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Research article

On normal curves and their characterizations in Lorentzian n-space

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Abstract: This paper deals with the generalization of null and non-null normal curves in Lorentzian n-space E_1^n. We reveal necessary and sufficient condition for a curve to be a normal curve in Lorentzian n-space E_1^n. We obtain the relationship between the curvatures for any arclength parametrized curve to be congruent to a normal curve in E_1^n. Moreover, we give differential equations by introducing a differentiable function $f(s)$ which can be solved explicitly for a curve to be congruent to a normal curve.

Keywords: Lorentzian n-space; null normal curves; non-null normal curves; Frenet equations; curvatures
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1. Introduction

In the Euclidean space E^3, three classes of curves exists, which are called rectifying, normal and osculating curves all satisfying Cesaro's fixed point condition [1]. The rectifying curve in E^3 is defined as a curve whose position vector always lies in its rectifying plane which is spanned by the tangent vector T and the binormal vector B [2]. The relationship between the rectifying curves and the notion of centrodes in mechanics was introduced in [3]. Many authors in their papers have investigated rectifying curves in Euclidean and Lorentz-Minkowski space [4–9].

Similarly, a normal curve in Minkowski 3-space E_1^3 is defined in [10] as a space curve whose position vector always lies in its normal plane which is spanned by the normal vector N and the binormal vector B of the curve. According to this definition, the position vector of a normal curve satisfies $\alpha(s)=\lambda(s)N(s)+\mu(s)B(s)$ for some differentiable functions $\lambda(s)$ and $\mu(s)$ in arclength function s. Spacelike, timelike and null normal curves in Minkowski space are studied in [10] and [11]. Spacelike and timelike normal curves in Minkowski space-time are investigated in [12]. The relations between rectifying and normal curves in Minkowski 3-space are obtained in [13]. The characterizations of normal curves in Galilean space are obtained in [14] and [15]. Moreover in [16], the definition and concept of a normal curve is extended to the general case E^n.

In this paper, by using similar methods as in [8], we introduce the normal curves in the Lorentzian n-space E_1^n. We characterize null and non-null normal curves in terms of their curvature functions and obtain necessary and sufficient conditions for any curve to be a normal curve.

2. Preliminaries

Let E_1^n denote the Lorentzian n-space. For vectors $X=(x_1,x_2,...,x_n)$ and $Y=(y_1,y_2,...,y_n)$ in E_1^n

$$\langle X,Y\rangle = -x_1y_1+\sum_{i=2}^{n} x_iy_i$$

is called Lorentzian inner product. Since $\langle , \rangle$ is an indefinite metric, recall that a vector $v\in E_1^n$ can have one of three causal characters; it can be spacelike $\langle v,v\rangle\rangle 0$ or $v=0$; timelike if $\langle v,v\rangle\langle 0$ and null (lightlike) if $\langle v,v\rangle=0$ and $v\neq 0$. The pseudo-norm (length) of a vector v is given by $\|v\|=\sqrt{|\langle v,v\rangle|}$ [17].

We define the curve of $\alpha=\alpha(s)$ to be an arclength parametrized non-null curve in E_1^n. Let $\{T(s),N(s),B_1(s),...,B_{n-2}(s)\}$ be the moving frame along α, where the vectors $T(s),N(s),$ $B_1(s),...,B_{n-2}(s)$ are mutually orthogonal vectors satisfying

$$\langle T,T\rangle=\varepsilon_1=\pm 1,\ \ \langle N,N\rangle=\varepsilon_2=\pm 1\ \ and\ \ \langle B_i,B_i\rangle=\varepsilon_{i+2}=\pm 1,\quad i=\{1,2,...,n-2\}. \tag{2.1}$$

Then the Frenet equations of the curve α are as follows [18]:

$$\begin{aligned}
&T'(s)=\varepsilon_2k_1(s)N(s),\\
&N'(s)=-\varepsilon_1k_1(s)T(s)+\varepsilon_3k_2(s)B_1(s),\\
&B_1'(s)=-\varepsilon_2k_2(s)N(s)+\varepsilon_4k_3(s)B_2(s),\\
&B_i'(s)=-\varepsilon_{i+1}k_{i+1}(s)B_{i-1}(s)+\varepsilon_{i+3}k_{i+2}(s)B_{i+1}(s),\\
&B_{n-2}'(s)=-\varepsilon_{n-1}k_{n-1}(s)B_{n-3}(s).
\end{aligned} \tag{2.2}$$

If the curve is not arclength parametrized, then the right-hand sides of (2.2) must be multiplied by the speed v of α. We recall the functions $k_i(s)$ are called the i-th curvatures of for $i=\{1,2,...,n-1\}$. All the curvatures satisfy $k_i(s)\rangle 0$ for all $s\in I$, $1\le i\le n-2$. If $k_{n-1}(s)=0$ for all $s\in I$, then $B_{n-2}(s)$ is a constant vector and the curve lies in a $(n-1)$ -dimensional affine subspace orthogonal to B_{n-2} which is isometric to the Lorentzian $(n-1)$-space E_1^{n-1}. Thus the curve lies in a hyperplane if and only if in every point the position vector of a curve lies in the orthogonal complement of B_{n-2}. Analogously, if in every point the position vector of an arclength parametrized curve lies in the orthogonal complement of the tangent vector T, then the curve α lies on some hyperquadrics. Indeed, we see that the derivative of $\langle\alpha,\alpha\rangle$ is zero; hence $\langle\alpha,\alpha\rangle$ is a constant and thus lies on some hyperquadrics. Here the converse is also true. From this reasoning, we study curves for which in every point the position vector of the curve lies in the orthogonal complement of the tangent vector T [8].

Definition 2.1. A curve $\alpha: I\subset\mathbb{R}\to E_1^n$ is a normal curve if the orthogonal complement of $T(s)$ contains a fixed point for all $s\in I$. Hereafter, since the orthogonal complement of $T(s)$ is defined by

$T^{\perp}(s)=\{v\in E_1^n \mid \langle v,T(s)\rangle =0\}$, the position vector of a normal curve holds (2.3) with $\lambda,\mu_1,\mu_2,...\mu_{n-2}$ differentiable real functions

$$\alpha(s)=\lambda(s)N(s)+\mu_1(s)B_1(s)+...+\mu_{n-2}(s)B_{n-2}(s)\,. \tag{2.3}$$

Let us note that the hyperquadrics S_1^{n-1} and H_0^{n-1} are defined by

$$S_1^{n-1}=\{v\in E_1^n \mid \langle v,v\rangle =1\},\quad H_0^{n-1}=\{v\in E_1^n \mid \langle v,v\rangle =-1\}$$

respectively. In the rest of this paper, we assume that all the curvatures of the curve are not identically zero.

3. Some characterizations of null normal curves in E_1^n

In this section we give some characterizations of null normal curves in E_1^n, $n\rangle 4$.

Let $\alpha: I\subset\mathbb{R}\to E_1^n$ be a null curve parametrized by the pseudo-arclength such that $\{\alpha'(s),\alpha''(s),...,\alpha^n(s)\}$ is a basis of $T_{\alpha(s)}E_1^n$ for all s. Then there exists only one Frenet frame satisfying the equations

$$\begin{aligned}
&T'=N,\\
&N'=k_1T-B_1,\\
&B_1'=-k_1N+k_2B_2,\\
&B_2'=-k_2T+k_3B_3,\\
&B_i'=-k_iB_{i-1}+k_{i+1}B_{i+1},\qquad i\in\{3,...,n-3\}\\
&B_{n-2}'=-k_{n-2}B_{n-3}
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
&\langle T,T\rangle=\langle B_1,B_1\rangle=0,\qquad \langle T,B_1\rangle=1,\\
&\langle N,N\rangle=\langle B_2,B_2\rangle=...=\langle B_{n-2},B_{n-2}\rangle=1.
\end{aligned}$$

Let $\alpha(s)$ be a null normal curve in E_1^n, parametrized by pseudo-arclength s. Then its position vector satisfies the equation

$$\alpha(s)=\lambda(s)T(s)+\mu_1(s)N(s)+\mu_2(s)B_2(s)+...+\mu_{n-2}(s)B_{n-2}(s) \tag{3.2}$$

for some differentiable functions $\lambda(s),\mu_1(s),\mu_2(s),...,\mu_{n-2}(s)$. Differentiating (3.2) with respect to s and by using (3.1), we obtain the system of equations

$$\lambda'+\mu_1k_1-\mu_2k_2=1, \tag{3.3}$$

$$\lambda+\mu_1'=0, \tag{3.4}$$

$$\mu_1 = 0, \tag{3.5}$$

$$\mu_2{}' - \mu_3 k_3 = 0, \tag{3.6}$$

$$\mu_i{}' - \mu_{i-1} k_i - \mu_{i+1} k_{i+1} = 0, \qquad i \in \{2,3...,n-3\}, \tag{3.7}$$

$$\mu'_{n-2} + \mu_{n-3} k_{n-2} = 0. \tag{3.8}$$

From the Eqs. (3.4) and (3.5) we get $\mu_1(s) = 0$ and $\lambda(s) = 0$. Considering the differentiable functions $\lambda(s), \mu_1(s), \mu_2(s), ..., \mu_{n-2}(s)$, we get the following theorem.

Theorem 3.1. Let $\alpha : I \subset \mathbb{R} \to E_1^n$ be a null normal curve in E_1^n, parametrized by pseudo-arclength s. Then the following statements are hold:

i. The components of the position vector of α are

$$\lambda(s) = 0, \qquad \mu_1(s) = 0,$$

$$\mu_i(s) = \sum_{i=0}^{i-2} \mu_{i,k}(s) \frac{\partial^k}{\partial s^k}\left(-\frac{1}{k_2(s)}\right) \qquad i \in \{2,3,...,n-2\}$$

where the functions $\mu_{i,k}$ can be inductively defined by

$$\mu_{1,0} = 0, \qquad \mu_{2,0} = 1$$

and for $i \in \{3,4,...,n-2\}$

$$\begin{aligned}
\mu_{i,0}(s) &= \frac{k_{i-1}(s)\mu_{i-2,0}(s) + \mu'_{i-1,0}(s)}{k_i(s)}, \\
\mu_{i,k}(s) &= \frac{k_{i-1}(s)\mu_{i-2,k}(s) + \mu_{i-1,k-1}(s) + \mu'_{i-1,k}(s)}{k_i(s)}, \\
\mu_{i,i-3}(s) &= \frac{\mu_{i-1,i-4}(s) + \mu_{i-1,i-3}(s)}{k_i(s)}, \\
\mu_{i,i-2}(s) &= \frac{\mu_{i-1,i-3}(s)}{k_i(s)}.
\end{aligned} \tag{3.9}$$

ii. If $k_2, k_3, ..., k_{n-2}$ are nonzero constants, then α lies in pseudosphere $S_1^{n-1}(r), r \in \mathbb{R}_0^+$.

Proof. **i.** Let $\alpha(s)$ be a null normal curve in E_1^n, parametrized by pseudo-arclength s. Then its position vector is given by (3.2). Then the equation system (3.3)–(3.8) gives

$$\lambda(s) = 0, \qquad \mu_1(s) = 0,$$

$$\mu_2(s) = -\frac{1}{k_2(s)}, \qquad \mu_3(s) = \frac{1}{k_3(s)}\left(-\frac{1}{k_2(s)}\right)'. \tag{3.10}$$

Considering the functions $\mu_{1,0}, \mu_{2,0}, \mu_{3,0}$ and $\mu_{3,1}$, we have

$$\mu_1(s) = \mu_{1,0}(s)\left(-\frac{1}{k_2(s)}\right), \qquad \mu_{1,0}(s) = 0,$$

$$\mu_2(s) = \mu_{2,0}(s)\left(-\frac{1}{k_2(s)}\right), \qquad \mu_{2,0}(s) = 1,$$

$$\mu_3(s) = \frac{1}{k_3(s)}\left(-\frac{1}{k_2(s)}\right)', \qquad \mu_{3,0}(s) = 0, \quad \mu_{3,1}(s) = \frac{1}{k_3(s)}.$$

By induction from (3.7), we obtain

$$\mu_i(s) = \sum_{k=0}^{i-2} \mu_{i,k}(s) \frac{\partial^k}{\partial s^k}\left(-\frac{1}{k_2(s)}\right), \qquad 2 \le i \le n-2. \tag{3.11}$$

Here the functions $\mu_{i,k}$ are defined by (3.9). This proves the statement (i).

ii. If $k_2, k_3, ..., k_{n-2}$ are nonzero constants, then the components $\mu_2, \mu_3 ..., \mu_{n-2}$ of the position vector of $\alpha(s)$ are constant numbers. Then the position vector of $\alpha(s)$ is

$$\alpha(s) = \mu_2 B_2(s) + \mu_3 B_3(s) + ... + \mu_{n-2} B_{n-2}(s).$$

From the last equation, we get

$$\langle \alpha(s), \alpha(s) \rangle = \mu_2^2 + \mu_3^2 + ... + \mu_{n-2}^2 = r^2, \qquad r \in \mathbb{R}_0^+,$$

which means that $\alpha(s)$ lies in $S_1^{n-1}(r)$ with center at the origin and the radius *r*. This proves the statement (ii).

Theorem 3.2. Let $\alpha : I \subset \mathbb{R} \to E_1^n$ be a null curve in E_1^n with nonzero curvatures. Then $\alpha(s)$ is congruent to a normal curve if and only if

$$\left(\sum_{k=0}^{n-4} \mu_{n-2,k}(s) \frac{\partial^k}{\partial s^k}\left(-\frac{1}{k_2(s)}\right)\right)' + k_{n-2}(s)\left(\sum_{k=0}^{n-5} \mu_{n-3,k}(s) \frac{\partial^k}{\partial s^k}\left(-\frac{1}{k_2(s)}\right)\right) = 0. \tag{3.12}$$

Proof. If $\alpha(s)$ is a null normal curve in E_1^n, writing (3.11) for $i = n-3$ and $i = n-2$ in (3.8), we obtain (3.12).

Conversely, assume that (3.12) holds. Then we define the vector $m(s) \in E_1^n$ given by

$$m(s) = \alpha(s) - \mu_2(s) B_2(s) - ... - \mu_{n-2}(s) B_{n-2}(s) \tag{3.13}$$

with $\mu_2, \mu_3, ..., \mu_{n-2}$ as in (3.10) and (3.11). If we differentiate (3.13) with respect to s and using (3.1)

$$m'(s)=\left(\left(\sum_{k=0}^{n-4}\mu_{n-2,k}(s)\frac{\partial^k}{\partial s^k}\left(-\frac{1}{k_2(s)}\right)\right)'+k_{n-2}(s)\left(\sum_{k=0}^{n-5}\mu_{n-3,k}(s)\frac{\partial^k}{\partial s^k}\left(-\frac{1}{k_2(s)}\right)\right)\right)B_{n-2}(s) \tag{3.14}$$

gives $m'(s)=0$. Then $m(s)$ is a constant vector and so $\alpha(s)$ is congruent to a null normal curve.

4. Some characterizations of non-null normal curves in E_1^n

In this section, we first characterize the non-null normal curves in terms of their curvatures.

Let $\alpha: I\subset\mathbb{R}\to E_1^n$ be an arclength parametrized non-null normal curve in E_1^n. The position vector of the curve satisfies (2.3) for smooth functions $\lambda,\mu_1,\mu_2,...\mu_{n-2}$. Differentiating (2.3) with respect to s and using (2.2), we have

$$\begin{aligned}T(s)=&-\varepsilon_1k_1(s)\lambda(s)T(s)+(\lambda'(s)-\varepsilon_2k_2(s)\mu_1(s))N(s)\\&+(\varepsilon_3k_2(s)\lambda(s)+\mu_1'(s)-\varepsilon_3k_3(s)\mu_2(s))B_1(s)\\&+\sum_{i=2}^{n-3}(\varepsilon_{i+2}k_{i+1}(s)\mu_{i-1}(s)+\mu_i'(s)-\varepsilon_{i+2}k_{i+2}(s)\mu_{i+1}(s))B_i(s)\\&+(\mu_{n-2}'(s)+\varepsilon_nk_{n-1}(s)\mu_{n-3}(s))B_{n-2}(s)\end{aligned}$$

It follows that

$$1+\varepsilon_1k_1(s)\lambda(s)=0, \tag{4.1}$$

$$\lambda'(s)-\varepsilon_2k_2(s)\mu_1(s)=0, \tag{4.2}$$

$$\varepsilon_3k_2(s)\lambda(s)+\mu_1'(s)-\varepsilon_3k_3(s)\mu_2(s)=0, \tag{4.3}$$

$$\varepsilon_{i+2}k_{i+1}(s)\mu_{i-1}(s)+\mu_i'(s)-\varepsilon_{i+2}k_{i+2}(s)\mu_{i+1}(s)=0, \qquad i\in\{2,3,...,n-3\} \tag{4.4}$$

$$\mu_{n-2}'(s)+\varepsilon_nk_{n-1}(s)\mu_{n-3}(s)=0\,. \tag{4.5}$$

This system consists of n equations and $(n-1)$ curvature functions, the function λ and $(n-2)$ functions μ_i. Thus the coefficient functions μ_i can be expressed in terms of the curvature functions, derivatives of the curvature functions and the function λ. From (4.1), we have

$$\lambda(s)=-\frac{\varepsilon_1}{k_1(s)}\,. \tag{4.6}$$

Using the coefficient (4.6) in (4.2), we get

$$\mu_1(s)=-\varepsilon_1\varepsilon_2\frac{1}{k_2(s)}\left(\frac{1}{k_1(s)}\right)'. \tag{4.7}$$

Similarly using the coefficient (4.7) in (4.3), we obtain

$$\mu_2(s) = -\frac{\varepsilon_1\varepsilon_3}{k_3(s)}\left[\varepsilon_3\frac{k_2(s)}{k_1(s)} + \varepsilon_2\left(\frac{1}{k_2(s)}\left(\frac{1}{k_1(s)}\right)'\right)'\right].$$

When the other coefficient functions are calculated, long and complex expressions with curvature functions appear. Considering the functions $\mu_{1,0}, \mu_{2,0}, \mu_{2,1}$ and $\mu_{2,2}$, we have

$$\mu_1(s) = \mu_{1,1}(s)\left(\frac{1}{k_1(s)}\right)', \qquad \mu_{1,1}(s) = -\varepsilon_1\varepsilon_2\frac{1}{k_2(s)}, \tag{4.8}$$

$$\mu_2(s) = \mu_{2,0}(s)\left(\frac{1}{k_1(s)}\right) + \mu_{2,1}(s)\left(\frac{1}{k_1(s)}\right)' + \mu_{2,2}(s)\left(\frac{1}{k_1(s)}\right)'' \tag{4.9}$$

where

$$\begin{aligned} &\mu_{2,0}(s) = -\varepsilon_1\frac{k_2(s)}{k_3(s)}, \qquad \mu_{2,1}(s) = -\varepsilon_1\varepsilon_2\varepsilon_3\left(\frac{1}{k_3(s)}\right)\left(\frac{1}{k_2(s)}\right)', \\ &\mu_{2,2}(s) = -\varepsilon_1\varepsilon_2\varepsilon_3\left(\frac{1}{k_3(s)}\right)\left(\frac{1}{k_2(s)}\right). \end{aligned} \tag{4.10}$$

Similarly introducing the functions $\mu_{3,0}, \mu_{3,1}, \mu_{3,2}$ and μ_{33} we have

$$\mu_3(s) = \mu_{3,0}(s)\left(\frac{1}{k_1(s)}\right) + \mu_{3,1}(s)\left(\frac{1}{k_1(s)}\right)' + +\mu_{3,2}(s)\left(\frac{1}{k_1(s)}\right)'' + \mu_{3,3}(s)\left(\frac{1}{k_1(s)}\right)''' \tag{4.11}$$

where

$$\begin{aligned} &\mu_{3,0}(s) = \frac{-\varepsilon_1\varepsilon_4}{k_4(s)}\left(\frac{k_2(s)}{k_3(s)}\right)', \\ &\mu_{3,1}(s) = \frac{-\varepsilon_1\varepsilon_4}{k_4(s)}\left[\varepsilon_2\varepsilon_4\left(\frac{k_3(s)}{k_2(s)}\right) + \left(\frac{k_2(s)}{k_3(s)}\right) + \varepsilon_2\varepsilon_3\left(\frac{1}{k_3(s)}\left(\frac{1}{k_2(s)}\right)'\right)'\right], \\ &\mu_{3,2}(s) = \frac{-\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4}{k_4(s)}\left[\frac{1}{k_3(s)}\left(\frac{1}{k_2(s)}\right)' + \left(\frac{1}{k_3(s)}\left(\frac{1}{k_2(s)}\right)\right)'\right], \\ &\mu_{3,3}(s) = \frac{-\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4}{k_4(s)}\left(\frac{1}{k_3(s)}\right)\left(\frac{1}{k_2(s)}\right). \end{aligned} \tag{4.12}$$

By induction from (4.4), we obtain

$$\mu_i(s) = \sum_{k=0}^{i}\mu_{i,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right), \qquad 1 \le i \le n-2. \tag{4.13}$$

Here the functions $\mu_{i,k}$ can be inductively defined by

$$\begin{cases} \mu_{1,0}(s)=0, \qquad \mu_{1,1}(s)=-\varepsilon_1\varepsilon_2\dfrac{1}{k_2(s)}, \qquad \mu_{2,0}(s)=-\varepsilon_1\dfrac{k_2(s)}{k_3(s)}, \\ \mu_{2,1}(s)=-\varepsilon_1\varepsilon_2\varepsilon_3\left(\dfrac{1}{k_3(s)}\right)\left(\dfrac{1}{k_2(s)}\right)', \qquad \mu_{2,2}(s)=-\varepsilon_1\varepsilon_2\varepsilon_3\left(\dfrac{1}{k_3(s)}\right)\left(\dfrac{1}{k_2(s)}\right), \\ \mu_{i,0}(s)=\dfrac{\varepsilon_{i+1}}{k_{i+1}(s)}\left(\varepsilon_{i+1}k_i(s)\mu_{i-2,0}(s)+\mu'_{i-1,0}(s)\right), \\ \mu_{i,k}(s)=\dfrac{\varepsilon_{i+1}}{k_{i+1}(s)}\left(\varepsilon_{i+1}k_i(s)\mu_{i-2,k}(s)+\mu'_{i-1,k}(s)+\mu_{i-1,k-1}(s)\right), \\ \mu_{i,i-1}(s)=\dfrac{\varepsilon_{i+1}}{k_{i+1}(s)}\left(\mu_{i-1,i-2}(s)+\mu'_{i-1,i-1}(s)\right), \\ \mu_{i,i}(s)=\dfrac{\varepsilon_{i+1}}{k_{i+1}(s)}\mu_{i-1,i-1}(s). \end{cases} \tag{4.14}$$

where $k\in\{1,2,...,i-3\}$ and $i\in\{3,4,...,n-2\}$. Substituting Eqs. (4.6) and (4.13) into (2.3), we get the position vector of the normal curve as:

$$\alpha(s)=-\frac{\varepsilon_1}{k_1(s)}N(s)+\sum_{i=1}^{n-2}\left(\sum_{k=0}^{i}\mu_{i,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right)B_i(s)\,. \tag{4.15}$$

Then based on the Eqs system (4.1)–(4.5), we state the following theorem:

Theorem 4.1. Let $\alpha: I\subset\mathbb{R}\to E_1^n$ be an arclength parametrized curve in E_1^n with nonzero curvatures. Then $\alpha(s)$ is congruent to a normal curve if and only if

$$\left(\sum_{k=0}^{n-2}\mu_{n-2,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right)'+\varepsilon_n k_{n-1}(s)\left(\sum_{k=0}^{n-3}\mu_{n-3,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right)=0 \tag{4.16}$$

with $\mu_{i,k}$ inductively defined by the system (4.14).

Proof. If $\alpha(s)$ is a normal curve, writing (4.13) for $i=n-3$ and $i=n-2$ in (4.5), we obtain (4.16).

Conversely, assume that (4.16) holds. Then we define the vector $m(s)\in E_1^n$ given by

$$m(s)=\alpha(s)-\lambda(s)N(s)-\mu_1(s)B_1(s)-...-\mu_{n-2}(s)B_{n-2}(s) \tag{4.17}$$

with $\lambda(s)=-\dfrac{\varepsilon_1}{k_1(s)}$ and $\mu_1(s),\mu_2(s),...,\mu_{n-2}(s)$ as in (4.8), (4.9) and (4.13). If we differentiate (4.17) with respect to s and by using (2.2)

$$m'(s)=\left(\left(\sum_{k=0}^{n-2}\mu_{n-2,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right)'+\varepsilon_n k_{n-1}(s)\left(\sum_{k=0}^{n-3}\mu_{n-3,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right)\right)B_{n-2}(s) \tag{4.18}$$

gives $m'(s)=0$. Then $m(s)$ is a constant vector and so $\alpha(s)$ is congruent to a normal curve.

Now, assume that all the curvature functions $k_1,k_2,...,k_{n-1}$ of normal curve are nonzero constants. Then, we give the following result:

Theorem 4.2. For odd n, there exists no normal curve with nonzero constant curvatures and for even n, every curve with nonzero constant curvatures is a normal curve in E_1^n.

Proof. Assume that there exists a normal curve with its nonzero constant curvatures $k_1,k_2,...,k_{n-1}$. From (4.1), (4.2), (4.3) and (4.4), it follows that

$$\lambda=-\frac{\varepsilon_1}{k_1}, \qquad \mu_1=0,$$
$$\mu_2=-\varepsilon_1\frac{k_2}{k_1.k_3}, \qquad \mu_3=0.$$

For $i\in\{4,5,...,n-2\}$ Eq. (4.4) gives

$$\mu_{i+1}(s)=\frac{\varepsilon_{i+2}}{k_{i+2}(s)}\left(\varepsilon_{i+2}k_{i+1}(s)\mu_{i-1}(s)+\mu_i'(s)\right).$$

By induction we obtain that

$$\mu_{2m-1}=0, \tag{4.19}$$

$$\mu_{2m}=-\varepsilon_1\frac{\prod_{i=1}^{m}k_{2i}}{\prod_{i=1}^{m+1}k_{2i-1}}. \tag{4.20}$$

For odd n , with the help of (4.19) and (4.20), Eq. (4.16) takes the following form

$$\varepsilon_1\varepsilon_n\frac{k_2k_4...k_{n-3}k_{n-1}}{k_1k_3...k_{n-4}k_{n-2}}=0.$$

However, since we assume all curvatures to be nonzero, this leads to a contradiction. For even n, according to Theorem 4.1., since the curvature functions obviously satisfy the relation (4.16) then $\alpha(s)$ is congruent to normal curve. Thus the proof is completed.

Example 4.1. A curve $\beta: I\subset\mathbb{R}\to E_1^n$ which has all its curvatures constant is parametrized by

$$\beta(s)=(a_1\cosh(b_1s),a_1\sinh(b_1s),a_2\cos(b_2s),a_2\sin(b_2s),...,a_t\cos(b_ts),a_t\sin(b_ts)) \tag{4.21}$$

for even $n=2t$ and by

$$\beta(s)=(a_1\cosh(b_1s),a_1\sinh(b_1s),a_2\cos(b_2s),a_2\sin(b_2s),...,a_t\cos(b_ts),a_t\sin(b_ts),cs) \tag{4.22}$$

for odd $n=2t+1$. Here $c,a_i,b_i\in\mathbb{R}$ and all b_i are different numbers for $i=\{1,2,...,t\}$.

From the parametrization (4.21), for even n, we obtain the derivative of $\langle \beta, \beta \rangle$ is zero. Then $\langle \beta, \beta \rangle = constant$ and thus β lies on some hyperquadrics in E_1^n. This means that in every point the position vector of the curve lies in the orthogonal complement of the tangent vector T. So the curve β is a normal curve. Also we can easily show that the curve β with all its curvatures constant is a normal curve since $\langle \beta(s), T(s) \rangle = 0$. From the parametrization (4.22), for odd n, the curve β with all its curvatures are constant is not a normal curve since $\langle \beta(s), T(s) \rangle \neq 0$.

Theorem 4.3. Let $\alpha = \alpha(s)$ be an arclength parametrized curve, lying fully in the n-dimensional Lorentzian space with nonzero curvatures. Then α is a normal curve if and only if α lies in some hyperquadrics in E_1^n.

Proof. First assume that $\alpha(s)$ is congruent to a normal curve. It follows, by straightforward calculations using Theorem 4.1., we obtain

$$
\begin{aligned}
&2\varepsilon_2\left(\frac{1}{k_1(s)}\right)\left(\frac{1}{k_1(s)}\right)' + 2\varepsilon_3\left(\frac{1}{k_2(s)}\left(\frac{1}{k_1(s)}\right)'\right)\left(\frac{1}{k_2(s)}\left(\frac{1}{k_1(s)}\right)'\right)' \\
&+2\varepsilon_4\left\{\frac{1}{k_3(s)}\left[\varepsilon_3\frac{k_2(s)}{k_1(s)}+\varepsilon_2\left(\frac{1}{k_2(s)}\left(\frac{1}{k_1(s)}\right)'\right)'\right]\right\}\times\left\{\frac{1}{k_3(s)}\left[\varepsilon_3\frac{k_2(s)}{k_1(s)}+\varepsilon_2\left(\frac{1}{k_2(s)}\left(\frac{1}{k_1(s)}\right)'\right)'\right]\right\}' \\
&+\ldots+2\varepsilon_{i+2}\left(\sum_{k=0}^{i}\mu_{i,k}\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right)\left(\sum_{k=0}^{i}\mu_{i,k}\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right)' \\
&+\ldots+2\varepsilon_n\left(\sum_{k=0}^{n-2}\mu_{n-2,k}\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right)\left(\sum_{k=0}^{n-2}\mu_{n-2,k}\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right)' = 0.
\end{aligned}
$$

On the other hand, the previous equation is differential of the equation

$$
\begin{aligned}
&\varepsilon_2\left(\frac{1}{k_1(s)}\right)^2 + \varepsilon_3\left(\frac{1}{k_2(s)}\left(\frac{1}{k_1(s)}\right)'\right)^2 + \varepsilon_4\left\{\frac{1}{k_3(s)}\left[\varepsilon_3\frac{k_2(s)}{k_1(s)}+\varepsilon_2\left(\frac{1}{k_2(s)}\left(\frac{1}{k_1(s)}\right)'\right)'\right]\right\}^2 \\
&+\ldots+\varepsilon_{i+2}\left(\sum_{k=0}^{i}\mu_{i,k}\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right)^2+\ldots+\varepsilon_n\left(\sum_{k=0}^{n-2}\mu_{n-2,k}\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right)^2 = r, \quad r\in\mathbb{R}.
\end{aligned}
\tag{4.23}
$$

Then using (4.6)–(4.13) in (4.17), we get $\langle \alpha(s)-m, \alpha(s)-m \rangle = r$. Consequently $\alpha(s)$ lies in some hyperquadrics in E_1^n.

Conversely, if $\alpha(s)$ lies in some hyperquadrics in E_1^n, then $\langle \alpha(s)-m, \alpha(s)-m \rangle = r$, $r\in\mathbb{R}$ where $m(s)\in E_1^n$ is a constant vector. By taking the derivative of the previous equation with respect to s, we obtain $\langle \alpha(s)-m, T(s) \rangle = 0$, which means that $\alpha(s)$ is a normal curve.

Theorem 4.4. Let $\alpha : I \subset \mathbb{R} \to E_1^n$ be an arclength parametrized normal curve in E_1^n with nonzero curvatures. Then the following statements are hold:

i. The normal component and the first binormal component of the position vector of the curve are given by

$$\langle \alpha(s), N(s) \rangle = -\frac{\varepsilon_1\varepsilon_2}{k_1(s)}, \qquad \langle \alpha(s), B_1(s) \rangle = -\frac{\varepsilon_1\varepsilon_2\varepsilon_3}{k_2(s)}\left(\frac{1}{k_1(s)}\right)'. \tag{4.24}$$

ii. The first binormal component and the second binormal component of the position vector of the curve are given by

$$\begin{aligned} \langle \alpha(s), B_1(s) \rangle &= -\frac{\varepsilon_1\varepsilon_2\varepsilon_3}{k_2(s)}\left(\frac{1}{k_1(s)}\right)', \\ \langle \alpha(s), B_2(s) \rangle &= -\frac{\varepsilon_1\varepsilon_3\varepsilon_4}{k_3(s)}\left[\varepsilon_3\frac{k_2(s)}{k_1(s)} + \varepsilon_2\left(\frac{1}{k_2(s)}\left(\frac{1}{k_1(s)}\right)'\right)'\right]. \end{aligned} \tag{4.25}$$

iii. The second binormal component and the third binormal component of the position vector of the curve are given by

$$\begin{aligned} \langle \alpha(s), B_2(s) \rangle &= -\frac{\varepsilon_1\varepsilon_3\varepsilon_4}{k_3(s)}\left[\varepsilon_3\frac{k_2(s)}{k_1(s)} + \varepsilon_2\left(\frac{1}{k_2(s)}\left(\frac{1}{k_1(s)}\right)'\right)'\right], \\ \langle \alpha(s), B_3(s) \rangle &= -\frac{\varepsilon_1\varepsilon_5}{k_4(s)}\left[\begin{array}{l} \varepsilon_2\left(\dfrac{k_3(s)}{k_2(s)}\left(\dfrac{1}{k_1(s)}\right)'\right) \\ +\varepsilon_4\left\{\dfrac{1}{k_3(s)}\left[\dfrac{k_2(s)}{k_1(s)} + \varepsilon_2\varepsilon_3\left(\dfrac{1}{k_2(s)}\left(\dfrac{1}{k_1(s)}\right)'\right)'\right]\right\}' \end{array}\right], \end{aligned} \tag{4.26}$$

iv. The jth binormal component and the (j+1)th binormal component of the position vector of the curve are given by

$$\begin{aligned} \langle \alpha(s), B_j(s) \rangle &= \varepsilon_{j+2}\sum_{k=0}^{j}\mu_{j,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right), \\ \langle \alpha(s), B_{j+1}(s) \rangle &= \varepsilon_{j+3}\sum_{k=0}^{j+1}\mu_{j+1,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right) \end{aligned} \tag{4.27}$$

where $3 \le j \le n-3$ and $\mu_{j,k}$ is introduced by (4.14).

v. The distance function $\rho(s) = \|\alpha(s)\|$ satisfies $\rho^2(s) = |a|$ for some $a \in \mathbb{R}$.

vi. The distance function $\rho(s) = \|\alpha(s)\|$ is constant and the binormal component $\alpha^B(s)$ of the position vector of the curve has nonconstant length.

Conversely, if $\alpha(s)$ is an arclength parametrized curve with nonzero curvatures and one of the above statements holds in E_1^n, then $\alpha(s)$ is congruent to a normal curve.

Proof. To prove (i), (ii), (iii) and (iv), assume that $\alpha(s)$ is an arclength parametrized normal curve in E_1^n. Taking the inner product of the two sides (4.15) with $N(s), B_1(s), B_2(s),..., B_j(s), B_{j+1}(s)$ where $3 \le j \le n-3$ respectively, we obtain the statements (i), (ii), (iii), (iv).

Conversely, assume that (i) is given. Differentiating $\langle \alpha(s), N(s)\rangle = -\frac{\varepsilon_1\varepsilon_2}{k_1(s)}$ with respect to s and by using (2.2), we get $\langle \alpha(s), T(s)\rangle = 0,$ which means $\alpha(s)$ is congruent to a normal curve. Similarly since the statements (ii), (iii), (iv) holds, then $\alpha(s)$ is a normal curve.

To prove (v), assume that $\alpha(s)$ is an arclength parametrized normal curve. Then multiplying (4.3), (4.4), and (4.5) with $\varepsilon_{i+2}\mu_i(s)$ where $i \in \{1,2,...,n-2\}$ respectively,

$$\varepsilon_3\mu_1(s)(\mu_1' - \varepsilon_3 k_3(s)\mu_2(s)) = -\varepsilon_2\left(\frac{1}{k_1(s)}\right)\left(\frac{1}{k_1(s)}\right)',$$
$$\varepsilon_{i+2}\mu_i(s)(\varepsilon_{i+2}k_{i+1}(s)\mu_{i-1}(s) + \mu_i' - \varepsilon_{i+2}k_{i+2}(s)\mu_{i+1}(s)) = 0,$$
$$\varepsilon_n\mu_{n-2}(s)(\varepsilon_n k_{n-1}(s)\mu_{n-3}(s) + \mu_{n-2}'(s)) = 0$$

and adding these equations we get $\sum_{i=1}^{n-2}\varepsilon_{i+2}\mu_i^2 = -\varepsilon_2\left(\frac{1}{k_1(s)}\right)^2 + a$ for $a \in \mathbb{R}$. From (4.15), we have

$$\rho^2(s) = \left|\langle \alpha(s), \alpha(s)\rangle\right| = \left|\varepsilon_2\lambda^2 + \sum_{i=1}^{n-2}\varepsilon_{i+2}\mu_i^2\right| = |a|.$$

Conversely, differentiating $\rho^2(s) = \left|\langle \alpha(s), \alpha(s)\rangle\right| = |a|$ with respect to s, we get $\langle \alpha(s), T(s)\rangle = 0.$ Thus, $\alpha(s)$ is congruent to a normal curve.

vi. Decompose the position vector of a curve $\alpha(s)$ in its normal and binormal component, i.e.,

$$\alpha(s) = \varepsilon_2\langle \alpha(s), N(s)\rangle N(s) + \alpha^B(s).$$

From $\alpha^B(s) = \sum_{i=1}^{n-2}\mu_i(s)B_i(s)$, we have $\left\|\alpha^B(s)\right\| = \sqrt{\left|-\varepsilon_2\left(\frac{1}{k_1(s)}\right)^2 + a\right|}$. Thus, the binormal component has nonconstant length. The distance function ρ is proved in (v).

Conversely since $\rho^2(s) = \left\|\alpha(s)\right\|^2$ is constant, $\langle \alpha(s), T(s)\rangle = 0.$ Hence, $\alpha(s)$ is congruent to a normal curve.

Lemma 4.1. Let $\alpha: I \subset \mathbb{R} \to E_1^n$ be an arclength parametrized curve with non-null vector fields $N, B_1, B_2, ..., B_{n-2}$, lying fully in E_1^n, then $\alpha(s)$ is congruent to a normal curve if and only if there exists a differentiable function $f(s)$ such that

$$f(s)k_{n-1}(s) = \left(\sum_{k=0}^{n-2}\mu_{n-2,k}(s)k_{n-1}(s)\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right),$$
$$f'(s) = -\varepsilon_n k_{n-1}(s)\left(\sum_{k=0}^{n-3}\mu_{n-3,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right) \tag{4.28}$$

Applying similar methods as in [19–21] together with Lemma 4.1., we obtain the following theorems for normal curves in E_1^n.

Theorem 4.5. Let $\alpha(s)$ be an arclength parametrized curve in E_1^n with nonzero curvatures and timelike principal binormal B_{n-2}. Then $\alpha(s)$ is congruent to a normal curve if and only if there exist constants $a_0, b_0 \in \mathbb{R}$ such that

$$\begin{aligned}\frac{f'(s)}{k_{n-1}(s)} &= \left\{a_0 - \int\left(\frac{f''(s)}{k_{n-1}(s)} - \frac{f'(s)}{k_{n-1}^2(s)}k_{n-1}'(s) - f(s)k_{n-1}(s)\right)\sinh\theta(s)ds\right\}\sinh\theta(s) \\ &\quad - \left\{b_0 - \int\left(\frac{f''(s)}{k_{n-1}(s)} - \frac{f'(s)}{k_{n-1}^2(s)}k_{n-1}'(s) - f(s)k_{n-1}(s)\right)\cosh\theta(s)ds\right\}\cosh\theta(s)\end{aligned} \tag{4.29}$$

where $\int_0^s k_{n-1}(s)ds = \theta(s)$.

Proof. Let $\alpha(s)$ is congruent to a normal curve with $\varepsilon_n = -1$. According to Lemma 4.1., there exists a differentiable function $f(s)$ such that the relation (4.28) holds. Let us determine the differentiable functions $\theta(s), a(s)$ and $b(s)$ by

$$\theta(s) = \int_0^s k_{n-1}(s)ds\,, \tag{4.30}$$

$$\begin{aligned}a(s) &= -\frac{f'(s)}{k_{n-1}(s)}\sinh\theta(s) + f(s)\cosh\theta(s) \\ &\quad + \int\left[\frac{f''(s)}{k_{n-1}(s)} - \frac{f'(s)}{k_{n-1}^2(s)}k_{n-1}'(s) - f(s)k_{n-1}(s)\right]\sinh\theta(s)ds,\end{aligned} \tag{4.31}$$

$$\begin{aligned}b(s) &= -\frac{f'(s)}{k_{n-1}(s)}\cosh\theta(s) + f(s)\sinh\theta(s) \\ &\quad + \int\left[\frac{f''(s)}{k_{n-1}(s)} - \frac{f'(s)}{k_{n-1}^2(s)}k_{n-1}'(s) - f(s)k_{n-1}(s)\right]\cosh\theta(s)ds.\end{aligned} \tag{4.32}$$

By using (4.28), we find $a'(s) = 0$ and $b'(s) = 0$. Thus $a(s) = a_0$ and $b(s) = b_0 \in \mathbb{R}$. Multiplying (4.31) and (4.32) respectively with $\sinh\theta(s)$ and $-\cosh\theta(s)$, adding the obtained equations, we get (4.29).

Conversely let $a_0, b_0 \in \mathbb{R}$ are constants such that the relation (4.29) holds. Let us define the differentiable function $f(s)$ by

$$f(s) = \frac{1}{k_{n-1}(s)}\left[\sum_{k=0}^{n-2}\mu_{n-2,k}(s)k_{n-1}(s)\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right].$$

By the derivative of (4.29) with respect to s, we obtain

$$\begin{aligned}f(s) &= \left\{a_0 - \int\left(\frac{f''(s)}{k_{n-1}(s)} - \frac{f'(s)}{k_{n-1}^2(s)}k_{n-1}'(s) - f(s)k_{n-1}(s)\right)\sinh\theta(s)ds\right\}\cosh\theta(s) \\ &\quad - \left\{b_0 - \int\left(\frac{f''(s)}{k_{n-1}(s)} - \frac{f'(s)}{k_{n-1}^2(s)}k_{n-1}'(s) - f(s)k_{n-1}(s)\right)\cosh\theta(s)ds\right\}\sinh\theta(s).\end{aligned}$$

As a result of this and (4.29), we obtain $f'(s)=\left(\sum_{k=0}^{n-3}\mu_{n-3,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{1}{k_1(s)}\right)\right)k_{n-1}(s)$. Thus Lemma 4.1. implies that $\alpha(s)$ is congruent to a normal curve.

For the curves with spacelike principal binormal B_{n-2}, we obtain the following theorem, which can be proved in a similar way as Theorem 4.5.

Theorem 4.6. Let $\alpha(s)$ be an arclength parametrized curve in E_1^n with nonzero curvatures and spacelike principal binormal B_{n-2}. Then $\alpha(s)$ is congruent to a normal curve if and only if there exist constants $a_0, b_0 \in \mathbb{R}$ such that

$$\begin{aligned}\frac{f'(s)}{k_{n-1}(s)} &= \left\{\int\left(\frac{f''(s)}{k_{n-1}(s)}-\frac{f'(s)}{k_{n-1}^2(s)}k'_{n-1}(s)-f(s)k_{n-1}(s)\right)\cos\theta(s)ds-a_0\right\}\cos\theta(s)\\ &-\left\{\int\left(\frac{f''(s)}{k_{n-1}(s)}-\frac{f'(s)}{k_{n-1}^2(s)}k'_{n-1}(s)-f(s)k_{n-1}(s)\right)\sin\theta(s)ds-b_0\right\}\sin\theta(s).\end{aligned} \tag{4.33}$$

5. Conclusion

This study gives normal curves and examines some characterizations of normal curves in Lorentzian n-space E_1^n. We determine necessary and sufficient condition for a null and non-null curve to be congruent to a normal curve in Lorentzian n-space E_1^n. We characterize normal curves in terms of their curvature functions. The results of this study may also be developed to other different spaces.

Conflict of interest

The author declares no conflicts of interest in this paper.

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