{N}$.

Let $n \in \mathbb{N}_0 := \{0, 1, 2, ...\}$, and $a, b \in \mathbb{R}$. We define the power functions as follows:

- The q-analogue of the power function

$$(a-b)^{\underline{0}}_q := 1, \qquad (a-b)^{\underline{n}}_q := \prod_{i=0}^{n-1}(a-bq^i).$$

- The q-symmetric analogue of the power function

$$(\widetilde{a-b})^{\underline{0}}_q := 1, \qquad (\widetilde{a-b})^{\underline{n}}_q := \prod_{i=0}^{n-1}(a-bq^{2i+1}).$$

- The q,ω-symmetric analogue of the power function

$$(\widetilde{a-b})^{\underline{0}}_{q,\omega} := 1, \qquad (\widetilde{a-b})^{\underline{n}}_{q,\omega} := \prod_{i=0}^{n-1}\left[a-\sigma^{2i+1}_{q,\omega}(b)\right].$$

Generally, for $\alpha \in \mathbb{R}$, we have

$$(a-b)^{\underline{\alpha}}_q = a^\alpha \prod_{i=0}^{\infty} \frac{1-\left(\frac{b}{a}\right)q^i}{1-\left(\frac{b}{a}\right)q^{\alpha+i}}, \; a \neq 0,$$

$$(\widetilde{a-b})^{\underline{\alpha}}_q = a^\alpha \prod_{i=0}^{\infty} \frac{1-\left(\frac{b}{a}\right)q^{2i+1}}{1-\left(\frac{b}{a}\right)q^{2(\alpha+i)+1}}, \; a \neq 0,$$

$$(\widetilde{a-b)})^{\underline{\alpha}}_{q,\omega} = \left((a-\omega_0)\widetilde{-(b}-\omega_0)\right)^{\underline{\alpha}}_q = (a-\omega_0)^\alpha \prod_{i=0}^{\infty} \frac{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^{2i+1}}{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^{2(\alpha+i)+1}}, \; a \neq \omega_0.$$

Particularly, we have $a^{\underline{\alpha}}_q = \widetilde{a}^{\underline{\alpha}}_q = a^\alpha$ and $(\widetilde{a-\omega_0})^{\underline{\alpha}}_{q,\omega} = (a-\omega_0)^\alpha$ if $b = 0$. If $a = b$, we define $(0)^{\underline{\alpha}}_q = \widetilde{(0)}^{\underline{\alpha}}_q = \widetilde{(\omega_0)}^{\underline{\alpha}}_{q,\omega} = 0$ for $\alpha > 0$.

The q-symmetric gamma and q-symmetric beta functions are defined as

$$\tilde{\Gamma}_q(x) \;:=\; \begin{cases} \frac{(1-q^2)_q^{\underline{x-1}}}{(1-q^2)^{x-1}} = \frac{\widetilde{(1-q)}_q^{\underline{x-1}}}{(1-q^2)^{x-1}}, & x \in \mathbb{R} \setminus \{0,-1,-2,...\} \\ \widetilde{[x-1]}_q!, & x \in \mathbb{N} \end{cases}$$

$$\tilde{B}_q(x,y) \;:=\; \int_0^1 \left(q^{-1}s\right)^{x-1} \widetilde{(1-s)}_q^{\underline{y-1}} \tilde{d}_q s = \frac{\tilde{\Gamma}_q(x)\tilde{\Gamma}_q(y)}{\tilde{\Gamma}_q(x+y)},$$

respectively.

Lemma 2.1. *[21] For $m,n \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$,*

(a) $\widetilde{(x-\sigma^n_{q,\omega}(x))}_{q,\omega}^{\underline{\alpha}} = (x-\omega_0)^k \widetilde{(1-q^n)}_q^{\underline{\alpha}}$,

(b) $\widetilde{(\sigma^m_{q,\omega}(x)-\sigma^n_{q,\omega}(x))}_{q,\omega}^{\underline{\alpha}} = q^{m\alpha}(x-\omega_0)^{\alpha}\widetilde{(1-q^{n-m})}_q^{\underline{\alpha}}$.

Definition 2.1. *[20] For $q \in (0,1)$, $\omega > 0$, and f is a function defined on $I^T_{q,\omega} \subseteq \mathbb{R}$, the symmetric Hahn difference of f is defined by*

$$\tilde{D}_{q,\omega} f(t) := \frac{f(\sigma_{q,\omega}(t)) - f(\rho_{q,\omega}(t))}{\sigma_{q,\omega}(t) - \rho_{q,\omega}(t)} \quad t \in I^T_{q,\omega} - \{\omega_0\},$$

$$\tilde{D}_{q,\omega} f(\omega_0) = f'(\omega_0) \;\; \textit{where } f \textit{ is differentiable at } \omega_0.$$

$\tilde{D}_{q,\omega} f$ is called q,ω-symmetric derivative of f, and f is q,ω-symmetric differentiable on $I^T_{q,\omega}$.

In addition, we define

$$\tilde{D}^0_{q,\omega} f(x) = f(x) \text{ and } \tilde{D}^N_{q,\omega} f(x) = \tilde{D}_{q,\omega}\tilde{D}^{N-1}_{q,\omega} f(x) \text{ where } N \in \mathbb{N}.$$

Remarks If f and g are q,ω-symmetric differentiable on $I^T_{q,\omega}$,

(a) $\tilde{D}_{q,\omega}[f(t)+g(t)] = \tilde{D}_{q,\omega} f(t) + \tilde{D}_{q,\omega} g(t)$,

(b) $\tilde{D}_{q,\omega}[f(t)g(t)] = f(\rho_{q,\omega}(t))\tilde{D}_{q,\omega} g(t) + g(\sigma_{q,\omega}(t))\tilde{D}_{q,\omega} f(t)$,

(c) $\tilde{D}_{q,\omega}\left[\frac{f(t)}{g(t)}\right] = \frac{g(\rho_{q,\omega}(t))\tilde{D}_{q,\omega} f(t) - f(\rho_{q,\omega}(t))\tilde{D}_{q,\omega} g(t)}{g(\rho_{q,\omega}(t))g(\sigma_{q,\omega}(t))}$, $g(\rho_{q,\omega}(t))g(\sigma_{q,\omega}(t)) \neq 0$,

(d) $\tilde{D}_{q,\omega}[C] = 0$ where C is constant.

Definition 2.2. *[20] Let I be any closed interval of $\mathbb{R}$ containing a, b and ω_0 and $f: I \to \mathbb{R}$ be a given function. The symmetric Hahn integral of f from a to b is defind by*

$$\int_a^b f(t)\tilde{d}_{q,\omega} t := \int_{\omega_0}^b f(t)\tilde{d}_{q,\omega} t - \int_{\omega_0}^a f(t)\tilde{d}_{q,\omega} t,$$

where

$$\tilde{\mathcal{I}}_{q,\omega} f(t) = \int_{\omega_0}^x f(t)\tilde{d}_{q,\omega} t := (1-q^2)(x-\omega_0)\sum_{k=0}^{\infty} q^{2k} f\left(\sigma^{2k+1}_{q,\omega}(x)\right), \quad x \in I.$$

Providing that the above series converges at $x = a$ and $x = b$, f is symmetric Hahn integrable on $[a,b]$. In addition, f is symmetric Hahn integrable on I if it is symmetric Hahn integrable on $[a,b]$ for all $a, b \in I$.

In addition,

$$\tilde{\mathcal{I}}^0_{q,\omega}f(x) = f(x),\ \ \tilde{\mathcal{I}}^N_{q,\omega}f(x) = \tilde{\mathcal{I}}_{q,\omega}\tilde{\mathcal{I}}^{N-1}_{q,\omega}f(x)\ \text{ where }\ N\in\mathbb{N},$$
$$\tilde{D}_{q,\omega}\tilde{\mathcal{I}}_{q,\omega}f(x) = f(x),\ \text{ and }\ \tilde{\mathcal{I}}_{q,\omega}\tilde{D}_{q,\omega}f(x) = f(x) - f(\omega_0).$$

Remarks [20] Let $a,b\in I^T_{q,\omega}$ and f,g be symmetric Hahn integrable on $I^T_{q,\omega}$. Then,

(a) $\int_a^a f(t)\tilde{d}_{q,\omega}t = 0$,
(b) $\int_a^b f(t)\tilde{d}_{q,\omega}t = -\int_b^a f(t)\tilde{d}_{q,\omega}t$,
(c) $\int_a^b f(t)\tilde{d}_{q,\omega}t = \int_c^b f(t)\tilde{d}_{q,\omega}t + \int_a^c f(t)\tilde{d}_{q,\omega}t,\ \ c\in I^T_{q,\omega},\ a<c<b$,
(d) $\int_a^b [\alpha f(t)+\beta g(t)]\,\tilde{d}_{q,\omega}t = \alpha\int_a^b f(t)\tilde{d}_{q,\omega}t + \beta\int_a^b g(t)\tilde{d}_{q,\omega}t,\ \ \alpha,\beta\in\mathbb{R}$,
(e) $\int_a^b \left[f(\rho_{q,\omega}(t))\tilde{D}_{q,\omega}g(t)\right]\tilde{d}_{q,\omega}t = [f(t)g(t)]_a^b - \int_a^b \left[g\left(\sigma_{q,\omega}(t)\right)\tilde{D}_{q,\omega}f(t)\right]\tilde{d}_{q,\omega}t$.

Lemma 2.2. *[20] **[Fundamental theorem of symmetric Hahn calculus]***
Let $f: I\to\mathbb{R}$ be continuous at ω_0. Then,

$$F(x) := \int_{\omega_0}^x f(t)\tilde{d}_{q,\omega}t,\quad x\in I$$

is continuous at ω_0 and $\tilde{D}_{q,\omega}F(x)$ exists for every $x\in\sigma_{q,\omega}(I) := \{qt+\omega : t\in I\}$ where

$$\tilde{D}_{q,\omega}F(x) = f(x).$$

In addition,

$$\int_a^b \tilde{D}_{q,\omega}f(t)\tilde{d}_{q,\omega}t = f(b) - f(a)\ \textit{for all}\ a,b\in I.$$

Lemma 2.3. *[21] Let $0<q<1$, $\omega>0$ and $f: I\to\mathbb{R}$ be continuous at ω_0. Then,*

$$\int_{\omega_0}^t\int_{\omega_0}^r f(s)\,\tilde{d}_{q,\omega}s\,\tilde{d}_{q,\omega}r \;=\; q\int_{\omega_0}^t\int_{qs+\omega}^t f(qs+\omega)\,\tilde{d}_{q,\omega}r\,\tilde{d}_{q,\omega}s.$$

Definition 2.3. *[21] Let $\alpha,\omega>0,\ 0<q<1$, and f be a function defined on $I^T_{q,\omega}$. The fractional symmetric Hahn integral is defined by*

$$\begin{aligned}
\widetilde{\mathcal{I}}^{\alpha}_{q,\omega}f(t) &:= \frac{q^{\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(\alpha)}\int_{\omega_0}^t \widetilde{(t-s)}^{\underline{\alpha-1}}_{q,\omega}\, f\left(\sigma^{\alpha-1}_{q,\omega}(s)\right)\tilde{d}_{q,\omega}s \\
&= \frac{(1-q^2)q^{\binom{\alpha}{2}}(t-\omega_0)}{\widetilde{\Gamma}_q(\alpha)}\sum_{k=0}^{\infty} q^{2k}\left(t-\sigma^{2k+1}_{q,\omega}(t)\right)^{\underline{\alpha-1}}_{q,\omega} f\left(\sigma^{2k+\alpha}_{q,\omega}(t)\right) \\
&= \frac{(1-q^2)q^{\binom{\alpha}{2}}(t-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(\alpha)}\sum_{k=0}^{\infty} q^{2k}\left(1-\widetilde{q^{2k+1}}\right)^{\underline{\alpha-1}}_{q} f\left(\sigma^{2k+\alpha}_{q,\omega}(t)\right)
\end{aligned}$$

and $\widetilde{\mathcal{I}}^0_{q,\omega}f)(t) = f(t)$.

Definition 2.4. *[21] For $\alpha, \omega > 0,\ 0 < q < 1$ and f defined on $I^T_{q,\omega}$, the fractional symmetric Hahn difference operator of Riemann-Liouville type of order α is defined by*

$$\begin{aligned}
\widetilde{D}^{\alpha}_{q,\omega} f(t) &:= \widetilde{D}^{N}_{q,\omega} \widetilde{\mathcal{I}}^{N-\alpha}_{q,\omega} f(t) \\
&= \frac{q^{\binom{-\alpha}{2}}}{\widetilde{\Gamma}_q(-\alpha)} \int_{\omega_0}^{t} \widetilde{(t-s)}^{\underline{-\alpha-1}}_{q,\omega} f\left(\sigma^{-\alpha-1}_{q,\omega}(s)\right) \tilde{d}_{q,\omega} s, \\
\widetilde{D}^{0}_{q,\omega} f(t) &= f(t)
\end{aligned}$$

where $N - 1 < \alpha < N,\ N \in \mathbb{N}$.

Lemma 2.4. *[21] Let $\alpha, \omega > 0,\ 0 < q < 1$ and $f : I^T_{q,\omega} \to \mathbb{R}$. Then,*

$$\tilde{\mathcal{I}}^{\alpha}_{q,\omega} \tilde{D}^{\alpha}_{q,\omega} f(t) = f(t) + C_1(t-\omega_0)^{\alpha-1} + C_2(t-\omega_0)^{\alpha-2} + ... + C_N(t-\omega_0)^{\alpha-N}$$

for some $C_i \in \mathbb{R}, i = 1, 2, ..., N$ and $N - 1 < \alpha < N$ for $N \in \mathbb{N}$.

Lemma 2.5. *[24] (Arzelá-Ascoli theorem) A set of function in $C[a,b]$ with the sup norm, is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 2.6. *[24] If a set is closed and relatively compact then it is compact.*

Lemma 2.7. *[25] (Schauder's fixed point theorem) Let (D,d) be a complete metric space, U be a closed convex subset of D, and $T : D \to D$ be the map such that the set $Tu : u \in U$ is relatively compact in D. Then the operator T has at least one fixed point $u^* \in U$: $Tu^* = u^*$.*

2.2. Auxiliary lemmas

In this section, we formulate some lemmas that will be used as a tool for our calculations as follows.

Lemma 2.8. *Let $q \in (0,1), \omega > 0$ and $n > 0$. Then,*

$$\int_{\omega_0}^{t} \tilde{d}_{q,\omega} s = t - \omega_0 \quad and \quad \int_{\omega_0}^{t} (s-\omega_0)^n \,\tilde{d}_{q,\omega} s = \frac{q^n}{\widetilde{[n+1]}_q} (t-\omega_0)^{n+1}.$$

Proof. Using the definition of symmetric Hahn integral, we have

$$\begin{aligned}
\int_{\omega_0}^{t} \tilde{d}_{q,\omega} s &= (1-q^2)(t-\omega_0) \sum_{k=0}^{\infty} q^{2k} \\
&= (1-q^2)(t-\omega_0) \left[\frac{1}{1-q^2}\right] = t - \omega_0,
\end{aligned}$$

and

$$\int_{\omega_0}^{t} (s-\omega_0)^n \,\tilde{d}_{q,\omega} s = (1-q^2)(t-\omega_0) \sum_{k=0}^{\infty} q^{2k} \left(\sigma^{2k+1}_{q,\omega}(t) - \omega_0\right)^n$$

$$
\begin{aligned}
&= q^n(1-q^2)\,(t-\omega_0)^{n+1}\sum_{k=0}^{\infty} q^{(n+1)2k}\\
&= q^n(1-q^2)\,(t-\omega_0)^{n+1}\left[\frac{1}{1-q^{2(n+1)}}\right]\\
&= \frac{q^n}{\widetilde{[n+1]}_q}\,(t-\omega_0)^{n+1}.
\end{aligned}
$$

The proof is complete.

Lemma 2.9. *Let* $\alpha,\beta>0,\ q\in(0,1)$ *and* $\omega>0$. *Then,*

$$
\begin{aligned}
&(i)\quad \int_{\omega_0}^{t}(\widetilde{t-s})_{q,\omega}^{\underline{\alpha-1}}\,\tilde{d}_{q,\omega}s \;=\; \frac{(t-\omega_0)^{\alpha}}{\widetilde{[\alpha]}_q},\\
&(ii)\quad \int_{\omega_0}^{t}(\widetilde{t-s})_{q,\omega}^{\underline{\alpha-1}}\left(\sigma_{q,\omega}^{\alpha-1}(s)-\omega_0\right)^{\beta}\,\tilde{d}_{q,\omega}s \;=\; q^{\alpha\beta}(t-\omega_0)^{\alpha+\beta}B_q(\beta+1,\alpha),\\
&(iii)\quad \int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{\alpha-1}(s)}(\widetilde{t-s})_{q,\omega}^{\underline{\alpha-1}}\Big(\widetilde{\sigma_{q,\omega}^{\alpha-1}(s)-r}\Big)_{q,\omega}^{\underline{\beta-1}}\,\tilde{d}_{q,\omega}r\,\tilde{d}_{p,\omega}s \;=\; \frac{q^{\alpha\beta}}{[\tilde{\beta}]_q}(t-\omega_0)^{\alpha+\beta}B_q(\beta+1,\alpha).
\end{aligned}
$$

Proof. From the definition of q,ω-symmetric analogue of the power function, Lemma 2.1 and Definition 2.2, we obtain

$$
\begin{aligned}
(i)\quad \int_{\omega_0}^{t}(\widetilde{t-s})_{q,\omega}^{\underline{\alpha-1}}\,\tilde{d}_{q,\omega}s &= (1-q^2)\,(t-\omega_0)\sum_{k=0}^{\infty}q^{2k}\Big(\widetilde{t-\sigma_{q,\omega}^{2k+1}(t)}\Big)_{q,\omega}^{\underline{\alpha-1}}\\
&= (1-q^2)\,(t-\omega_0)^{\alpha}\sum_{k=0}^{\infty}q^{2k}(\widetilde{1-q^{2k+1}})_q^{\underline{\alpha-1}}\\
&= (1-q^2)\,(t-\omega_0)^{\alpha}\sum_{k=0}^{\infty}q^{2k}\left[\prod_{i=0}^{\infty}\frac{1-q^{2k+2i+2}}{1-q^{2k+2i+2\alpha}}\right]\\
&= \frac{(t-\omega_0)^{\alpha}}{\widetilde{[\alpha]}_q},
\end{aligned}
$$

$$
\begin{aligned}
(ii)\quad &\int_{\omega_0}^{t}(\widetilde{t-s})_{q,\omega}^{\underline{\alpha-1}}\,(\sigma_{q,\omega}^{\alpha-1}(s)-\omega_0)^{\beta}\,\tilde{d}_{q,\omega}s\\
&= (1-q^2)\,(t-\omega_0)\sum_{k=0}^{\infty}q^{2k}\Big(\widetilde{t-\sigma_{q,\omega}^{2k+1}(t)}\Big)_{q,\omega}^{\underline{\alpha-1}}\Big(q^{\alpha-1}\left(\sigma_{q,\omega}^{2k+1}(t)-\omega_0\right)\Big)^{\beta}\\
&= q^{\alpha\beta}(1-q^2)\,(t-\omega_0)^{\alpha+\beta}\sum_{k=0}^{\infty}q^{2k}(\widetilde{1-q^{2k+1}})_q^{\underline{\alpha-1}}(q^{-1}q^{2k+1})^{\beta}\\
&= q^{\alpha\beta}\,(t-\omega_0)^{\alpha+\beta}\int_{\omega_0}^{1}(\widetilde{1-s})_{q,\omega}^{\underline{\alpha-1}}\left(q^{-1}s\right)^{\beta}\,\tilde{d}_{q,\omega}s\\
&= q^{\alpha\beta}\,(t-\omega_0)^{\alpha+\beta}\,\tilde{B}_q(\beta+1,\alpha).
\end{aligned}
$$

Using (i) and (ii), we have

$$\begin{aligned}&\int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{\alpha-1}(s)}\widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}\left(\widetilde{\sigma_{q,\omega}^{\alpha-1}(s)-r}\right)_{q,\omega}^{\underline{\beta-1}}\tilde{d}_{q,\omega}r\,\tilde{d}_{p,\omega}s\\ &\qquad=\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}\left[\int_{\omega_0}^{\sigma_{q,\omega}^{\alpha-1}(s)}\left(\widetilde{\sigma_{q,\omega}^{\alpha-1}(s)-r}\right)_{q,\omega}^{\underline{\beta-1}}\tilde{d}_{q,\omega}r\right]\tilde{d}_{p,\omega}s\\ &\qquad=\frac{1}{[\tilde{\beta}]_q}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}\left(\sigma_{q,\omega}^{\alpha-1}(s)-\omega_0\right)^{\beta}\,\tilde{d}_{p,\omega}s\\ &\qquad=\frac{q^{\alpha\beta}}{[\tilde{\beta}]_q}(t-\omega_0)^{\alpha+\beta}\tilde{B}_q(\beta+1,\alpha).\end{aligned}$$

2.3. Lemma for linear variant form

The following lemma present a solution of a linear variant form of the problem (1.1).

Lemma 2.10. *Let* $\Lambda\neq 0$; $\omega>0$; $q\in(0,1)$*;* $\alpha\in(1,2]$; $\theta_1,\theta_2\in(0,1]$; $\lambda_1,\lambda_2\in\mathbb{R}^+$*;* $h\in C(I_{q,\omega}^{T},\mathbb{R})$ *and* $g_1,g_2\in C(I_{q,\omega}^{T},\mathbb{R}^+)$ *be given functions. Then the linear variant form*

$$\begin{aligned}\tilde{D}_{q,\omega}^{\alpha}u(t)&=h(t),\quad t\in I_{q,\omega}^{T},\\ u(\omega_0)&=\lambda_1\tilde{\mathcal{I}}_{q,\omega}^{\theta_1}g(\eta_1)u(\eta_1),\\ u(T)&=\lambda_2\tilde{\mathcal{I}}_{q,\omega}^{\theta_2}g(\eta_2)u(\eta_2),\quad \eta_1,\eta_2\in I_{q,\omega}^{T}-\{\omega_0,T\},\end{aligned}\tag{2.1}$$

has the unique solution which is

$$\begin{aligned}u(t)=&\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}h(\sigma_{q,\omega}^{\alpha-1}(s))\,\tilde{d}_{q,\omega}s\\ &+\frac{(t-\omega_0)^{\alpha-1}}{\Lambda}\Big\{\boldsymbol{B}_2\Phi_1[h]+\boldsymbol{A}_2\Phi_2[h]\Big\}\\ &-\frac{(t-\omega_0)^{\alpha-2}}{\boldsymbol{A}_2}\Big\{\left(1+\frac{\boldsymbol{A}_1\boldsymbol{B}_2}{\Lambda}\right)\Phi_1[h]+\frac{\boldsymbol{A}_1\boldsymbol{A}_2}{\Lambda}\Phi_2[h]\Big\}\end{aligned}\tag{2.2}$$

where the functionals $\Phi_1[h]$, $\Phi_2[h]$ *are defined by*

$$\begin{aligned}\Phi_1[h]:=&\frac{\lambda_1 q^{\binom{\theta_1}{2}+\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\theta_1)\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{\eta_1}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_1-1}(s)}\widetilde{(\eta_1-s)}_{q,\omega}^{\underline{\theta_1-1}}\left(\widetilde{\sigma_{q,\omega}^{\theta_1-1}(s)-r}\right)_{q,\omega}^{\underline{\alpha-1}}g_1\left(\sigma_{q,\omega}^{\theta_1-1}(s)\right)\times\\ &h(\sigma_{q,\omega}^{\alpha-1}(r))\,\tilde{d}_{q,\omega}r\tilde{d}_{q,\omega}s,\end{aligned}\tag{2.3}$$

$$\begin{aligned}\Phi_2[h]:=&\frac{\lambda_2 q^{\binom{\theta_2}{2}+\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\theta_2)\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{\eta_2}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_2-1}(s)}\widetilde{(\eta_2-s)}_{q,\omega}^{\underline{\theta_2-1}}\left(\widetilde{\sigma_{q,\omega}^{\theta_2-1}(s)-r}\right)_{q,\omega}^{\underline{\alpha-1}}g_2\left(\sigma_{q,\omega}^{\theta_2-1}(s)\right)\times\\ &h(\sigma_{q,\omega}^{\alpha-1}(r))\,\tilde{d}_{q,\omega}r\tilde{d}_{q,\omega}s-\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{T}\widetilde{(T-s)}_{q,\omega}^{\underline{\alpha-1}}h\left(\sigma_{q,\omega}^{\alpha-1}(s)\right)\tilde{d}_{q,\omega}s,\end{aligned}\tag{2.4}$$

and the constants $\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{B}_1,\boldsymbol{B}_2$ *and* Λ *are defined by*

$$\boldsymbol{A}_1:=\frac{\lambda_1 q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\theta_1)}\int_{\omega_0}^{\eta_1}\widetilde{(\eta_1-s)}_{q,\omega}^{\underline{\theta_1-1}}g_1\left(\sigma_{q,\omega}^{\theta_1-1}(s)\right)\left(\sigma_{q,\omega}^{\theta_1-1}(s)-\omega_0\right)^{\alpha-1}\tilde{d}_{q,\omega}s,\tag{2.5}$$

$$\boldsymbol{A}_2 := \frac{\lambda_1 q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\theta_1)} \int_{\omega_0}^{\eta_1} (\widetilde{\eta_1 - s})_{q,\omega}^{\underline{\theta_1 - 1}} g_1\left(\sigma_{q,\omega}^{\theta_1 - 1}(s)\right)\left(\sigma_{q,\omega}^{\theta_1 - 1}(s) - \omega_0\right)^{\alpha - 2} \tilde{d}_{q,\omega}s, \tag{2.6}$$

$$\boldsymbol{B}_1 := (T - \omega_0)^{\alpha - 1} - \frac{\lambda_2 q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\theta_2)} \int_{\omega_0}^{\eta_2} (\widetilde{\eta_2 - s})_{q,\omega}^{\underline{\theta_2 - 1}} g_2\left(\sigma_{q,\omega}^{\theta_2 - 1}(s)\right)\left(\sigma_{q,\omega}^{\theta_2 - 1}(s) - \omega_0\right)^{\alpha - 1} \tilde{d}_{q,\omega}s, \tag{2.7}$$

$$\boldsymbol{B}_2 := (T - \omega_0)^{\alpha - 2} - \frac{\lambda_2 q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\theta_2)} \int_{\omega_0}^{\eta_2} (\widetilde{\eta_2 - s})_{q,\omega}^{\underline{\theta_2 - 1}} g_2\left(\sigma_{q,\omega}^{\theta_2 - 1}(s)\right)\left(\sigma_{q,\omega}^{\theta_2 - 1}(s) - \omega_0\right)^{\alpha - 2} \tilde{d}_{q,\omega}s, \tag{2.8}$$

$$\Lambda := \boldsymbol{A}_2\boldsymbol{B}_1 - \boldsymbol{A}_1\boldsymbol{B}_2. \tag{2.9}$$

Proof. Taking fractional symmetric Hahn integral of order α for (2.1), we obtain

$$u(t) = C_1(t - \omega_0)^{\alpha - 1} + C_2(t - \omega_0)^{\alpha - 2} + \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{t} (\widetilde{t - s})_{q,\omega}^{\underline{\alpha - 1}} h\left(\sigma_{q,\omega}^{\alpha - 1}(s)\right) \tilde{d}_{q,\omega}s. \tag{2.10}$$

Taking fractional symmetric Hahn integral of order $\theta_i,\ i = 1, 2$ for (2.10), we get

$$\begin{aligned} \mathcal{I}_{q,\omega}^{\theta_i} u(t) = {} & \frac{q^{\binom{\theta_i}{2}}}{\tilde{\Gamma}_q(\theta_i)} \int_{\omega_0}^{t} (\widetilde{t - s})_{q,\omega}^{\underline{\theta_i - 1}} \Big[C_1(\sigma_{q,\omega}^{\theta_i - 1}(s) - \omega_0)^{\alpha - 1} + C_2(\sigma_{q,\omega}^{\theta_i - 1}(s) - \omega_0)^{\alpha - 2} \Big] \tilde{d}_{q,\omega}s \\ & + \frac{q^{\binom{\theta_i}{2} + \binom{\alpha}{2}}}{\tilde{\Gamma}_q(\theta_i)\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{t} \int_{\omega_0}^{\sigma_{q,\omega}^{\theta_i - 1}(s)} (\widetilde{t - s})_{q,\omega}^{\underline{\theta_i - 1}} \left(\widetilde{\sigma_{q,\omega}^{\theta_i - 1}(s) - r}\right)_{q,\omega}^{\underline{\alpha - 1}} h\left(\sigma_{q,\omega}^{\alpha - 1}(r)\right) \tilde{d}_{q,\omega}r \tilde{d}_{q,\omega}s. \end{aligned} \tag{2.11}$$

After substituting $i = 1$ into (2.11) and employing the first condition of (2.1), we have

$$\mathbf{A}_1 C_1 + \mathbf{A}_2 C_2 = -\Phi_1[h]. \tag{2.12}$$

Taking $i = 2$ into (2.11) and employing the second condition of (2.1), we have

$$\mathbf{B}_1 C_1 + \mathbf{B}_2 C_2 = \Phi_2[h], \tag{2.13}$$

where $\Phi_1[h], \Phi_2[h], \mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1$ and $\mathbf{B}_2$ are defined as (2.3) – (2.8), respectively.

Solving the system of Eqs (2.12) – (2.13), we have

$$C_1 = \frac{\mathbf{B}_2\Phi_1[h] + \mathbf{A}_2\Phi_2[h]}{\Lambda} \quad \text{and} \quad C_2 = -\frac{1}{\mathbf{A}_2}\left\{\left(1 + \frac{\mathbf{A}_1\mathbf{B}_2}{\Lambda}\right)\Phi_1[h] + \frac{\mathbf{A}_1\mathbf{A}_2}{\mathit{\Lambda}}\Phi_2[h]\right\},$$

where $\mathit{\Lambda}$ is defined as (2.9). Substituting the constants C_1 and C_2 into (2.10), we obtain (2.2).

3. Main results

In this section, we prove the existence and uniqueness of solution of the problem (1.1). Furthermore, we show the existence of at least one solution of problem (1.1).

3.1. Existence and uniqueness result

In this section, we consider the existence and uniqueness result for the problem (1.1). Let $C = C\left(I^T_{q,\omega}, \mathbb{R}\right)$ be a Banach space of all function u with the norm defined by

$$\|u\|_C = \max_{t\in I^T_{q,\omega}} \left\{|u(t)|, \left|\tilde{D}^{\beta}_{q,\omega}u(t)\right|\right\},$$

where $\alpha \in (1,2];\ \beta,\gamma,\theta_1,\theta_2 \in (0,1];\ \omega > 0;\ q \in (0,1);\ \lambda_1,\lambda_2 \in \mathbb{R}^+$. We define an operator $\mathcal{F} : C \to C$ as

$$\begin{aligned}(\mathcal{F}u)(t) := &\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{t} \widetilde{(t-s)}^{\underline{\alpha-1}}_{q,\omega} \times \\ &F\Big(\sigma^{\alpha-1}_{q,\omega}(s), u\left(\sigma^{\alpha-1}_{q,\omega}(s)\right), \tilde{D}^{\beta}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(s)\right), \tilde{\Psi}^{\gamma}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(s)\right)\Big)\,\tilde{d}_{q,\omega}s \\ &+\frac{(t-\omega_0)^{\alpha-1}}{\Lambda}\Big\{\mathbf{B}_2\Phi_1[f(u)] + \mathbf{A}_2\Phi_2[f(u)]\Big\} \\ &-\frac{(t-\omega_0)^{\alpha-2}}{\mathbf{A}_2}\left\{\left(1+\frac{\mathbf{A}_1\mathbf{B}_2}{\Lambda}\right)\Phi_1[f(u)] + \frac{\mathbf{A}_1\mathbf{A}_2}{\Lambda}\Phi_2[f(u)]\right\} \end{aligned} \tag{3.1}$$

where the functionals $\Phi_1[F(u)],\ \Phi_2[F(u)]$ are given by

$$\begin{aligned}\Phi_1[F(u)] := &\frac{\lambda_1 q^{\binom{\theta_1}{2}+\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\theta_1)\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{\eta_1}\int_{\omega_0}^{\sigma^{\theta_1-1}_{q,\omega}(s)} \widetilde{(\eta_1-s)}^{\underline{\theta_1-1}}_{q,\omega}\Big(\widetilde{\sigma^{\theta_1-1}_{q,\omega}(s)}-r\Big)^{\underline{\alpha-1}}_{q,\omega} g_1\left(\sigma^{\theta_1-1}_{q,\omega}(s)\right)\times \\ &F\Big(\sigma^{\alpha-1}_{q,\omega}(r), u\left(\sigma^{\alpha-1}_{q,\omega}(r)\right), \tilde{D}^{\beta}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(r)\right), \tilde{\Psi}^{\gamma}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(r)\right)\Big)\tilde{d}_{q,\omega}r\tilde{d}_{q,\omega}s, \end{aligned} \tag{3.2}$$

$$\begin{aligned}\Phi_2[F(u)] := &\frac{\lambda_2 q^{\binom{\theta_2}{2}+\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\theta_2)\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{\eta_2}\int_{\omega_0}^{\sigma^{\theta_2-1}_{q,\omega}(s)} \widetilde{(\eta_2-s)}^{\underline{\theta_2-1}}_{q,\omega}\Big(\widetilde{\sigma^{\theta_2-1}_{q,\omega}(s)}-r\Big)^{\underline{\alpha-1}}_{q,\omega} g_2\left(\sigma^{\theta_2-1}_{q,\omega}(s)\right)\times \\ &F\Big(\sigma^{\alpha-1}_{q,\omega}(r), u\left(\sigma^{\alpha-1}_{q,\omega}(r)\right), \tilde{D}^{\beta}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(r)\right), \tilde{\Psi}^{\gamma}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(r)\right)\Big)\tilde{d}_{q,\omega}r\tilde{d}_{q,\omega}s \\ &-\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{T}\widetilde{(T-s)}^{\underline{\alpha-1}}_{q,\omega}\times \\ &F\Big(\sigma^{\alpha-1}_{q,\omega}(r), u\left(\sigma^{\alpha-1}_{q,\omega}(r)\right), \tilde{D}^{\beta}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(r)\right), \tilde{\Psi}^{\gamma}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(r)\right)\Big)\tilde{d}_{q,\omega}s, \end{aligned} \tag{3.3}$$

and the constants $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ and Λ are defined by (2.5)-(2.9), respectively.

We find that the problem (1.1) has solution if and only if the operator $\mathcal{F}$ has fixed point.

Theorem 3.1. *Assume that $F : I^T_{q,\omega} \times \mathbb{R}^3 \to \mathbb{R}$, and $g_1, g_2 : I^T_{q,\omega} \to \mathbb{R}^+$ are continuous, and $\varphi : I^T_{q,\omega} \times I^T_{q,\omega} \to [0,\infty)$ is continuous with $\varphi_0 = \max\{\varphi(t,s) : (t,s) \in I^T_{q,\omega} \times I^T_{q,\omega}\}$. Suppose that the following conditions hold:*

(H_1) *There exist constants $\ell_1, \ell_2, \ell_3 > 0$ such that for each $t \in I^T_{q,\omega}$ and $u, v \in \mathbb{R}$,*

$$\begin{aligned}&\left|F\left(t,u,\tilde{D}^{\beta}_{q,\omega}u, \tilde{\Psi}^{\gamma}_{q,\omega}u\right) - F\left(t,v,\tilde{D}^{\beta}_{q,\omega}v, \tilde{\Psi}^{\gamma}_{q,\omega}v\right)\right| \\ \leq\ &\ell_1\left|u-v\right| + \ell_2\left|\tilde{D}^{\beta}_{q,\omega}u - \tilde{D}^{\beta}_{q,\omega}v\right| + \ell_3\left|\tilde{\Psi}^{\gamma}_{q,\omega}u - \tilde{\Psi}^{\gamma}_{q,\omega}v\right|.\end{aligned}$$

(H_2) *There exist constants* $g_i, G_i > 0$ *where* $i = 1, 2$ *such that for each* $t \in I^T_{q,\omega}$,

$$0 < g_i < g_i(t) < G_i.$$

(H_3) $\Theta < 1$,

where

$$\Omega_1 := \frac{\lambda_1 G_1 q^{\binom{\theta_1}{2}+\binom{\alpha}{2}+\theta_1+\alpha}}{\tilde{\Gamma}_q(\theta_1+\alpha+1)}(\eta_1-\omega_0)^{\theta_1+\alpha} \tag{3.4}$$

$$\Omega_2 := \frac{\lambda_2 G_2 q^{\binom{\theta_2}{2}+\binom{\alpha}{2}+\theta_2+\alpha}}{\tilde{\Gamma}_q(\theta_2+\alpha+1)}(\eta_2-\omega_0)^{\theta_2+\alpha} + \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha+1)}(T-\omega_0)^{\alpha} \tag{3.5}$$

$$\overline{\boldsymbol{A}}_1 := \frac{\lambda_1 G_1 q^{\binom{\theta_1}{2}+\theta_1(\alpha-1)}\tilde{\Gamma}_q(\alpha)}{\tilde{\Gamma}_q(\theta_1+\alpha)}(\eta_1-\omega_0)^{\theta_1+\alpha-1} > \boldsymbol{A}_1 \tag{3.6}$$

$$\overline{\boldsymbol{A}}_2 := \frac{\lambda_1 G_1 q^{\binom{\theta_1}{2}+\theta_1(\alpha-2)}\tilde{\Gamma}_q(\alpha-1)}{\tilde{\Gamma}_q(\theta_1+\alpha-1)}(\eta_1-\omega_0)^{\theta_1+\alpha-2} > \boldsymbol{A}_2 \tag{3.7}$$

$$\boldsymbol{A}^*_2 := \frac{\lambda_1 g_1 q^{\binom{\theta_1}{2}+\theta_1(\alpha-2)}\tilde{\Gamma}_q(\alpha-1)}{\tilde{\Gamma}_q(\theta_1+\alpha-1)}(\eta_1-\omega_0)^{\theta_1+\alpha-2} < \boldsymbol{A}_2 \tag{3.8}$$

$$\overline{\boldsymbol{B}}_1 := (T-\omega_0)^{\alpha-1} + \frac{\lambda_2 G_2 q^{\binom{\theta_2}{2}+\theta_2(\alpha-1)}\tilde{\Gamma}_q(\alpha)}{\tilde{\Gamma}_q(\theta_2+\alpha)}(\eta_2-\omega_0)^{\theta_2+\alpha-1} > \boldsymbol{B}_1 \tag{3.9}$$

$$\overline{\boldsymbol{B}}_2 := (T-\omega_0)^{\alpha-2} + \frac{\lambda_2 G_2 q^{\binom{\theta_2}{2}+\theta_2(\alpha-2)}\tilde{\Gamma}_q(\alpha-1)}{\tilde{\Gamma}_q(\theta_2+\alpha-1)}(\eta_2-\omega_0)^{\theta_2+\alpha-2} > \boldsymbol{B}_2 \tag{3.10}$$

$$\Lambda^* := g_1 g_2 \left|\overline{\boldsymbol{A}}_2\overline{\boldsymbol{B}}_1 - \overline{\boldsymbol{A}}_1\overline{\boldsymbol{B}}_2\right| < |\Lambda| \tag{3.11}$$

and

$$\begin{aligned}\Theta := &\left(\ell_1+\ell_2+\ell_3\varphi_0\frac{(T+\omega_0)^{\gamma}}{\tilde{\Gamma}_q(\gamma+1)}\right)\Bigg\{\frac{(T-\omega_0)^{\alpha-1}}{\Lambda^*}\left[\overline{\boldsymbol{B}}_2\Omega_1+\overline{\boldsymbol{A}}_2\Omega_2\right]\\ &+\frac{(t-\omega_0)^{\alpha-2}}{\boldsymbol{A}^*_2}\left[\left(1+\frac{\overline{\boldsymbol{A}}_1\overline{\boldsymbol{B}}_2}{\Lambda^*}\right)\Omega_1+\frac{\overline{\boldsymbol{A}}_1\overline{\boldsymbol{A}}_2}{\Lambda^*}\Omega_2\right]+\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(\alpha+1)}\Bigg\}.\end{aligned} \tag{3.12}$$

Then the problem (1.1) *has a unique solution in* $I^T_{q,\omega}$.

Proof. To show that F is contraction, we first consider

$$\mathcal{H}|u-v|(t) := \left|F\left(t,u(t),\tilde{D}^{\beta}_{q,\omega}u(t),\tilde{\Psi}^{\gamma}_{q,\omega}u(t)\right) - F\left(t,v(t),\tilde{D}^{\beta}_{q,\omega}v(t),\tilde{\Psi}^{\gamma}_{q,\omega}v(t)\right)\right|,$$

for each $t \in I^T_{q,\omega}$ and $u, v \in C$. We find that

$$\begin{aligned}&\left|\Phi_1[F(u)] - \Phi_1[F(v)]\right|\\ &\le \frac{\lambda_1 G_1 q^{\binom{\theta_1}{2}+\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\theta_1)\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{\eta_1}\int_{\omega_0}^{\sigma^{\theta_1-1}_{q,\omega}(s)}\widetilde{(\eta_1-s)}^{\underline{\theta_1-1}}_{q,\omega}\left(\widetilde{\sigma^{\theta_1-1}_{q,\omega}(s)}-r\right)^{\underline{\alpha-1}}_{q,\omega}\mathcal{H}|u-v|(r)\tilde{d}_{q,\omega}r\tilde{d}_{q,\omega}s\end{aligned}$$

$$
\begin{aligned}
&\le \Big(\ell_1\big|u-v\big| + \ell_2\Big|\tilde{D}^{\beta}_{q,\omega}u - \tilde{D}^{\beta}_{q,\omega}v\Big| + \ell_3\big|\tilde{\Psi}^{\gamma}_{q,\omega}u - \tilde{\Psi}^{\gamma}_{q,\omega}v\big|\Big)\times \\
&\quad \frac{\lambda_1 G_1 q^{\binom{\theta_1}{2}+\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\theta_1)\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{\eta_1}\int_{\omega_0}^{\sigma^{\theta_1-1}_{q,\omega}(s)}(\widetilde{\eta_1 - s})^{\underline{\theta_1-1}}_{q,\omega}\Big(\widetilde{\sigma^{\theta_1-1}_{q,\omega}(s)} - r\Big)^{\underline{\alpha-1}}_{q,\omega}\tilde{d}_{q,\omega}r\tilde{d}_{q,\omega}s \\
&\le \left[\left(\ell_1 + \ell_3\varphi_0\frac{(T+\omega_0)^{\gamma}}{\tilde{\Gamma}_q(\gamma+1)}\right)\big|u-v\big| + \ell_2\Big|\tilde{D}^{\beta}_{q,\omega}u - \tilde{D}^{\beta}_{q,\omega}v\Big|\right]\Omega_1 \\
&\le \|u-v\|_C\left(\ell_1+\ell_2+\ell_3\varphi_0\frac{(T+\omega_0)^{\gamma}}{\tilde{\Gamma}_q(\gamma+1)}\right)\Omega_1.
\end{aligned}
$$

Similary,

$$
\Big|\Phi_2[F(u)] - \Phi_2[F(v)]\Big| \le \|u-v\|_C\left(\ell_1+\ell_2+\ell_3\varphi_0\frac{(T+\omega_0)^{\gamma}}{\tilde{\Gamma}_q(\gamma+1)}\right)\Omega_2.
$$

In addition, we find that

$$
\begin{aligned}
&\Big|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\Big| \\
&\le \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{t}(\widetilde{t-s})^{\underline{\alpha-1}}_{q,\omega}\mathcal{H}|u-v|(s)\,\tilde{d}_{q,\omega}s \\
&\quad + \|u-v\|_C\left(\ell_1+\ell_2+\ell_3\varphi_0\frac{(T+\omega_0)^{\gamma}}{\tilde{\Gamma}_q(\gamma+1)}\right)\frac{(T-\omega_0)^{\alpha-1}}{|\Lambda|}\Big\{\mathbf{B}_2\Omega_1 + \mathbf{A}_2\Omega_2\Big\} \\
&\quad + \|u-v\|_C\left(\ell_1+\ell_2+\ell_3\varphi_0\frac{(T+\omega_0)^{\gamma}}{\tilde{\Gamma}_q(\gamma+1)}\right)\frac{(T-\omega_0)^{\alpha-2}}{\mathbf{A}_2}\left\{\left(1+\frac{\mathbf{A}_1\mathbf{B}_2}{|\Lambda|}\right)\Omega_1 + \frac{\mathbf{A}_1\mathbf{A}_2}{|\Lambda|}\Omega_2\right\} \\
&\le \|u-v\|_C\left(\ell_1+\ell_2+\ell_3\varphi_0\frac{(T+\omega_0)^{\gamma}}{\tilde{\Gamma}_q(\gamma+1)}\right)\left\{\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(\alpha+1)} + \frac{(T-\omega_0)^{\alpha-1}}{\Lambda^*}\Big[\overline{\mathbf{B}}_2\Omega_1 + \overline{\mathbf{A}}_2\Omega_2\Big]\right. \\
&\quad \left. + \frac{(t-\omega_0)^{\alpha-2}}{\mathbf{A}_2^*}\left[\left(1+\frac{\overline{\mathbf{A}}_1\overline{\mathbf{B}}_2}{\Lambda^*}\right)\Omega_1 + \frac{\overline{\mathbf{A}}_1\overline{\mathbf{A}}_2}{\Lambda^*}\Omega_2\right]\right\} \\
&= \|u-v\|_C\Theta.
\end{aligned}
\tag{3.13}
$$

Taking fractional symmetric Hahn difference of order γ for (3.1), we obtain

$$
\begin{aligned}
&(\tilde{D}^{\beta}_{q,\omega}\mathcal{F}u)(t) \\
&\quad = \frac{q^{\binom{-\beta}{2}+\binom{\alpha}{2}}}{\tilde{\Gamma}_q(-\beta)\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{t}\int_{\omega_0}^{\sigma^{-\beta-1}_{q,\omega}(s)}(\widetilde{t-s})^{\underline{-\beta-1}}_{q,\omega}\Big(\widetilde{\sigma^{-\beta-1}_{q,\omega}(s)} - r\Big)^{\underline{\alpha-1}}_{q,\omega}\times \\
&\qquad F\Big(\sigma^{\alpha-1}_{q,\omega}(r), u\left(\sigma^{\alpha-1}_{q,\omega}(r)\right), \tilde{D}^{\beta}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(r)\right), \tilde{\Psi}^{\gamma}_{q,\omega}u\left(\sigma^{\alpha-1}_{q,\omega}(r)\right)\Big)\tilde{d}_{q,\omega}r\tilde{d}_{q,\omega}s \\
&\qquad + \left[\frac{\mathbf{B}_2\Phi_1[f(u)] + \mathbf{A}_2\Phi_2[f(u)]}{\Lambda}\right]\frac{q^{\binom{-\beta}{2}}}{\tilde{\Gamma}_q(-\beta)}\int_{\omega_0}^{t}(\widetilde{t-s})^{\underline{-\beta-1}}_{q,\omega}\left(\sigma^{-\beta-1}_{q,\omega}(s) - \omega_0\right)^{\alpha-1}\tilde{d}_{q,\omega}s \\
&\qquad - \frac{1}{\mathbf{A}_2}\left[\left(1+\frac{\mathbf{A}_1\mathbf{B}_2}{\Lambda}\right)\Phi_1[f(u)] + \frac{\mathbf{A}_1\mathbf{A}_2}{\Lambda}\Phi_2[f(u)]\right]\times
\end{aligned}
$$

$$\frac{q^{\binom{-\beta}{2}}}{\tilde{\Gamma}_q(-\beta)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{-\beta-1}}\left(\sigma_{q,\omega}^{-\beta-1}(s)-\omega_0\right)^{\alpha-2}\tilde{d}_{q,\omega}s. \tag{3.14}$$

Similary, we have

$$\left|(\tilde{D}_{q,\omega}^{\beta}\mathcal{F}u)(t)-(\tilde{D}_{q,\omega}^{\beta}\mathcal{F}v)(t)\right|<\|u-v\|_C\Theta. \tag{3.15}$$

From (3.13) and (3.15), we get

$$\|\mathcal{F}u-\mathcal{F}v\|_C\leq\|u-v\|_C\Theta.$$

Using (H_3) we can conclude that $\mathcal{F}$ is a contraction. Based on Banach fixed point theorem, $\mathcal{F}$ has a fixed point which is a unique solution of problem (1.1) on $I_{q,\omega}^{T}$.

3.2. Existence of at least one solution

In this section, we particularly study the existence of at least one solution of (1.1) by using the Schauder's fixed point theorem as follows

Theorem 3.2. *Suppose that* (H_1) *and* (H_3) *defined in Theorem 3.1 hold. Then, problem* (1.1) *has at least one solution on* $I_{q,\omega}^{T}$.

Proof. The proof is established as the following structures.

Step I. Verify $\mathcal{F}$ map bounded sets into bounded sets in B_R. Let $B_R=\{u\in C(I_{q,\omega}^{T}):\|u\|_C\leq R\}$, $\max_{t\in I_{q,\omega}^{T}}|F(t,0,0,0)|=M$ and choose a constant

$$R\geq\frac{M\left\{\frac{(T-\omega_0)^{\alpha-1}}{\Lambda^*}[\overline{\mathbf{B}}_2\Omega_1+\overline{\mathbf{A}}_2\Omega_2]+\frac{(t-\omega_0)^{\alpha-2}}{\mathbf{A}_2^*}\left[\left(1+\frac{\overline{\mathbf{A}}_1\overline{\mathbf{B}}_2}{\Lambda^*}\right)\Omega_1+\frac{\overline{\mathbf{A}}_1\overline{\mathbf{A}}_2}{\Lambda^*}\Omega_2\right]+\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(\alpha+1)}\right\}}{1-\Theta}. \tag{3.16}$$

Denote that

$$\left|\mathcal{K}(t,u,0)\right|=\left|F(t,u(t),\tilde{D}_{q,\omega}^{\beta}u(t),\tilde{\Psi}_{q,\omega}^{\gamma}u(t))-F(t,0,0,0)\right|+\left|F(t,0,0,0)\right|.$$

We find that

$$\begin{aligned}
&\left|\Phi_1[F(u)]\right|\\
\leq\;&\frac{\lambda_1G_1q^{\binom{\theta_1}{2}+\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\theta_1)\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{\eta_1}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_1-1}(s)}\widetilde{(\eta_1-s)}_{q,\omega}^{\underline{\theta_1-1}}\left(\widetilde{\sigma_{q,\omega}^{\theta_1-1}(s)}-r\right)_{q,\omega}^{\underline{\alpha-1}}\left|\mathcal{K}(t,u,0)\right|\tilde{d}_{q,\omega}r\tilde{d}_{q,\omega}s\\
\leq\;&\left[\left(\ell_1|u|+\ell_2\left|\tilde{D}_{q,\omega}^{\beta}u\right|+\ell_3\left|\tilde{\Psi}_{q,\omega}^{\gamma}u\right|\right)+M\right]\Omega_1\\
\leq\;&\left[\left(\ell_1+\ell_2+\ell_3\varphi_0\frac{(T+\omega_0)^{\gamma}}{\tilde{\Gamma}_q(\gamma+1)}\right)\|u-v\|_C+M\right]\Omega_1\\
\leq\;&\left[\left(\ell_1+\ell_2+\ell_3\varphi_0\frac{(T+\omega_0)^{\gamma}}{\tilde{\Gamma}_q(\gamma+1)}\right)R+M\right]\Omega_1,
\end{aligned} \tag{3.17}$$

where $t \in I^T_{q,\omega}$ and $u \in B_R$.
Similary,

$$\left|\Phi_2[F(u)]\right| \leq \left[\left(\ell_1 + \ell_2 + \ell_3\varphi_0 \frac{(T+\omega_0)^\gamma}{\tilde{\Gamma}_q(\gamma+1)}\right)R + M\right]\Omega_2. \tag{3.18}$$

Employing (3.17) and (3.18), we find that

$$\begin{aligned}\left|(\mathcal{F}u)(t)\right| \leq & \left[\left(\ell_1 + \ell_2 + \ell_3\varphi_0 \frac{(T+\omega_0)^\gamma}{\tilde{\Gamma}_q(\gamma+1)}\right)R + M\right]\left\{\frac{(T-\omega_0)^{\alpha-1}}{\Lambda^*}[\overline{\mathbf{B}}_2\Omega_1 + \overline{\mathbf{A}}_2\Omega_2]\right. \\ & \left. + \frac{(t-\omega_0)^{\alpha-2}}{\mathbf{A}_2^*}\left[\left(1 + \frac{\overline{\mathbf{A}}_1\overline{\mathbf{B}}_2}{\Lambda^*}\right)\Omega_1 + \frac{\overline{\mathbf{A}}_1\overline{\mathbf{A}}_2}{\Lambda^*}\Omega_2\right] + \frac{q^{\binom{\alpha}{2}}(T-\omega_0)^\alpha}{\tilde{\Gamma}_q(\alpha+1)}\right\} \\ \leq & \ R. \end{aligned} \tag{3.19}$$

Since

$$\left|\left(\tilde{D}^\beta_{q,\omega}\mathcal{F}u\right)(t)\right| < R. \tag{3.20}$$

Therefore, $\|\mathcal{F}u\|_C \leq R$. Hence, $\mathcal{F}$ is uniformly bounded.

Step II. That the operator $\mathcal{F}$ is continuous on B_R since the continuity of F.

Step III. Examine that $\mathcal{F}$ is equicontinuous on B_R.
For any $t_1, t_2 \in I^T_{q,\omega}$ with $t_1 < t_2$, by Lemma 2.9 we have

$$\begin{aligned}&\left|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\right| \\ &\quad \leq \frac{q^{\binom{\alpha}{2}}\|F\|}{\tilde{\Gamma}_q(\alpha+1)}\left|(t_2-\omega_0)^\alpha - (t_1-\omega_0)^\alpha\right| \\ &\quad\quad + \frac{\left(\overline{\mathbf{B}}_2\Omega_1 + \overline{\mathbf{A}}_2\Omega_2\right)\|F\|}{\Lambda^*}\left|(t_2-\omega_0)^{\alpha-1} - (t_1-\omega_0)^{\alpha-1}\right| \\ &\quad\quad + \frac{\|F\|}{\mathbf{A}_2^*}\left[\left(1 + \frac{\overline{\mathbf{A}}_1\overline{\mathbf{B}}_2}{\Lambda^*}\right)\Omega_1 + \frac{\overline{\mathbf{A}}_1\overline{\mathbf{A}}_2}{\Lambda^*}\Omega_2\right]\left|(t_2-\omega_0)^{\alpha-2} - (t_1-\omega_0)^{\alpha-2}\right| \end{aligned} \tag{3.21}$$

and

$$\begin{aligned}&\left|\left(\tilde{D}^\beta_{q,\omega}\mathcal{F}u\right)(t_1) - \left(\tilde{D}^\beta_{q,\omega}\mathcal{F}u\right)(t_2)\right| \\ &\leq \frac{q^{\binom{\alpha}{2}+\binom{-\beta}{2}-\alpha\beta}\|F\|}{\tilde{\Gamma}_q(\alpha-\beta+1)}\left|(t_2-\omega_0)^{\alpha-\beta} - (t_1-\omega_0)^{\alpha-\beta}\right| \\ &\quad + \frac{q^{\binom{-\beta}{2}-\alpha\beta}\,\tilde{\Gamma}_q(\alpha)\|F\|}{\Lambda^*\tilde{\Gamma}_q(\alpha-\beta)}\left(\overline{\mathbf{B}}_2\Omega_1 + \overline{\mathbf{A}}_2\Omega_2\right)\left|(t_2-\omega_0)^{\alpha-\beta-1} - (t_1-\omega_0)^{\alpha-\beta-1}\right| \\ &\quad + \frac{q^{\binom{-\beta}{2}-\alpha\beta}\,\tilde{\Gamma}_q(\alpha-1)\|F\|}{\mathbf{A}_2^*\tilde{\Gamma}_q(\alpha-\beta-1)}\left[\left(1 + \frac{\overline{\mathbf{A}}_1\overline{\mathbf{B}}_2}{\Lambda^*}\right)\Omega_1 + \frac{\overline{\mathbf{A}}_1\overline{\mathbf{A}}_2}{\Lambda^*}\Omega_2\right]\times\end{aligned}$$

$$\left|(t_2-\omega_0)^{\alpha-\beta-2}-(t_1-\omega_0)^{\alpha-\beta-2}\right|. \tag{3.22}$$

Since the right-hand side of (3.22) tends to be zero when $|t_2-t_1|\to 0$, $\mathcal{F}$ is relatively compact on B_R. Therefore, the set $\mathcal{F}(B_R)$ is an equicontinuous set. From Steps I to III together with the Arzelá-Ascoli theorem, $\mathcal{F}: C\to C$ is completely continuous. By Schauder's fixed point theorem, we can conclude that problem (1.1) has at least one solution.

4. Example

Thoroughly, we provide the boundary value problem for fractional Hahn difference equation

$$\begin{aligned}
\tilde{D}^{\frac{5}{3}}_{\frac{1}{2},\frac{2}{3}}u(t) &= \frac{1}{(100e^2+t^3)(1+|u(t)|)}\Bigg[e^{-3t}\left(u^2+2|u|\right)+e^{-(\pi+\cos^2\pi t)}\left|\tilde{D}^{\frac{2}{5}}_{\frac{1}{2},\frac{2}{3}}u(t)\right| \\
&\quad +e^{-(1+\sin^2\pi t)}\left|\tilde{\Psi}^{\frac{3}{4}}_{\frac{1}{2},\frac{2}{3}}u(t)\right|\Bigg] \\
u\left(\frac{4}{3}\right) &= 2\tilde{\mathcal{I}}^{\frac{3}{4}}_{\frac{1}{2},\frac{2}{3}}e^{\cos\left(\frac{15}{6}\pi\right)}u\left(\frac{15}{6}\right) \\
u(10) &= 3\tilde{\mathcal{I}}^{\frac{1}{3}}_{\frac{1}{2},\frac{2}{3}}e^{2\sin\left(\frac{47}{32}\pi\right)}u\left(\frac{47}{32}\right),
\end{aligned} \tag{4.1}$$

where $t\in I^{10}_{\frac{1}{2},\frac{2}{3}}$ and $\varphi(t,s)=\frac{e^{-s}}{(t+10)^3}$.

We let $\alpha=\frac{5}{3}$, $\beta=\frac{2}{5}$, $\gamma=\frac{3}{4}$, $\theta_1=\frac{3}{4}$, $\theta_2=\frac{1}{3}$, $q=\frac{1}{2}$, $\omega=\frac{2}{3}$, $\omega_0=\frac{\omega}{1-q}=\frac{4}{3}$, $T=10$, $\eta_1=10\left(\frac{1}{2}\right)^4+\frac{2}{3}[4]_{\frac{1}{2}}=\frac{15}{8}$, $\eta_2=10\left(\frac{1}{2}\right)^6+\frac{2}{3}[6]_{\frac{1}{2}}=\frac{47}{32}$, $\lambda_1=2$, $\lambda_2=3$, $g_1(t)=e^{\cos(\pi t)}$, $g_2(t)=e^{2\sin(\pi t)}$ and $\varphi_0=\max\{\varphi(t,s)\}=\frac{27}{1156e^{\frac{4}{3}}}$.

For all $t\in I^{10}_{\frac{1}{2},\frac{2}{3}}$ and $u,v\in\mathbb{R}$, we have

$$\begin{aligned}
&\left|F\left(t,u,\tilde{D}^{\beta}_{q,\omega}u,\tilde{\Psi}^{\gamma}_{q,\omega}u\right)-F\left(t,v,\tilde{D}^{\beta}_{q,\omega}v,\tilde{\Psi}^{\gamma}_{q,\omega}v\right)\right| \\
&\le \frac{1}{e^4\left(100e^2+\frac{64}{27}\right)}|u-v|+\frac{1}{e^\pi\left(100e^2+100e^2+\frac{64}{27}\right)}\left|\tilde{D}^{\beta}_{q,\omega}u-\tilde{D}^{\beta}_{q,\omega}v\right| \\
&\quad +\frac{1}{e\left(100e^2+100e^2+\frac{64}{27}\right)}\left|\tilde{\Psi}^{\gamma}_{q,\omega}u-\tilde{\Psi}^{\gamma}_{q,\omega}v\right|,
\end{aligned}$$

and $\frac{1}{e}<g_1(t)<e$, $\frac{1}{e^2}<g_2(t)<e^2$

Thus, (H_1) and (H_2) hold with $\ell_1=0.0000247$, $\ell_2=0.0000583$, $\ell_3=0.000496$ and $g_1=\frac{1}{e}$, $g_2=\frac{1}{e^2}$, $G_1=e$, $G_2=e^2$.

Since

$$\Omega_1=0.161,\quad \Omega_2=21.708,\quad \overline{\mathbf{A}}_1=1.518,\quad \overline{\mathbf{A}}_2=6.717,\quad \mathbf{A}_2^*=0.909,$$

$$\overline{\mathbf{B}}_1 = 3.241, \quad \overline{\mathbf{B}}_2 = 30.841 \quad \text{and} \quad \Lambda^* = 1.247,$$

therefore, (H_3) holds with

$$\Theta = 0.063 < 1.$$

Hence, by Theorem 3.1 problem (4.1) has a unique solution. □

5. Conclusion

The new problem containing two fractional symmetric Hahn difference operators and three fractional symmetric Hahn integral with different numbers of order was proposed. The new concepts of fractional symmetric Hanh calculus were used in the study of existence results of the govern problem. The Banach fixed point and Schauder's fixed point theorems were also employed in this study.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

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