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Research article

On Ricci curvature of submanifolds in statistical manifolds of constant (quasi-constant) curvature

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Abstract: In 1999, B. Y. Chen established a sharp inequality between the Ricci curvature and the squared mean curvature for an arbitrary Riemannian submanifold of a real space form. This inequality was extended in 2015 by M. E. Aydin et al. to the case of statistical submanifolds in a statistical manifold of constant curvature, obtaining a lower bound for the Ricci curvature of the dual connections. Also, the similar inequality for submanifolds in statistical manifolds of quasi-constant curvature studied by H. Aytimur and C. Ozgur in their recent article. In the present paper, we give a different proof of the same inequality but working with the statistical curvature tensor field, instead of the curvature tensor fields with respect to the dual connections. A geometric inequality can be treated as an optimization problem. The new proof is based on a simple technique, known as Oprea's optimization method on submanifolds, namely analyzing a suitable constrained extremum problem. We also provide some examples. This paper finishes with some conclusions and remarks.

Keywords: statistical manifolds; quasi-constant curvature; Ricci curvature; Chen-Ricci inequality; statistical immersion
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1. Introduction

The curvature invariants play the most fundamental role in Riemannian geometry. They provide the intrinsic characteristics of Riemannian manifolds, which affect the behavior in general of the Riemannian manifold. They are the main Riemannian invariants and the most natural ones. They are widely used in the field of differential geometry and in physics also. The innovative work of Kaluza-Klein in general relativity and string theory in particle physics has inspired the mathematicians and physicists to do work on submanifolds of (pseudo-)Riemannian manifolds. Intrinsic and extrinsic invariants are very powerful tools to study submanifolds of Riemannian manifolds. The Ricci curvature is the essential term in the Einstein field equations, which plays a key role in general relativity. It is immensely studied in differential geometry as it gives a way of measuring the degree to which the geometry determined by a given Riemannian metric might differ from that of ordinary Euclidean q-space. A Riemannian manifold is said to be an Einstein manifold if the Ricci tensor satisfies the vacuum Einstein equation. The lower bounds on the Ricci tensor on a Riemannian manifold enable one to find global geometric and topological information by comparison with the geometry of a constant curvature space form.

In the study of Riemannian submanifolds, it is a fundamental problem for the geometers to establish some relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold. B. Y. Chen, in his initial papers, obtained some useful inequalities between the scalar curvature, the sectional curvature and the squared norm of the mean curvature of a submanifold in a real space form. He also talked about the inequalities between k-Ricci curvature, the squared mean curvature and the shape operator of a submanifold with arbitrary codimension of the same ambient space [1]. Since then different geometers found the similar relationships for different submanifolds and ambient spaces (for example [2–4]).

Differential geometry is a traditional yet currently very active branch of pure mathematics with applications notably in a number of areas of physics. Until recently applications in the theory of statistics were fairly limited, but within the last few years there has been intensive interest in the subject. The notion of a statistical manifold has arisen from the study of statistical distribution. A differential geometric approach for a statistical model of discrete probability distribution was introduced in 1985 by Amari [5]. Statistical manifolds have many applications in affine differential geometry, Hessian geometry and information geometry. In 1989, Vos [6] introduced and studied the notion of statistical submanifolds. Later, Furuhata [7] studied statistical hypersurfaces in the space of Hessian curvature zero and provided some examples as well. Though, till the date it has made very little progress due to the hardness to find classical differential geometric approaches for study of statistical submanifolds. Geometry of statistical submanifolds is still young and efforts are on, so it is growing.

Generally, one cannot define a sectional curvature with respect to the dual connections (which are not metric) by the standard definitions. However, B. Opozda [8, 9] defined a sectional curvature on a statistical manifold. Suppose that $(\hat{B}, \hat{\nabla}, \hat{g})$ is a q-dimensional statistical manifold and X_1 is a unit vector such that $\|X_1\| = 1$. We choose an orthonormal frame $\{e_1, \ldots, e_q\}$ of $T\hat{B}$ such that $e_1 = X_1$. Then the Ricci curvature at X_1 is given by

$$\begin{aligned} \hat{Ric}^{\hat{\nabla},\hat{\nabla}^*}(X_1) &= \sum_{i=2}^{q} \hat{\mathcal{K}}^{\hat{\nabla},\hat{\nabla}^*}(X_1 \wedge e_i) \\ &= \frac{1}{2}\Big\{ \sum_{i=2}^{q} \hat{\mathcal{K}}(X_1 \wedge e_i) + \sum_{i=2}^{q} \hat{\mathcal{K}}^*(X_1 \wedge e_i) \Big\}, \end{aligned}$$

where $\hat{\mathcal{K}}^{\hat{\nabla},\hat{\nabla}^*}(e_i \wedge e_j)$ denotes the sectional curvature, with respect to $\hat{\nabla}$ and $\hat{\nabla}^*$, of the 2-plane section spanned by e_i and e_j.

We denote the Ricci tensors of the induced connections ∇ and ∇^0 respectively by Ric and Ric^0. $\hat{\mathcal{K}}^0(X_1 \wedge \cdot)$ is the sectional curvature function of a statistical manifold with respect to the Levi-Civita connection restricted to 2-plane sections of the tangent space which are tangent to X_1. In [10], M.

E. Aydin et al. proved the following Chen-Ricci inequality for a p-dimensional submanifold B in a statistical manifold $\hat{B}$ of constant curvature $\hat{c}$.

$$\begin{aligned} Ric(X_1) \geq\ & 2Ric^0(X_1) - \frac{p^2}{8}g(H,H) - \frac{p^2}{8}g(H^*,H^*) + \hat{c}(p-1) \\ & -2(p-1)\max \hat{\mathcal{K}}^0(X_1 \wedge \cdot). \end{aligned} \tag{1.1}$$

Recently, in [11], H. Aytimur et al. obtained the same inequality for a p-dimensional statistical submanifold B in a statistical manifold $\hat{B}$ of quasi-constant curvature.

$$\begin{aligned} Ric(X_1) \geq\ & 2Ric^0(X_1) - \frac{p^2}{8}g(H,H) - \frac{p^2}{8}g(H^*,H^*) \\ & +\hat{a}(p-1) + \hat{b} + \hat{b}(p-2)F(X_1)F(X_1) \\ & -2\sum_{i=2}^{p} \hat{\mathcal{K}}^0(X_1 \wedge e_i). \end{aligned} \tag{1.2}$$

Remark that if $\hat{b} = 0$, then $\hat{B}$ becomes a statistical manifold of constant curvature and inequality (1.2) turns into (1.1).

Optimization on manifolds is about exploiting tools of differential geometry to build optimization schemes on abstract manifolds, then turning these abstract geometric algorithms into practical numerical methods for specific manifolds, with applications to problems that can be rephrased as optimizing a differentiable function over a manifold. This research program has shed new light on existing algorithms and produced novel methods backed by a strong convergence analysis. Here, we point out that optimization of real-valued functions on manifolds is not the only place where optimization and differential geometry meet and also is the Riemannian geometry of the central path in linear programming. As applications to the area of optimization on manifolds, T. Oprea [12] derived Chen-Ricci inequality by using optimization technique applied in the setup of Riemannian geometry. The purpose of this paper is to adopt this technique to give another demonstration for the inequalities (1.1) and (1.2) including the Ricci curvature.

2. Statistical manifolds and submanifolds

Definition 2.1. [5, 7] A Riemannian manifold $(\hat{B}, \hat{g})$ with an affine connection $\hat{\nabla}$ is said to be a statistical manifold $(\hat{B}, \hat{g}, \hat{\nabla})$ if $\hat{\nabla}$ is a torsion free connection on $\hat{B}$ and the covariant derivative $\hat{\nabla}\hat{g}$ is symmetric.

A statistical manifold is a Riemannian manifold $(\hat{B}, \hat{g})$ endowed with a pair of torsion-free affine connections $\hat{\nabla}$ and $\hat{\nabla}^*$ satisfying [5, 7]

$$X_1 g(Y_1, X_2) = \hat{g}(\hat{\nabla}_{X_1} Y_1, X_2) + \hat{g}(Y_1, \hat{\nabla}^*_{X_1} X_2),$$

for any $X_1, Y_1, X_2 \in \Gamma(T\hat{B})$. Here the connection $\hat{\nabla}^*$ is called the conjugate (or dual) connection. This concept was widely studied in information geometry. Also, $(\hat{\nabla}^*)^* = \hat{\nabla}$. If $(\hat{\nabla}, \hat{g})$ is a statistical structure on $\hat{B}$, then $(\hat{\nabla}^*, \hat{g})$ is also a statistical structure. Moreover, a dual connection of any torsion free affine connection $\hat{\nabla}$ is given by [5, 7]

$$2\hat{\nabla}^0 = \hat{\nabla} + \hat{\nabla}^*, \tag{2.1}$$

where $\hat{\nabla}^0$ is the Levi-Civita connection on $\hat{B}$.

Let $(\hat{B}, \hat{\nabla}, \hat{g})$ be a statistical manifold and $f : B \to \hat{B}$ an immersion. define g and ∇ on B by [7]

$$g = f^*\hat{g}, \quad \text{and} \quad g(\nabla_{X_1} Y_1, X_2) = \hat{g}(\hat{\nabla}_{X_1} f_* Y_1, f_* X_2), \tag{2.2}$$

for any $X_1, Y_1, X_2 \in \Gamma(TB)$, where the connection induced from $\hat{\nabla}$ by f on the induced bundle $f^* : T\hat{B} \to B$ is denoted by $\hat{\nabla}$. Then the pair (∇, g) is called an induced statistical structure on B by f from $(\hat{\nabla}, \hat{g})$.

Definition 2.2. [7] Let $(\hat{B}, \hat{\nabla}, \hat{g})$ and (B, ∇, g) be two statistical manifolds. An immersion $f : B \to \hat{B}$ is called a statistical immersion if (∇, g) coincides with the induced statistical structure, that is, (2.2) holds. Thus, (B, ∇, g) is called a statistical submanifold of $(\hat{B}, \hat{\nabla}, \hat{g})$.

Let $(\hat{B}, \hat{\nabla}, \hat{g})$ be a statistical manifold and B be a statistical submanifold of $\hat{B}$. By $T_x^{\perp}B$, we denote the normal space of B, that is, $T_x^{\perp}B = \{v \in T_x\hat{B} \big| g(u, v) = 0, u \in T_xB\}$. Then the Gauss and Weingarten formulae are as follows [6]:

$$\hat{\nabla}_{X_1} Y_1 = \nabla_{X_1} Y_1 + h(X_1, Y_1), \quad \hat{\nabla}^*_{X_1} Y_1 = \nabla^*_{X_1} Y_1 + h^*(X_1, Y_1),$$

and

$$\hat{\nabla}_{X_1} V = -A_V(X_1) + \nabla^{\perp}_{X_1} V, \quad \hat{\nabla}^*_{X_1} V = -A^*_V(X_1) + \nabla^{\perp *}_{X_1} V,$$

for any $X_1, Y_1 \in \Gamma(TB)$ and $V \in \Gamma(T^{\perp}B)$. Here $\hat{\nabla}$ and $\hat{\nabla}^*$ (respectively, ∇ and ∇^*) are the dual connections on $\hat{B}$ (respectively, on B), h and h^* are symmetric and bilinear, called the imbedding curvature tensor of B in $\hat{B}$ for $\hat{\nabla}$ and the imbedding curvature tensor of B in $\hat{B}$ for $\hat{\nabla}^*$, respectively. Since h and h^* are bilinear, the linear transformations A_V and A^*_V are related to the imbedding curvature tensors by [6]

$$\hat{g}(h(X_1, Y_1), V) = g(A^*_V(X_1), Y_1), \quad \text{and} \quad \hat{g}(h^*(X_1, Y_1), V) = g(A_V(X_1), Y_1),$$

for any $X_1, Y_1 \in \Gamma(TB)$ and $V \in \Gamma(T^{\perp}B)$.

Suppose that $dim(B) = p$ and $dim(\hat{B}) = q$. We consider a local orthonormal tangent frame $\{e_1, \ldots, e_p\}$ of TB and a local orthonormal normal frame $\{e_{p+1}, \ldots, e_q\}$ of $T^{\perp}B$ in $\hat{B}$. Then the mean curvature vectors H and H^* of B in $\hat{B}$ are

$$H = \frac{1}{p}\sum_{i=1}^{p} h(e_i, e_i), \quad \text{and} \quad H^* = \frac{1}{p}\sum_{i=1}^{p} h^*(e_i, e_i).$$

Also, we set

$$h^r_{ij} = g(h(e_i, e_j), e_r), \quad \text{and} \quad h^{*r}_{ij} = g(h^*(e_i, e_j), e_r),$$

for $i, j \in \{1, \ldots, p\}$, $r \in \{p + 1, \ldots, q\}$.

Let $\hat{R}$ and R be the curvature tensor fields with respect to $\hat{\nabla}$ and ∇, respectively. Similarly, $\hat{R}^*$ and R^* are respectively the curvature tensor fields with respect to $\hat{\nabla}^*$ and ∇^*. Then the Gauss equation with respect to $\hat{\nabla}$ and the dual connection $\hat{\nabla}^*$ on $\hat{B}$ are respectively defined by [6]

$$\begin{aligned}\hat{g}(\hat{R}(X_1,Y_1)X_2,Y_2) &= g(R(X_1,Y_1)X_2,Y_2)+g(h(X_1,X_2),h^*(Y_1,Y_2))\\ &\quad -g(h^*(X_1,Y_2),h(Y_1,X_2)),\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}\hat{g}(\hat{R}^*(X_1,Y_1)X_2,Y_2) &= g(R^*(X_1,Y_1)X_2,Y_2)+g(h^*(X_1,X_2),h(Y_1,Y_2))\\ &\quad -g(h(X_1,Y_2),h^*(Y_1,X_2)),\end{aligned}\tag{2.4}$$

for any $X_1,Y_1,X_2,Y_2\in\Gamma(TB)$.

The statistical curvature tensor fields of $\hat{B}$ and B are respectively given by

$$2\hat{\mathcal{S}}=\hat{R}+\hat{R}^*,\quad \text{and}\quad 2\mathcal{S}=R+R^*.\tag{2.5}$$

A statistical manifold $(\hat{B},\hat{\nabla},\hat{g})$ is said to be of constant curvature $\hat{c}\in\mathbb{R}$ if the following curvature equation holds [7]

$$\hat{\mathcal{S}}(X_1,Y_1)X_2=\hat{c}(\hat{g}(Y_1,X_2)X_1-\hat{g}(X_1,X_2)Y_1),\tag{2.6}$$

for any $X_1,Y_1,X_2\in\Gamma(T\hat{B})$. It is denoted by $\hat{B}(\hat{c})$, called a statistical manifold of constant curvature.

A statistical structure $(\hat{B},\hat{\nabla},\hat{g})$ is said to be of quasi-constant curvature if the following curvature equation holds [11]

$$\begin{aligned}\hat{\mathcal{S}}(X_1,Y_1)X_2 &= \hat{a}[\hat{g}(Y_1,X_2)X_1-\hat{g}(X_1,X_2)Y_1]\\ &\quad +\hat{b}[F(Y_1)F(X_2)X_1-\hat{g}(X_1,X_2)F(Y_1)\mathcal{P}\\ &\quad +\hat{g}(Y_1,X_2)F(X_1)\mathcal{P}-F(X_1)F(X_2)Y_1],\end{aligned}\tag{2.7}$$

where $\hat{a},\hat{b}$ are scalar functions, $\mathcal{P}$ is a unit vector filed, and F is a 1-form defined by

$$\hat{g}(X_1,\mathcal{P})=F(X_1),$$

for any $X_1,Y_1,X_2\in\Gamma(T\hat{B})$. It is called a statistical manifold of quasi-constant curvature.

3. Main inequalities

In this section, we prove the statistical version of well known Chen-Ricci inequality for statistical submanifolds in statistical manifolds of constant (quasi-constant) curvature by optimization technique.

Optimizations on submanifolds: Let (B,g) be a Riemannian submanifold of a Riemannian manifold $(\hat{B},\hat{g})$ and $\phi:\hat{B}\to\mathbb{R}$ be a differentiable function. Following [13], we have

Theorem 3.1. *If $x\in B$ is a solution of the constrained extremum problem $\min\limits_{x_0\in B}\phi(x_0)$, then* *(a)* $(grad\ \phi)(x) \in T_x^{\perp}B$,

(b) the bilinear form $\pi : T_xB \times T_xB \to \mathbb{R}$,

$$\pi(X_1, Y_1) = Hess_{\phi}(X_1, Y_1) + \hat{g}(h^{'}(X_1, Y_1), (grad\ \phi)(x))$$

is positive semi-definite, where $h^{'}$ is the second fundamental form of B in $\hat{B}$, grad ϕ denotes the gradient of ϕ.

Theorem 3.2. *Let (B, ∇, g) be a p-dimensional submanifold in a statistical manifold $\hat{B}(\hat{c})$ of constant curvature $\hat{c}$.*

(a) For each unit vector $X_1 \in T_{\wp}B$, $\wp \in B$, we have

$$Ric^{\nabla,\nabla^*}(X_1) \geq 2Ric^0(X_1) - \hat{c}(p-1) - \frac{p^2}{8}\Big[||H||^2 + ||H^*||^2\Big]. \tag{3.1}$$

(b) Moreover, the equality holds in the inequality (3.1) if and only if

$$h(X_1, X_1) = \frac{p}{2}H(\wp), \quad h^*(X_1, X_1) = \frac{p}{2}H^*(\wp),$$

and

$$h(X_1, Y_1) = 0, \quad h^*(X_1, Y_1) = 0,$$

for all $Y_1 \in T_{\wp}B$ orthogonal to X_1.

Proof. We choose $\{e_1, \ldots, e_p\}$ as the orthonormal frame of $T_{\wp}B$ such that $e_1 = X_1$ and $||X_1|| = 1$, and $\{e_{p+1}, \ldots, e_q\}$ as the the orthonormal frame of $T_{\wp}B$ in $\hat{B}$. Then by (2.3), (2.4) and (2.5), we have

$$\begin{aligned}
2\hat{\mathcal{S}}(e_1, e_i, e_1, e_i) &= 2\mathcal{S}(e_1, e_i, e_1, e_i) - g(h(e_1, e_1), h^*(e_i, e_i)) \\
&\quad -g(h^*(e_1, e_1), h(e_i, e_i)) + 2g(h(e_1, e_i), h^*(e_1, e_i)) \\
&= 2\mathcal{S}(e_1, e_i, e_1, e_i) - \{4g(h^0(e_1, e_1), h^0(e_i, e_i)) \\
&\quad -g(h(e_1, e_1), h(e_i, e_i)) - g(h^*(e_1, e_1), h^*(e_i, e_i)) \\
&\quad -4g(h^0(e_1, e_i), h^0(e_1, e_i)) + g(h(e_1, e_i), h(e_1, e_i)) \\
&\quad +g(h^*(e_1, e_i), h^*(e_1, e_i))\} \\
&= 2\mathcal{S}(e_1, e_i, e_1, e_i) - 4\sum_{r=p+1}^{q}(h_{11}^{0r}h_{ii}^{0r} - (h_{1i}^{0r})^2) \\
&\quad + \sum_{r=p+1}^{q}(h_{11}^{r}h_{ii}^{r} - (h_{1i}^{r})^2) + \sum_{r=p+1}^{q}(h_{11}^{*r}h_{ii}^{*r} - (h_{1i}^{*r})^2),
\end{aligned}$$

where we have used the notations $\hat{\mathcal{S}}(X_1, Y_1, X_2, Y_2) = g(\hat{\mathcal{S}}(X_1, Y_1)Y_2, X_2)$ and $2h^0 = h + h^*$ (see (2.1)).

Summing over $2 \leq i \leq p$ and using (2.6), we have

$$2\hat{c}(p-1) = 2Ric^{\nabla,\nabla^*}(X_1) - 4\sum_{r=p+1}^{q}\sum_{i=2}^{p}(h_{11}^{0r}h_{ii}^{0r} - (h_{1i}^{0r})^2)$$
$$+\sum_{r=p+1}^{q}\sum_{i=2}^{p}(h_{11}^{r}h_{ii}^{r} - (h_{1i}^{r})^2) + \sum_{r=p+1}^{q}\sum_{i=2}^{p}(h_{11}^{*r}h_{ii}^{*r} - (h_{1i}^{*r})^2),$$

where $Ric^{\nabla,\nabla^*}(X_1)$ denotes the Ricci curvature of B with respect to ∇ and ∇^* at $\wp$. Further, we derive

$$\begin{aligned}2Ric^{\nabla,\nabla^*}(X_1) - 2\hat{c}(p-1) &= 4\sum_{r=p+1}^{q}\sum_{i=2}^{p}(h_{11}^{0r}h_{ii}^{0r} - (h_{1i}^{0r})^2)\\ &\quad -\sum_{r=p+1}^{q}\sum_{i=2}^{p}(h_{11}^{r}h_{ii}^{r} - (h_{1i}^{r})^2)\\ &\quad -\sum_{r=p+1}^{q}\sum_{i=2}^{p}(h_{11}^{*r}h_{ii}^{*r} - (h_{1i}^{*r})^2).\end{aligned} \tag{3.2}$$

By Gauss equation with respect to Levi-Civita connection, it follows that

$$Ric^0(X_1) - \hat{c}(p-1) = \sum_{r=p+1}^{q}\sum_{i=2}^{p}(h_{11}^{0r}h_{ii}^{0r} - (h_{1i}^{0r})^2).$$

Substituting into (3.2), we arrive at

$$\begin{aligned}2Ric^{\nabla,\nabla^*}(X_1) - 2\hat{c}(p-1) &= 4[Ric^0(X_1) - \hat{c}(p-1)]\\ &\quad -\sum_{r=p+1}^{q}\sum_{i=2}^{p}(h_{11}^{r}h_{ii}^{r} - (h_{1i}^{r})^2)\\ &\quad -\sum_{r=p+1}^{q}\sum_{i=2}^{p}(h_{11}^{*r}h_{ii}^{*r} - (h_{1i}^{*r})^2).\end{aligned}$$

On simplifying the previous relation, we get

$$\begin{aligned}&-2Ric^{\nabla,\nabla^*}(X_1) - 2\hat{c}(p-1) + 4Ric^0(X_1)\\ =\; &\sum_{r=p+1}^{q}\sum_{i=2}^{p}(h_{11}^{r}h_{ii}^{r} - (h_{1i}^{r})^2)\\ &+\sum_{r=p+1}^{q}\sum_{i=2}^{p}(h_{11}^{*r}h_{ii}^{*r} - (h_{1i}^{*r})^2)\\ \leq\; &\sum_{r=p+1}^{q}\sum_{i=2}^{p}h_{11}^{r}h_{ii}^{r} + \sum_{r=p+1}^{q}\sum_{i=2}^{p}h_{11}^{*r}h_{ii}^{*r}.\end{aligned} \tag{3.3}$$

Let us define the quadratic form $\phi_r, \phi_r^* : \mathbb{R}^p \to \mathbb{R}$ by

$$\phi_r(h^r_{11}, h^r_{22}, \dots, h^r_{pp}) = \sum_{r=p+1}^{q} \sum_{i=2}^{p} h^r_{11} h^r_{ii},$$

and

$$\phi^*_r(h^{*r}_{11}, h^{*r}_{22}, \dots, h^{*r}_{pp}) = \sum_{r=p+1}^{q} \sum_{i=2}^{p} h^{*r}_{11} h^{*r}_{ii}.$$

We consider the constrained extremum problem max ϕ_r subject to

$$Q : \sum_{i=1}^{p} h^r_{ii} = \alpha^r,$$

where α^r is a real constant. The gradient vector field of the function ϕ_r is given by

$$grad\ \phi_r = (\sum_{i=2}^{p} h^r_{ii}, h^r_{11}, h^r_{11}, \dots, h^r_{11}).$$

For an optimal solution $a = (h^r_{11}, h^r_{22}, \dots h^r_{pp})$ of the problem in question, the vector $grad\ \phi_r$ is normal to Q at the point a. It follows that

$$h^r_{11} = \sum_{i=2}^{p} h^r_{ii} = \frac{\alpha^r}{2}.$$

Now, we fix $x \in Q$. The bilinear form $\pi : T_xQ \times T_xQ \to \mathbb{R}$ has the following expression:

$$\pi(X_1, Y_1) = Hess_{\phi_r}(X_1, Y_1) + < h^{'}(X_1, Y_1), (grad\ \phi_r)(x) >,$$

where $h^{'}$ denotes the second fundamental form of Q in $\mathbb{R}^p$ and $< \cdot, \cdot >$ denotes the standard inner product on $\mathbb{R}^p$. The Hessian matrix of ϕ_r is given by

$$Hess_{\phi_r} = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

We consider a vector $X_1 \in T_xQ$, which satisfies a relation

$$\sum_{i=2}^{p} X^i_1 = -X^1_1.$$

As $h^{'} = 0$ in $\mathbb{R}^p$, we get

$$\begin{aligned}\pi(X_1,X_1)=Hess_{\phi_r}(X_1,X_1) &= 2\sum_{i=2}^{p}X_1^1X_1^i\\ &= (X_1^1+\sum_{i=2}^{p}X_1^i)^2-(X_1^1)^2-(\sum_{i=2}^{p}X_1^i)^2\\ &= -2(X_1^1)^2\leq 0.\end{aligned}$$

However, the point p is the only optimal solution, that is, the global maximum point of problem. Thus, we obtain

$$\phi_r \leq \frac{1}{4}(\sum_{i=1}^{p}h_{ii}^r)^2=\frac{p^2}{4}(H^r)^2. \tag{3.4}$$

Next, we deal with the constrained extremum problem max ϕ_r^* subject to

$$Q^*:\sum_{i=1}^{p}h_{ii}^{*r}=\alpha^{*r},$$

where α^{*r} is a real constant. By similar arguments as above, we find

$$\phi_r^* \leq \frac{1}{4}(\sum_{i=1}^{p}h_{ii}^{*r})^2=\frac{p^2}{4}(H^{*r})^2. \tag{3.5}$$

On combining (3.3), (3.4) and (3.5), we get our desired inequality (3.1). Moreover, the vector field X_1 satisfies the equality case if and only if

$$h_{1i}^r=0,\quad h_{1i}^{*r}=0,\quad i\in\{2,\dots,p\},$$

and

$$h_{11}^r=\sum_{i=2}^{p}h_{ii}^r,\quad h_{11}^{*r}=\sum_{i=2}^{p}h_{ii}^{*r},\quad r\in\{p+1,\dots,q\},$$

which can be rewritten as

$$h_{11}^r=\frac{p}{2}H,$$

and

$$h_{11}^{*r}=\frac{p}{2}H^*.$$

Thus, it proves our assertion. □

Corollary 1. *Let (B,∇,g) be a p-dimensional submanifold in a statistical manifold $\hat{B}(\hat{c})$ of constant curvature $\hat{c}$. For each unit vector $X_1\in T_{\wp}B$, $\wp\in B$, we have*

$$Ric^{\nabla,\nabla^*}(X_1)\geq 2Ric^0(X_1)-\hat{c}(p-1)-\frac{p^2}{2}\|H^0\|^2+\frac{p^2}{4}g(H,H^*).$$

Corollary 2. *Let (B,∇,g) be a p-dimensional submanifold in a statistical manifold $\hat{B}(\hat{c})$ of constant curvature $\hat{c}$. If B is minimal with respect to Levi-Civita connection, then for each unit vector $X_1 \in T_\wp B$, $\wp \in B$, we have*

$$Ric^{\nabla,\nabla^*}(X_1) \geq 2Ric^0(X_1) - \hat{c}(p-1) + \frac{p^2}{4}g(H,H^*).$$

By similar arguments as in Theorem 3.2, one can obtain the following inequality for any submanifold in a statistical manifold of quasi-constant curvature.

Theorem 3.3. *Let (B,∇,g) be a p-dimensional submanifold in a statistical manifold $\hat{B}$ of quasi-constant curvature.*

(a) For each unit vector $X_1 \in T_\wp B$, $\wp \in B$, we have

$$\begin{aligned} Ric^{\nabla,\nabla^*}(X_1) \quad \geq \quad & 2Ric^0(X_1) - [\hat{a}(p-1) + \hat{b} + \hat{b}(p-2)F(X_1)F(X_1)] \\ & -\frac{p^2}{8}\Big[\|H\|^2 + \|H^*\|^2\Big]. \end{aligned} \tag{3.6}$$

(b) Moreover, the equality holds in the inequality (3.6) if and only if

$$h(X_1,X_1) = \frac{p}{2}H(\wp), \quad h^*(X_1,X_1) = \frac{p}{2}H^*(\wp),$$

and

$$h(X_1,Y_1) = 0, \quad h^*(X_1,Y_1) = 0,$$

for all $Y_1 \in T_\wp B$ orthogonal to X_1.

Corollary 3. *Let (B,∇,g) be a p-dimensional submanifold in a statistical manifold $\hat{B}$ of quasi-constant curvature. For each unit vector $X_1 \in T_\wp B$, $\wp \in B$, we have*

$$\begin{aligned} Ric^{\nabla,\nabla^*}(X_1) \quad \geq \quad & 2Ric^0(X_1) - [\hat{a}(p-1) + \hat{b} + \hat{b}(p-2)F(X_1)F(X_1)] \\ & -\frac{p^2}{2}\|H^0\|^2 + \frac{p^2}{4}g(H,H^*). \end{aligned}$$

Corollary 4. *Let (B,∇,g) be a p-dimensional submanifold in a statistical manifold $\hat{B}$ of quasi-constant curvature. If B is minimal with respect to Levi-Civita connection, then for each unit vector $X_1 \in T_\wp B$, $\wp \in B$, we have*

$$Ric^{\nabla,\nabla^*}(X_1) \geq 2Ric^0(X_1) - [\hat{a}(p-1) + \hat{b} + \hat{b}(p-2)F(X_1)F(X_1)] + \frac{p^2}{4}g(H,H^*).$$

4. Some examples

The family D of distribution represented by the pdf $\theta(u,\Phi)$ is called an n-dimensional statistical model $D = \{\theta(u,\Phi) | \Phi \in \Theta \subset \mathbb{R}^n\}$. There are many examples of statistical models, such as Poisson distribution, Normal distribution or Gaussian distribution, inverse Gamma distribution, Weibull distribution and Pareto distribution (see [14, 15] for details).

Example 4.1. *Let $(\hat{B}, \hat{g})$ be a family of exponential distributions of mean* 0*:*

$$\hat{B} := \{\theta(u,\Phi)|\theta(u,\Phi) = \Phi e^{-\Phi u}, u \in [0,\infty), \Phi \in (0,\infty)\},$$

a Riemannian metric is given by

$$\hat{g} := \Phi^{-2}(d\Phi)^2,$$

and α-connection on $\hat{B}$ is defined by

$$\hat{\nabla}^{\alpha}_{\frac{\partial}{\partial\Phi}} \frac{\partial}{\partial\Phi} = (\alpha - 1)\Phi^{-1}\frac{\partial}{\partial\Phi}.$$

Then, $(\hat{B}, \hat{\nabla}^{\alpha}, \hat{g})$ is a 1-dimensional statistical manifold.

We remark that one can also construct examples for higher dimension by defining Fisher information metric and α-connection on a family of statistical distribution (cf. [7]*).*

Example 4.2. ([16]) *The set $\mathcal{F}$ of Freund bivariate mixture exponential density functions,*

$$f(x,y) = \begin{cases} \alpha_1\beta_2\, e^{-\beta_2 y-(\alpha_1+\alpha_2-\beta_2)x} & for \quad 0 < x < y, \\ \alpha_2\beta_1\, e^{-\beta_1 x-(\alpha_1+\alpha_2-\beta_1)y} & for \quad 0 < y < x \end{cases}$$

where parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, is a 4-manifold with Fisher information metric

$$[\hat{g}_{ij}] = \begin{bmatrix} \frac{1}{\alpha_1^2+\alpha_1\alpha_2} & 0 & 0 & 0 \\ 0 & \frac{\alpha_2}{\beta_1^2(\alpha_1+\alpha_2)} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_2^2+\alpha_1\alpha_2} & 0 \\ 0 & 0 & 0 & \frac{\alpha_1}{\beta_2^2(\alpha_1+\alpha_2)} \end{bmatrix}.$$

The α-connection ($\alpha \in \mathbb{R}$) $\hat{\nabla}^{\alpha}$ and the curvature tensors $\hat{R}$ with respect to $\hat{\nabla}^{\alpha}$ are respectively given on pages-77 and 78 of [16]. Thus, $(\mathcal{F}, \hat{g}, \hat{\nabla}^{\alpha})$ is a 4-dimensional statistical manifold of constant curvature $\frac{1-\alpha^2}{4}$. Especially, $(\mathcal{F}, \hat{g}, \hat{\nabla}^{\pm 1})$ is flat, respectively. The constant scalar curvature of $\mathcal{F}$ with respect to $\hat{\nabla}^{\alpha}$ is $\frac{3}{2}(1-\alpha^2)$. The α-mean curvatures with respect to $\hat{\nabla}^{\alpha}$ are given by

$$\begin{aligned} H_1^{\alpha} &= \frac{(1-\alpha^2)\alpha_2}{6(\alpha_1+\alpha_2)}, \quad H_2^{\alpha} = \frac{1-\alpha^2}{6}, \\ H_3^{\alpha} &= \frac{(1-\alpha^2)\alpha_1}{6(\alpha_1+\alpha_2)}, \quad H_4^{\alpha} = H_2^{\alpha}. \end{aligned}$$

The Freund submanifold $\mathcal{F}_2$ of dimension 2 in $\mathcal{F}$ is defined by $\mathcal{F}_2 \subset \mathcal{F} : \alpha_1 = \alpha_2, \beta_1 = \beta_2$. The density functions are of form:

$$f(x,y) = \begin{cases} \alpha_1\beta_1\, e^{-\beta_1 y-(2\alpha_1-\beta_1)x} & for \quad 0 < x < y, \\ \alpha_1\beta_1\, e^{-\beta_1 x-(2\alpha_1-\beta_1)y} & for \quad 0 < y < x \end{cases}$$

with $\alpha_1, \beta_1 > 0$. *The Fisher metric tensor on* $\mathcal{F}_2$ *is given by*

$$[g_{ij}] = \begin{bmatrix} \frac{1}{\alpha_1^2} & 0 \\ 0 & \frac{1}{\beta_1^2} \end{bmatrix}.$$

It is easy to see that $(\mathcal{F}_2, g, \nabla^\alpha)$ *is a statistical submanifold of* $\mathcal{F}$. *The sectional, Ricci and scalar curvatures with respect to* ∇^α *of* $\mathcal{F}_2$ *are zero.*

Example 4.3. *Following are the other trivial examples for the submanifolds of* $\mathcal{F}$ [16]*:*

(a) The Freund submanifold $\mathcal{F}_1$ *of dimension 2 in* $\mathcal{F}$ *is defined by* $\mathcal{F}_1 \subset \mathcal{F} : \beta_1 = \alpha_1, \beta_2 = \alpha_2$. *The space* $\mathcal{F}_1$ *is the direct product of the two corresponding Riemannian spaces* $\{\alpha_1\, e^{-\alpha_1 x}, \alpha_1 > 0\}$ *and* $\{-\alpha_2\, e^{-\alpha_2 y}, \alpha_2 > 0\}$. *The Fisher metric tensor on* $\mathcal{F}_1$ *is given by*

$$[g_{ij}] = \begin{bmatrix} \frac{1}{\alpha_1^2} & 0 \\ 0 & \frac{1}{\alpha_2^2} \end{bmatrix}.$$

(b) The Freund submanifold $\mathcal{F}_3$ *of dimension 2 in* $\mathcal{F}$ *is defined by* $\mathcal{F}_3 \subset \mathcal{F} : \beta_1 == \beta_2 = \alpha_1 + \alpha_2$. *The density functions are of form:*

$$f(x,y) = \begin{cases} \alpha_1(\alpha_1+\alpha_2)\, e^{-(\alpha_1+\alpha_1)y} & \textit{for} \quad 0 < x < y, \\ \alpha_2(\alpha_1+\alpha_2)\, e^{-(\alpha_1+\alpha_1)x} & \textit{for} \quad 0 < y < x \end{cases}$$

with $\alpha_1, \alpha_2 > 0$. *The Fisher metric tensor on* $\mathcal{F}_2$ *is given by*

$$[g_{ij}] = \begin{bmatrix} \frac{\alpha_2+2\alpha_1}{\alpha_1(\alpha_1+\alpha_2)^2} & \frac{1}{(\alpha_1+\alpha_2)^2} \\ \frac{1}{(\alpha_1+\alpha_2)^2} & \frac{\alpha_1+2\alpha_2}{\alpha_2(\alpha_1+\alpha_2)^2} \end{bmatrix}.$$

It is easy to see that $(\mathcal{F}_1, g, \nabla^\alpha)$ *and* $(\mathcal{F}_3, g, \nabla^\alpha)$ *are statistical submanifolds of* $\mathcal{F}$. *The sectional, Ricci and scalar curvatures with respect to* ∇^α *of* $\mathcal{F}_1$ *and* $\mathcal{F}_3$ *are zero.*

Now, we provide a non-trivial example for the submanifold of $\mathcal{F}$ [16]:

Example 4.4. *The submanifold* $\mathcal{F}_4 \subset \mathcal{F}$ *with the density functions:*

$$f(x,y) = \begin{cases} \frac{\lambda_1\lambda(\lambda_{12}+\lambda_2)}{\lambda_1+\lambda_2}\, e^{-\lambda_1 x-(\lambda_2+\lambda_{12})y} & \textit{for} \quad 0 < x < y, \\ \\ \frac{\lambda_2\lambda(\lambda_{12}+\lambda_1)}{\lambda_1+\lambda_2}\, e^{-\lambda_2 y-(\lambda_1+\lambda_{12})x} & \textit{for} \quad 0 < y < x \end{cases}$$

where $\lambda_1, \lambda_{12}, \lambda_2 > 0$ *and* $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. *The metric tensor in the coordinate system* $\lambda_1, \lambda_{12}, \lambda_2$ *is*

$$[g_{ij}] = \begin{bmatrix} \frac{\lambda_2\left(\frac{1}{\lambda_1}+\frac{\lambda_1+\lambda_2}{(\lambda_1+\lambda_{12})^2}\right)}{(\lambda_1+\lambda_2)^2} + \frac{1}{\lambda^2} & \frac{\lambda_2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_{12})^2} + \frac{1}{\lambda^2} & \frac{-1}{(\lambda_1+\lambda_2)^2} + \frac{1}{\lambda^2} \\ \frac{\lambda_2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_{12})^2} + \frac{1}{\lambda^2} & \frac{\frac{\lambda_2}{(\lambda_1+\lambda_{12})^2}+\frac{\lambda_1}{(\lambda_1+\lambda_{12})^2}}{\lambda_1+\lambda_2} + \frac{1}{\lambda^2} & \frac{\lambda_1}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_{12})^2} + \frac{1}{\lambda^2} \\ \frac{-1}{(\lambda_1+\lambda_2)^2} + \frac{1}{\lambda^2} & \frac{\lambda_1}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_{12})^2} + \frac{1}{\lambda^2} & \frac{\lambda_1\left(\frac{1}{\lambda_2}+\frac{\lambda_1+\lambda_2}{(\lambda_2+\lambda_{12})^2}\right)}{(\lambda_1+\lambda_2)^2} + \frac{1}{\lambda^2} \end{bmatrix}$$

The α-connections and the curvatures with respect to ∇^{α} *were computed in* [16].

Note that the above example can be studied for $\lambda_1 = \lambda_2$. *In this case, the curvature tensor, Ricci curvature, and scalar curvature with respect to* ∇^{α} *are zero.*

Example 4.5. *Let* $\{e_1, e_2, e_3\}$ *be an orthonormal frame field on a statistical manifold* $(\hat{B} = \{(x, y, z) \in \mathbb{R}^3\}, \hat{\nabla}, \hat{g} = dx^2 + dy^2 + dz^2)$. *Then an affine connection* $\hat{\nabla}$ *on* $\hat{B}$ *is given as* [11]

$$\begin{aligned}
\hat{\nabla}_{e_1} e_1 &= \beta e_1, \quad \hat{\nabla}_{e_2} e_2 = \hat{\nabla}_{e_3} e_3 = \frac{\beta}{2} e_1, \\
\hat{\nabla}_{e_1} e_2 &= \hat{\nabla}_{e_2} e_1 = \frac{\beta}{2} e_2, \\
\hat{\nabla}_{e_1} e_3 &= \hat{\nabla}_{e_3} e_1 = \frac{\beta}{2} e_3, \\
\hat{\nabla}_{e_2} e_3 &= \hat{\nabla}_{e_3} e_2 = 0,
\end{aligned}$$

where β *is some constant. Thus,* $(B, \nabla, \hat{g})$ *is a statistical manifold of constant curvature* $\frac{\beta^2}{4}$. *The scalar curvature of* $\hat{B}$ *is* $\frac{3\beta^2}{2}$.

Example 4.6. *We consider* $(\mathbb{R}, \nabla^{\mathbb{R}}, g_1 = dz^2)$ *a trivial statistical manifold and* $(\mathbb{R}^2(-1), \nabla^{\mathbb{R}^2}, g_2 = dx^2 + dy^2)$ *a 2-dimensional statistical manifold of constant curvature* -1. *Then, the scalar curvature of* $\mathbb{R}^2$ *is* -2. *The dualistic structure on a product of two statistical manifolds* $\hat{B} = \mathbb{R} \times \mathbb{R}^2$ *is as follows:*

$$\begin{aligned}
\hat{\nabla}_{\partial z} \partial z &= \partial z, \quad \hat{\nabla}^*_{\partial z} \partial z = -\partial z, \\
\hat{\nabla}_{\partial z} \partial x &= \hat{\nabla}_{\partial x} \partial z = \hat{\nabla}^*_{\partial z} \partial x = \hat{\nabla}^*_{\partial x} \partial z = 0, \\
\hat{\nabla}_{\partial z} \partial y &= \hat{\nabla}_{\partial y} \partial z = \hat{\nabla}^*_{\partial z} \partial y = \hat{\nabla}^*_{\partial y} \partial z = 0, \\
\hat{\nabla}_{\partial x} \partial x &= \partial y, \quad \hat{\nabla}_{\partial y} \partial y = 0, \quad \hat{\nabla}_{\partial x} \partial y = \hat{\nabla}_{\partial y} \partial x = \partial x, \\
\hat{\nabla}^*_{\partial x} \partial x &= -\partial y, \quad \hat{\nabla}^*_{\partial y} \partial y = 0, \quad \hat{\nabla}^*_{\partial x} \partial y = \hat{\nabla}^*_{\partial y} \partial x = -\partial x.
\end{aligned}$$

Thus, $(\hat{B} = \mathbb{R} \times \mathbb{R}^2(-1), \hat{\nabla}, \hat{g} = g_1 + g_2)$ *is a statistical manifold. By* [11], *we conclude that* $\hat{B} = \mathbb{R} \times \mathbb{R}^2(-1)$ *is a statistical manifold of quasi-constant curvature with constant functions* $\hat{a} = \hat{b} = -1$.

As we know that the Levi-Civita connection of a Riemannian metric has symmetric Ricci tensor, but this property is not viable for any torsion-free affine connection. In fact, this property is so related to the idea of parallel volume element. Now, this is the known fact that any torsion-free affine connection on a simply connected q-manifold has symmetric Ricci tensor if and only if it is locally equiaffine (that is, a nonvanishing q-form is a parallel volume element or the connection preserves a volume q-form). Let $(\hat{B}(\hat{c}), \hat{\nabla}, \hat{g})$ be a q-dimensional statistical manifold with constant curvature $\hat{c}$. Then, for any $X_1, Y_1 \in \Gamma(T\hat{B})$, we have

$$\hat{Ric}^{\hat{\nabla}, \hat{\nabla}^*}(X_1, Y_1) = (q-1)\hat{c}\hat{g}(X_1, Y_1).$$

Moreover, if $\hat{B}$ is an equiaffine, then we can say that it is an Einstein statistical manifold. In particular, $\hat{B}$ is a Ricci-flat if $\hat{c} = 0$.

Example 4.7. *Let* $\mathbb{H}^{q+1} = \{(x_1, \ldots, x_{q+1}) \in \mathbb{R}^{q+1} | x_{q+1} > 0\}$ *be an upper half space of constant curvature* -1 *with metric*

$$\hat{g} = \frac{\sum_{i=1}^{q+1} dx_i^2}{x_{q+1}^2}.$$

We can easily verify that $(\mathbb{H}^{q+1}, \hat{\nabla}, \hat{g})$ *is a statistical manifold of constant curvature* 0 *(see* [10]*). Thus,* $(\mathbb{H}^{q+1}, \hat{\nabla}, \hat{g})$ *is a Ricci-flat manifold.*

5. Conclusions and remarks

Here we note some conclusions and remarks from this work:

(a) In [10], M. E. Aydin et al. found a lower bound for Ricci curvatures Ric and Ric^* respectively with respect to ∇ and ∇^* of a submanifold in a statistical manifold of constant curvature. Recently, Aytimur et al. [11] derived the similar inequality for a submanifold in a statistical manifold of quasi-constant curvature. In the present work, we have used the statistical curvature tensor fields (that is, $\hat{\mathcal{S}}$ with respect to $\hat{\nabla}$ and $\hat{\nabla}^*$, and $\mathcal{S}$ with respect to ∇ and ∇^*) and applied Theorem 3.1 to show that the Ricci curvature with respect to ∇ and ∇^* is bounded below by the squared norm of the mean curvature with respect to $\hat{\nabla}$ and $\hat{\nabla}^*$. The characterisation of equality cases is also discussed here. More nice applications of Theorem 3.1 can be found in [12].

(b) Theorem 3.2 can work for finding the sharp estimates of the squared mean curvature (with respect to Levi-Civita connection) of any submanifold with arbitrary codimension when H and H^* are orthogonal.
For instance, let (B, ∇, g) be a p-dimensional submanifold in a statistical manifold $(\mathbb{H}^{q+1}, \hat{\nabla}, \hat{g})$ of constant curvature 0. For each unit vector $X_1 \in T_\wp B$, $\wp \in B$, we have

$$\|H^0\|^2 \geq \frac{4}{p^2} Ric^0(X_1) - \frac{2}{p^2} Ric^{\nabla,\nabla^*}(X_1).$$

In addition, $(\mathbb{H}^{q+1}, \hat{\nabla}, \hat{g})$ is a Ricci-flat manifold (from relation (4.1)).

(c) From Theorem 3.2, we remark that the relation in (3.1) is the statistical version of well known Chen-Ricci inequality for a Riemannian submanifold of a real space form given by B.-Y. Chen in [1].

(d) We hope that the results stated here will open the door for the researcher and motivate further studies to obtain such inequality, which has the great geometric importance, for different ambient statistical manifolds by using an optimization technique (see [4]). For instance, by following [11] and the similar arguments in the proof of Theorem 3.2, one can easily derive the inequality (3.6) (see Theorem 3.3).

(e) The forthcoming challenge is to improve such geometric inequalities for different ambient statistical manifolds by adopting different techniques. Note that such geometric inequalities can also be proved via algebraic techniques.

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Conflict of interest

The authors declare that they have no conflict of interest.

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