Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(4): 3472–3479.
DOI:10.3934/math.2020225
Received: 28 December 2019
Accepted: 31 March 2020
Published: 08 April 2020

Research article

On an identity involving generalized derivations and Lie ideals of prime rings

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Abstract: Let R be a prime ring, U the Utumi quotient ring of R, C the extended centroid of R and L a noncentral Lie ideal of R. If R admits a generalized derivation F associated with a derivation δ of R such that for some fixed integers $m, n \geq 1$, $F([u, v])^m = [u, v]_n$ for all $u, v \in L$, then one of the following holds true:

(i) R satisfies s_4, the standard identity in four variables.
(ii) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$. Moreover, if $n = 1$, then $\lambda^m = 1$ and if $n > 1$, then $F = 0$.

Keywords: prime ring; generalized derivation; Lie ideal; GPIs
Mathematics Subject Classification: 16W25, 16N60, 16R50

1. Introduction

The standard identity s_4 in four variables is defined as

$$s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma X_{\sigma(1)} X_{\sigma(2)} X_{\sigma(3)} X_{\sigma(4)},$$

where $(-1)^\sigma$ is the sign of permutation σ. In everything that follows, R denotes an associative prime ring with center $Z(R)$ and the extended centroid C, U denotes the Utumi quotient ring of R (for construction and properties of U and C we refer the reader to [3]). For any $x, y \in R$, the symbol $[x, y]$ denotes the commutator $xy - yx$. We set $[x, y]_0 = x$, $[x, y]_1 = xy - yx$ and inductively $[x, y]_n = [[x, y]_{n-1}, y]$ for any integer $n > 1$. Further an *Engel condition* is a polynomial

$$[x, y]_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} y^i x y^{n-i}$$

in noncommutative indeterminates x and y. A nonempty subset L of R which is a subgroup of $(R, +)$ and satisfy the condition $[u, r] \in L$ for all $u \in L$ and $r \in R$; is called a Lie ideal of R. Note that every two-sided ideal is a Lie ideal but the converse is not true. Recall that a ring R is known as prime if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$ and it is called semiprime if $aRa = (0)$ implies $a = 0$. A mapping $\delta : R \to R$ is said to be a derivation of R if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$. The very first example of derivation is the mapping $x \mapsto [a, x]$ for all $x \in R$ and a is a fixed element of R. Such a mapping is called the inner derivation of R. More generally, if δ is a derivation of R and $F : R \to R$ be a mapping such that $F(xy) = F(x)y + x\delta(y)$ for all $x, y \in R$, then F is called a generalized derivation of R with the associated derivation δ. For fixed $a, b \in R$, a typical example of a generalized derivation is the mapping $x \mapsto ax + xb$, which is called the generalized inner derivation induced by a and b, with associated derivation $x \mapsto [x, b]$. Further, in a very systematic paper [13], Lee extended the notion of generalized derivation.

During the last few decades there has been an ongoing interest in the study of relationship between the commutative structure of associative rings and certain types of derivations defined on them. In this vein, Daif and Bell [6] studied derivations of semiprime rings that fix the commutators of appropriate subsets. Precisely, they proved that if R is a semiprime ring, I a nonzero ideal of R and d a derivation of R such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then I is contained in $Z(R)$. Ashraf and Rehman [1] examined the same identity on square-closed Lie ideals of prime rings. In [17], Quadri et al. extended this result to the class of generalized derivations and proved that if R is a prime ring, I a nonzero ideal of R and R admits a generalized derivation F associated with nonzero derivation d such that $F([x, y]) = [x, y]$ for all $x, y \in I$, then R is commutative. In [8], Filippis and Huang proved the following result: *Let R be prime ring, I a nonzero ideal of R and n a fixed positive integer. If R admits a generalized derivation F associated with a nonzero derivation d such that $F([x, y])^n = [x, y]$ for all $x, y \in I$, then either R is commutative or $n = 1, d = 0$ and F is the identity map on R.* Very recently, Huang and Rehman [10] generalized this result and proved the following: *Let R be a prime ring, I a nonzero ideal of R and m, n are the fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $F([x, y])^m = [x, y]_n$ for all $x, y \in I$, then R is commutative*. Motivated by the above cited papers, in this paper, we establish a more general result. More precisely we prove the following theorem:

Theorem 1.1. *Let R be a prime ring, U the Utumi quotient ring of R, C the extended centroid of R and L a noncentral Lie ideal of R. If R admits a generalized derivation F associated with a derivation δ of R such that for some fixed integers $m, n \geq 1$; $F([u, v])^m = [u, v]_n$ for all $u, v \in L$, then one of the following holds true:*

(i) R satisfies s_4, the standard identity in four variables.
(ii) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$. Moreover, if $n = 1$, then $\lambda^m = 1$ and if $n > 1$, then $F = 0$.

In order to prove this result, we need the following remarks:

Remark 1.1 ([13], Theorem 3). Every generalized derivation of R can be uniquely extended to a generalized derivation of U and assumes the form that $F(x) = ax + \delta(x)$ for some $a \in U$ and a derivation δ of U.

Remark 1.2 ([5], Theorem 2). Let I be a two-sided ideal of R. Then I, R and U satisfy the same generalized polynomial identities with coefficients in U.

Remark 1.3. Let L be a noncentral Lie ideal of R. If $\mathrm{char}(R) \neq 2$, then by Lemma 1 of [4], there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. Moreover, if $\mathrm{char}(R) = 2$ and $\dim_C(RC) > 4$ (i.e., R does not satisfy s_4), then by Theorem 13 of [14], there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. Thus we may conclude that there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$ unless $\mathrm{char}(R) = 2$ and R satisfies s_4.

2. Main results

2.1. The inner case

In this section, we assume that F is an inner generalized derivation, i.e., there exist some fixed elements $a, b \in U$ such that $F(x) = ax + xb$ for all $x \in R$. Then by our assumption, we have $(a[u, v] + [u, v]b)^m = [u, v]_n$ for all $u, v \in L$. In light of Remark 1.3, we find that there exists a nonzero ideal I of R that satisfies

$$(a[[x_1, x_2], [y_1, y_2]] + [[x_1, x_2], [y_1, y_2]]b)^m = [[x_1, x_2], [y_1, y_2]]_n$$

Moreover, by Chuang ([5], Theorem 2), we can assume that R satisfies the following generalized polynomial identity:

$$\Lambda(x_1, x_2, y_1, y_2) = (a[[x_1, x_2], [y_1, y_2]] + [[x_1, x_2], [y_1, y_2]]b)^m - [[x_1, x_2], [y_1, y_2]]_n.$$

In the development of our results in this section, we also need the following remark.

Remark 2.1. In view of our assumption, we have

$$(a[[x_1, x_2], [y_1, y_2]] + [[x_1, x_2], [y_1, y_2]]b)^m - [[x_1, x_2], [y_1, y_2]]_n = 0$$

for all $x_1, x_2, y_1, y_2 \in R$. In addition, for any inner automorphism φ of R, we have that

$$(\varphi(a)[[x_1, x_2], [y_1, y_2]] + [[x_1, x_2], [y_1, y_2]]\varphi(b))^m - [[x_1, x_2], [y_1, y_2]]_n = 0$$

for all $x_1, x_2, y_1, y_2 \in R$. Now it is clear that $a, b, a+b, a-b$ are central in R if and only if $\varphi(a), \varphi(b), \varphi(a+b), \varphi(a-b)$ are central in R. Thus we can replace a, b by $\varphi(a), \varphi(b)$ respectively, whenever it is needed.

Lemma 2.1. *Let $R = M_k(C)$, where $k \geq 3$, the ring of all $k \times k$ matrices over the extended centroid C of R. If for some $q \in R$ and fixed integers $m, n \geq 1$; $(q[[x_1, x_2], [y_1, y_2]])^m = [[x_1, x_2], [y_1, y_2]]_n$ for all $x_1, x_2, y_1, y_2 \in R$, then $q \in C$. Moreover, if $n = 1$, then $q^m = 1$ and if $n > 1$, then $q = 0$.*

Proof. Let $q \in R$, i.e., $q = \sum_{r,s=1}^{k} q_{rs}e_{rs}$, where $q_{rs} \in C$ and e_{rs} denote the usual unit matrices with (r, s)−entry 1 and 0 elsewhere. For $i \neq j \neq k \neq i$, let us choose $x_1 = e_{ik} - e_{kj}$, $x_2 = e_{kk}$, $y_1 = e_{ki}$, $y_2 = e_{ij}$ and so $[x_1, x_2] = e_{ik} + e_{kj} \neq 0$ and $[y_1, y_2] = e_{kj} \neq 0$. By our assumption, we notice that

$$(qe_{ij})^m = [e_{ik} + e_{kj}, e_{kj}]_n.$$

Left multiplying by e_{ij}, we find

$$q_{ji}^m e_{ij} = 0 \text{ implies } q_{ji} = 0 \text{ for any } i \neq j.$$

This shows that q is a diagonal matrix. Now, let φ be an automorphism of R induced by $P = (1 + e_{ji})$, i.e., $\varphi(x) = PxP^{-1}$ for all $x \in R$. By Remark 2.1, $\varphi(q)$ is also a diagonal matrix and hence $(j, i)-$entry of $\varphi(q)$ is zero. With this we have

$$\begin{aligned} 0 &= [\varphi(q)]_{ji} \\ &= q_{ji} - q_{jj} + q_{ii} - q_{ij} \\ &= q_{ii} - q_{jj}. \end{aligned}$$

That means $q_{ii} = q_{jj}$. Hence $q \in C$. By our hypothesis

$$q^m([[x_1, x_2], [y_1, y_2]])^m = [[x_1, x_2], [y_1, y_2]]_n \text{ for all } x_1, x_2, y_1, y_2 \in R. \tag{2.1}$$

Clearly it is a polynomial identity on R. Therefore, we choose x_1, x_2, y_1, y_2 in such a way that $[x_1, x_2] = e_{ij} - e_{ji}$ and $[y_1, y_2] = e_{ji}$ for any $i \neq j$. It implies that

$$q^m([e_{ij} - e_{ji}, e_{ji}])^m = [e_{ij} - e_{ji}, e_{ji}]_n. \tag{2.2}$$

If $n = 1$, Eq. (2.2) implies that

$$q^m(e_{ii} - e_{jj})^m = (e_{ii} - e_{jj}).$$

Right multiplying this expression by e_{ii}, we get $q^m e_{ii} = e_{ii}$ implying $q^m = 1$.
In case $n \geq 2$, by (2.2), we get

$$q^m(e_{ii} - e_{jj})^m = [e_{ii} - e_{jj}, e_{ji}]_{n-1} = -2e_{ji}.$$

Left multiplying the above expression by e_{ii}, we get $q^m e_{ii} = 0$ implying $q = 0$. □

Proposition 2.1. *Let R be a noncommutative prime ring, $m, n \geq 1$ some fixed integers, and F a generalized inner derivation of R induced by $a, b \in R$ such that*

$$F([u, v])^m = [u, v]_n \text{ for all } u, v \in [R, R].$$

Then either R satisfies s_4 or $a, b \in C$ and there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$. Moreover if $n = 1$, then $\lambda^m = 1$ otherwise $F = 0$.

Proof. Suppose that R does not satisfy s_4. By our hypothesis, R satisfies the generalized polynomial identity

$$\Lambda(x_1, x_2, y_1, y_2) = (a[[x_1, x_2], [y_1, y_2]] + [[x_1, x_2], [y_1, y_2]]b)^m - [[x_1, x_2], [y_1, y_2]]_n.$$

In light of a theorem of Beidar ([2], Theorem 2), $\Lambda(x_1, x_2, y_1, y_2)$ is also a generalized polynomial identity for U. In case C is infinite, then $\Lambda(x_1, x_2, y_1, y_2) = 0$ for all $x_1, x_2, y_1, y_2 \in U \otimes_C \overline{C}$, where $\overline{C}$ denotes the algebraic closure of C. Since U and $U \otimes_C \overline{C}$ are centrally closed (see [7], Theorem 2.5, Theorem 3.5), we may replace R by U or $U \otimes_C \overline{C}$ according as C is finite or infinite. Therefore, we may assume that R is centrally closed over C, which is either finite or algebraically closed. If both $a, b \in C$, then we have nothing to prove. Thus we assume that at least one of a and b is not in C. Then by Chuang [5], $\Lambda(x_1, x_2, y_1, y_2)$ is a nontrivial generalized polynomial identity for R. Now, with the aid of Martindale's theorem [16], R is a primitive ring having nonzero socle $\mathcal{H}$ with C as associated division ring. In this sequel, a result due to Jacobson ([11], pg. 75) yields that R is isomorphic to a dense ring of linear transformations on some vector space V over C. For some positive integer k, let $\dim_C(V) = k < \infty$, then by density of R on V, $R \cong M_k(C)$. In view of our assumption $\dim_C(V) \neq 1$. Moreover, in case $\dim_C(V) = 2$, i.e., $R \cong M_2(C)$, R satisfies s_4, a contradiction.

Now, let us consider the case when $\dim_C(V) \geq 3$. For any $v \in V$, we first show that the vectors v and bv are linearly $C-$dependent. In this view, we suppose that for some $0 \neq v$, the set $\{v, bv\}$ is linearly $C-$independent and show that a contradiction follows. Since $\dim_C(V) \geq 3$, there exists some $w \in V$ such that the set $\{v, bv, w\}$ is linearly independent over C. By the density of R, there exist $x_1, x_2, y_1, y_2 \in R$ such that

$$x_1 v = 0; \quad x_2 v = 0; \quad y_1 v = 0; \quad y_2 v = 0;$$

$$x_1 bv = v; \quad x_2 bv = 0; \quad y_1 bv = 0; \quad y_2 bv = w;$$

$$x_1 w = v; \quad x_2 w = bv; \quad y_1 w = w; \quad y_2 v = 0.$$

With all this, our hypothesis implies that

$$\begin{aligned} 0 &= ((a[[x_1, x_2], [y_1, y_2]] + [[x_1, x_2], [y_1, y_2]]b)^m - [[x_1, x_2], [y_1, y_2]]_n)v \\ &= (a[[x_1, x_2], [y_1, y_2]] + [[x_1, x_2], [y_1, y_2]]b)^m \\ &= v, \end{aligned}$$

which is a contradiction. It forces that for any $v \in V$, the vectors v and bv are linearly $C-$dependent. Thus there exists some $\tau_v \in C$ such that $bv = \tau_v v$ for all $v \in V$. By a standard argument, one can easily check that τ_v is not depending on the choice of v, i.e., $bv = \tau v$ for all $v \in V$. In this view, we have

$$\begin{aligned} [b, u]v &= (bu)v - u(bv) \\ &= \tau uv - u\tau v \\ &= 0 \end{aligned}$$

for all $v \in V$. This argument shows that for each $u \in V$, $[b, u]$ acts faithfully as a linear transformation on the vector space V, and hence $[b, u] = 0$, i.e., $b \in Z(R)$. Thus we get

$$((a + b)[[x_1, x_2], [y_1, y_2]])^m = [[x_1, x_2], [y_1, y_2]]_n \text{ for all } x_1, x_2, y_1, y_2 \in R.$$

In light of Lemma 2.1, $a + b \in C$ and hence $a \in C$.

In case $\dim_C(V) = \infty$, by Wong ([18], Lemma 2), R satisfies the polynomial identity

$$(a[x, y] + [x, y]b)^m = [x, y]_n$$

Let $v \in V$ such that the set $\{v, bv\}$ is linearly independent over C. By density of R, there exist $x, y \in R$ such that

$$xv = 0; \quad xbv = v; \quad yv = 0; \quad ybv = bv.$$

In this view, we have $0 = ((a[x, y] + [x, y]b)^m - [x, y]_n)v = v \neq 0$, again a contradiction. Therefore, the set $\{v, bv\}$ is linearly $C-$dependent for any $v \in V$. In this case R satisfies a nontrivial generalized polynomial identity $(a[x, y] + [x, y]b)^m = [x, y]_n$, which contradicts the infinite dimensionality of V over C. □

2.2. *The general case*

In this section, we consider F a generalized derivation of R. In order to prove our main result, we assume that there exists $a \in U$ and δ a derivation of R such that $F(x) = ax + \delta(x)$ (see [13], Theorem 3).

Proof of Theorem 1.1: Assume that R does not satisfy s_4. As above, assume that there exists $a \in U$ and a derivation δ of R such that $F(x) = ax + \delta(x)$ for all $x \in R$. By hypothesis, we have

$$(a[u,v] + \delta([u,v]))^m = [u,v]_n \text{ for all } u,v \in L. \tag{2.3}$$

By Remark 1.3, there exists a nonzero ideal I of R such that $0 \neq [I,R] \subseteq L$. By this fact, we re-write (2.3) as

$$(a[u,v] + \delta([u,v]))^m = [u,v]_n \text{ for all } u,v \in [I,R].$$

In view of Remark 1.2, we have

$$(a[[x_1,x_2],[y_1,y_2]] + \delta([[x_1,x_2],[y_1,y_2]]))^m = [[x_1,x_2],[y_1,y_2]]_n \tag{2.4}$$

for all $x_1, x_2, y_1, y_2 \in U$. Now, we apply Kharchenko's theory of differential identities ([12], Theorem 2) and split the proof into the following cases:

<u>The inner case</u>: Let δ be an inner derivation of R induced by some $b \in U$, i.e., $\delta(x) = [b,x]$ for all $x \in R$. Thus $F(x) = (a+b)x + x(-b)$ for all $x \in R$, so that U satisfies

$$((a+b)[[x_1,x_2],[y_1,y_2]] + [[x_1,x_2],[y_1,y_2]](-b))^m = [[x_1,x_2],[y_1,y_2]]_n.$$

The conclusion follows from Proposition 2.1.

<u>The outer case</u>: Let δ be an outer derivation. By Eq. (2.4), we have

$$(a[[x_1,x_2],[y_1,y_2]] + [[\delta(x_1),x_2],[y_1,y_2]] + [[x_1,\delta(x_2)],[y_1,y_2]]+$$
$$[[x_1,x_2],[\delta(y_1),y_2]] + [[x_1,x_2],[y_1,\delta(y_2)]])^m = [[x_1,x_2],[y_1,y_2]]_n.$$

By a result of Kharchenko [12], U satisfies the generalized polynomial identity

$$(a[[x_1,x_2],[y_1,y_2]] + [[u_1,x_2],[y_1,y_2]] + [[x_1,u_2],[y_1,y_2]]+$$
$$[[x_1,x_2],[v_1,y_2]] + [[x_1,x_2],[y_1,v_2]])^m = [[x_1,x_2],[y_1,y_2]]_n.$$

In particular for $x_1 = 0$, U satisfies the blended component

$$([[u_1,x_2],[y_1,y_2]])^m = 0. \tag{2.5}$$

Since (2.5) is a polynomial identity for U, then by Lanski ([15], Lemma 1), $U \cong M_t(F)$, for some suitable field and $t \geq 2$. Moreover, $M_t(F)$ satisfies (2.5); in this case we consider (2.5) with $u_1 = e_{ji}, x_2 = y_1 = e_{ii}$ and $y_2 = e_{ij}$. Thus we have

$$0 = ([[u_1,x_2],[y_1,y_2]])^m = (e_{jj} - e_{ii})^m.$$

Right multiplying by e_{jj}, we get $e_{jj} = 0$, which is not possible. It completes the proof.

By applying the same technique, the following theorem can be easily obtained which extends a known result of Huang and Davvaz ([9], Theorem 2.1).

Theorem 2.1. *Let R be a prime ring, U the Utumi quotient ring of R, C the extended centroid of R and L a noncentral Lie ideal of R. If R admits a generalized derivation F associated with a derivation δ of R such that for some fixed integers $m, n \geq 1$; $F([u, v])^m = [u, v]^n$ for all $u, v \in L$, then one of the following holds true:*

(i) R satisfies s_4, the standard identity in four variables.
(ii) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$. Moreover, if $n = 1$, then $\lambda^m = 1$ and if $n > 1$, then $F = 0$.

Acknowledgment

The author would like to thank the learned referees for their insightful comments and suggestions that immensely improved the presentation of this article.

Conflict of interest

The author declares no conflicts of interest in this paper.

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