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Research article

Completely monotonic integer degrees for a class of special functions

Ling Zhu*

Department of Mathematics, Zhejiang Gongshang University, Hangzhou 310018, P. R. China

* **Correspondence:** Email: zhuling0571@163.com; Tel: +8657188802322; Fax: +8657188802322.

Abstract: Let $f_n(x)$ $(n = 0, 1, \cdots)$ be the remainders for the asymptotic formula of $\ln\Gamma(x)$ and $R_n(x) = (-1)^n f_n(x)$. This paper introduced the concept of completely monotonic integer degree and discussed the ones for the functions $(-1)^m R_n^{(m)}(x)$, then demonstrated the correctness of the existing conjectures by using a elementary simple method. Finally, we propose some operational conjectures which involve the completely monotonic integer degrees for the functions $(-1)^m R_n^{(m)}(x)$ for $m = 0, 1, 2, \cdots$.

Keywords: completely monotonic function; completely monotonic integer degree; gamma function
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1. Introduction

It is well known that the convexity [1–6,8,9,11,15,16,40,55,63,64], monotonicity [7,12–14,41–53] and complete monotonicity [58, 59, 61, 62] have widely applications in many branches of pure and applied mathematics [19, 24, 28, 32, 35, 38, 65]. In particular, many important inequalities [20, 25, 30, 33, 37, 39, 69] can be discovered by use of the convexity, monotonicity and complete monotonicity. The concept of complete monotonicity can be traced back to 1920s [18]. Recently, the complete monotonicity has attracted the attention of many researchers [23, 34, 56, 67] due to it has become an important tool to study geometric function theory [26, 31, 36], its definition can be simply stated as follows.

Definition 1.1. Let $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \mapsto \mathbb{R}$ is said to be completely monotonic on I if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \tag{1.1}$$

for all $x \in I$ and $n = 0, 1, 2, \cdots$.

If $I = (0, \infty)$, then a necessary and sufficient condition for the complete monotonicity can be found in the literature [54]: the real-valued function $f : (0, \infty) \mapsto \mathbb{R}$ is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t) \tag{1.2}$$

is a Laplace transform, where $\alpha(t)$ is non-decreasing and such that the integral of (1.2) converges for $0 < x < \infty$.

In 1997, Alzer [10] studied a class of completely monotonic functions involving the classical Euler gamma function [21, 22, 60, 66, 68] and obtained the following result.

Theorem 1.1. Let $n \geq 0$ be an integer, $\kappa(x)$ and $f_n(x)$ be defined on $(0, \infty)$ by

$$\kappa(x) = \ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x + x - \frac{1}{2} \ln(2\pi) \tag{1.3}$$

and

$$f_n(x) = \begin{cases} \kappa(x) - \sum_{k=1}^{n} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}, & n \geq 1, \\ \kappa(x), & n = 0, \end{cases} \tag{1.4}$$

where B_n denotes the Bernoulli number. Then both the functions $x \mapsto f_{2n}(x)$ and $x \mapsto -f_{2n+1}(x)$ are strictly completely monotonic on $(0, \infty)$.

In 2009, Koumandos and Pedersen [27] first introduced the concept of completely monotonic functions of order r. In 2012, Guo and Qi [17] proposed the concept of completely monotonic degree of nonnegative functions on $(0, \infty)$. Since the completely monotonic degrees of many functions are integers, in this paper we introduce the concept of the completely monotonic integer degree as follows.

Definition 1.2. Let $f(x)$ be a completely monotonic function on $(0, \infty)$ and denote $f(\infty) = \lim_{x\to\infty} f(x)$. If there is a most non-negative integer k ($\leq \infty$) such that the function $x^k[f(x) - f(\infty)]$ is completely monotonic on $(0, \infty)$, then k is called the completely monotonic integer degree of $f(x)$ and denoted as $\deg_{cmi}^x [f(x)] = k$.

Recently, Qi and Liu [29] gave a number of conjectures about the completely monotonic degrees of these fairly broad classes of functions. Based on thirty six figures of the completely monotonic degrees, the following conjectures for the functions $(-1)^m R_n^{(m)}(x) = (-1)^m [(-1)^n f_n(x)]^{(m)} = (-1)^{m+n} f_n^{(m)}(x)$ are shown in [29]:

(i) If $m = 0$, then

$$\deg_{cmi}^x [R_n(x)] = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ 2(n-1), & \text{if } n \geq 2 \end{cases}; \tag{1.5}$$

(ii) If $m = 1$, then

$$\deg_{cmi}^x [-R_n'(x)] = \begin{cases} 1, & \text{if } n = 0 \\ 2, & \text{if } n = 1 \\ 2n-1, & \text{if } n \geq 2 \end{cases}; \tag{1.6}$$

(iii) If $m \geq 1$, then

$$\deg_{cmi}^x \left[(-1)^m R_n^{(m)}(x)\right] = \begin{cases} m-1, & \text{if } n = 0 \\ m, & \text{if } n = 1 \\ m + 2(n-1), & \text{if } n \geq 2 \end{cases}. \tag{1.7}$$

In this paper, we get the complete monotonicity of lower-order derivative and lower-scalar functions $(-1)^m R_n^{(m)}(x)$ and their completely monotonic integer degrees using the Definition 1.2 and a common sense in Laplace transform that the original function has the one-to-one correspondence with the image function, and demonstrated the correctness of the existing conjectures by using a elementary simple method. The negative conclusion to the second clause of (1.7) is given. Finally, we propose some operational conjectures which involve the completely monotonic integer degrees for the functions $(-1)^m R_n^{(m)}(x)$ for $m = 0, 1, 2, \cdots$.

2. Lemmas

In order to prove our main results, we need several lemmas and a corollary which we present in this section.

Lemma 2.1. If the function $x^n f(x)$ $(n \geq 1)$ is completely monotonic on $(0, \infty)$, so is the function $x^{n-1} f(x)$.

Proof. Since the function $1/x$ is completely monotonic on $(0, \infty)$, we have $x^{n-1} f(x) = (1/x)\left[x^n f(x)\right]$ is completely monotonic on $(0, \infty)$ too. □

Corollary 2.1. Let $\alpha(t) \geq 0$ be given in (1.2). Then the functions $x^{i-1} f(x)$ for $i = n, n-1, \cdots, 2, 1$ are completely monotonic on $(0, \infty)$ if the function $x^n f(x)$ $(n \in \mathbb{N})$ is completely monotonic on $(0, \infty)$.

The above Corollary 2.1 is a theoretical cornerstone to find the completely monotonic integer degree of a function $f(x)$. According to this theory and Definition 1.2, we only need to find a nonnegative integer k such that $x^k f(x)$ is completely monotonic on $(0, \infty)$ and $x^{k+1} f(x)$ is not, then $\deg_{cmi}^{x}[f(x)] = k$.

The following lemma comes from Yang [57]:

Lemma 2.2. Let $f_n(x)$ be defined as (1.4). Then $f_n(x)$ can be written as

$$f_n(x) = \frac{1}{4}\int_0^\infty p_n\left(\frac{t}{2}\right)e^{-xt}dt, \tag{2.1}$$

where

$$p_n(t) = \frac{\coth t}{t} - \sum_{k=0}^{n} \frac{2^{2k}B_{2k}}{(2k)!}t^{2k-2}. \tag{2.2}$$

Lemma 2.3. Let $m, r \geq 0$, $n \geq 1$, $f_n(x)$ and $p_n(t)$ be defined as (2.1) and (2.2). Then

$$x^r(-1)^m R_n^{(m)}(x) = x^r(-1)^{m+n} f_n^{(m)}(x) = \frac{1}{4}\int_0^\infty \left[(-1)^n t^m p_n\left(\frac{t}{2}\right)\right]^{(r)} e^{-xt}dt. \tag{2.3}$$

Proof. It follows from (2.1) that

$$\begin{aligned}
x(-1)^m R_n^{(m)}(x) &= x(-1)^{m+n} f_n^{(m)}(x) = x(-1)^{m+n}\frac{1}{4}\int_0^\infty (-t)^m p_n\left(\frac{t}{2}\right)e^{-xt}dt \\
&= x(-1)^n\frac{1}{4}\int_0^\infty t^m p_n\left(\frac{t}{2}\right)e^{-xt}dt = (-1)^{n-1}\frac{1}{4}\int_0^\infty t^m p_n\left(\frac{t}{2}\right)de^{-xt} \\
&= (-1)^{n-1}\frac{1}{4}\left\{\left[t^m p_n\left(\frac{t}{2}\right)e^{-xt}\right]_{t=0}^\infty - \int_0^\infty \left[t^m p_n\left(\frac{t}{2}\right)\right]' e^{-xt}dt\right\}
\end{aligned}$$

$$= \quad (-1)^n \frac{1}{4} \int_0^\infty \left[t^m p_n \left(\frac{t}{2} \right) \right]' e^{-xt} dt.$$

Repeat above process. Then we come to the conclusion that

$$x^r (-1)^m R_n^{(m)}(x) = (-1)^n \frac{1}{4} \int_0^\infty \left[t^m p_n \left(\frac{t}{2} \right) \right]^{(r)} e^{-xt} dt,$$

which completes the proof of Lemma 2.3. □

3. Complete monotonicity of the functions $R_n(x)$ and their completely monotonic integer degrees

In recent paper [70] the reslut $\deg_{cmi}^x [R_1(x)] = \deg_{cmi}^x [-f_1(x)] = 1$ was proved. In this section, we mainly discuss $\deg_{cmi}^x [R_2(x)]$ and $\deg_{cmi}^x [R_3(x)]$. Then discuss whether the most general conclusion exists about $\deg_{cmi}^x [R_n(x)]$.

Theorem 3.1. The function $x^3 R_2(x)$ is not completely monotonic on $(0, \infty)$, and

$$\deg_{cmi}^x [R_2(x)] = \deg_{cmi}^x [f_2(x)] = 2.$$

Proof. Note that the function $x^2 R_2(x)$ is completely monotonic on $(0, \infty)$ due to

$$\begin{aligned} x^2 R_2(x) &= \frac{1}{4} \int_0^\infty p_2^{(2)} \left(\frac{t}{2} \right) e^{-xt} dt, \\ p_2(t) &= \frac{\coth t}{t} + \frac{1}{45} t^2 - \frac{1}{t^2} - \frac{1}{3}, \\ p_2'(t) &= \frac{2}{45} t - \frac{1}{t \sinh^2 t} + \frac{2}{t^3} - \frac{1}{t^2} \frac{\cosh t}{\sinh t}, \\ p_2''(t) &= \frac{45A(t) + 180t^2 B(t) + t^4 C(t)}{90t^4 \sinh^3 t} > 0, \end{aligned}$$

where

$$\begin{aligned} A(t) &= t \cosh 3t - 3 \sinh 3t + 9 \sinh t - t \cosh t \\ &= \sum_{n=5}^\infty \frac{2(n-4)\left(3^{2n} - 1\right)}{(2n+1)!} t^{2n+1} > 0, \\ B(t) &= t \cosh t + \sinh t > 0, \\ C(t) &= \sinh 3t - 3 \sinh t > 0. \end{aligned}$$

So $\deg_{cmi}^x [R_2(x)] \geq 2$.

On the other hand, we can prove that the function $x^3 f_2(x) = x^3 R_2(x)$ is not completely monotonic on $(0, \infty)$. By (2.3) we have

$$x^3 f_2(x) = \frac{1}{4} \int_0^\infty p_2^{(3)} \left(\frac{t}{2} \right) e^{-xt} dt,$$

then by (1.2), we can complete the staged argument since we can verify

$$p_2^{(3)} \left(\frac{t}{2} \right) > 0 \Longleftrightarrow p_2^{(3)}(t) > 0$$

is not true for all $t > 0$ due to

$$\begin{aligned} p_2'''(t) &= \frac{2}{t\sinh^2 t} - \frac{2}{t^3\sinh^2 t} + \frac{4}{t^3} + \frac{24}{t^5} - \frac{6}{t}\frac{\cosh^2 t}{\sinh^4 t} - \frac{4}{t^3}\frac{\cosh^2 t}{\sinh^2 t} - \frac{2}{t^2}\frac{\cosh^3 t}{\sinh^3 t} \\ &\quad + \frac{2}{t^2}\frac{\cosh t}{\sinh t} - \frac{4}{t^2}\frac{\cosh t}{\sinh^3 t} - \frac{6}{t^4}\frac{\cosh t}{\sinh t} \end{aligned}$$

with $p_2'''(10) = -0.000\,36\cdots$. □

Theorem 3.2. The function $x^4 R_3(x)$ is completely monotonic on $(0, \infty)$, and

$$\deg^x_{cmi}[R_3(x)] = \deg^x_{cmi}[-f_3(x)] = 4.$$

Proof. By (2.3) we obtain that

$$x^4 R_3(x) = \int_0^\infty \left[-p_3^{(4)}\left(\frac{t}{2}\right)\right] e^{-xt} dt.$$

From (2.2) we clearly see that

$$p_3(t) = \frac{\coth t}{t} - \frac{1}{t^2} + \frac{1}{45}t^2 - \frac{2}{945}t^4 - \frac{1}{3},$$

$$p_3^{(4)}(t) =: -\frac{1}{630}\frac{H(t)}{t^6\sinh^5 t},$$

or

$$-p_3^{(4)}(t) = \frac{1}{630}\frac{H(t)}{t^6\sinh^5 t},$$

where

$$\begin{aligned} H(t) &= \left(2t^6 + 4725\right)\sinh 5t - 945t\cosh 5t - \left(1260t^5 + 3780t^3 - 2835t\right)\cosh 3t \\ &\quad - \left(10t^6 + 2520t^4 + 3780t^2 + 23\,625\right)\sinh 3t - \left(13\,860t^5 - 3780t^3 + 1890t\right)\cosh t \\ &\quad + \left(20t^6 - 7560t^4 + 11\,340t^2 + 47\,250\right)\sinh t \\ &: = \sum_{n=5}^{\infty} \frac{h_n}{(2n+3)!} t^{2n+3} \end{aligned}$$

with

$$\begin{aligned} h_n &= \frac{2}{125}\left[64n^6 + 96n^5 - 80n^4 - 120n^3 + 16n^2 - 2953\,101n + 32\,484\,375\right]5^{2n} \\ &\quad - \frac{10}{27}\left[64n^6 + 96n^5 + 5968n^4 + 105\,720n^3 + 393\,136n^2 + 400\,515n + 1760\,535\right]3^{2n} \\ &\quad + 20\left[64n^6 + 96n^5 - 11\,168n^4 - 9696n^3 + 9592n^2 + 13\,191n + 7749\right] \\ &> 0 \end{aligned}$$

for all $n \geq 5$. So $x^4 R_3(x)$ is completely monotonic on $(0, \infty)$, which implies $\deg^x_{cmi}[R_3(x)] \geq 4$.

Then we shall prove $x^5 R_3(x) = -x^5 f_3(x)$ is not completely monotonic on $(0, \infty)$. Since

$$x^5 R_3(x) = \int_0^\infty \left[-p_3^{(5)}\left(\frac{t}{2}\right)\right] e^{-xt} dt,$$

and

$$-p_3^{(5)}(t) = \frac{1}{4} \frac{K(t)}{t^7 \sinh^6 t},$$

where

$$\begin{aligned} K(t) \;=\; & 540\cosh 4t - 1350\cosh 2t - 90\cosh 6t - 240t^2\cosh 2t + 60t^2\cosh 4t \\ & +80t^4\cosh 2t + 40t^4\cosh 4t + 208t^6\cosh 2t + 8t^6\cosh 4t - 120t^3\sinh 2t \\ & +60t^3\sinh 4t + 200t^5\sinh 2t + 20t^5\sinh 4t + 75t\sinh 2t - 60t\sinh 4t \\ & +15t\sinh 6t + 180t^2 - 120t^4 + 264t^6 + 900. \end{aligned}$$

We find $K(5) \approx -2.6315 \times 10^{13} < 0$, which means $-p_3^{(5)}(5) < 0$. So the function $x^5 R_3(x) = -x^5 f_3(x)$ is not completely monotonic on $(0, \infty)$.

In a word, $\deg_{cmi}^x [R_3(x)] = \deg_{cmi}^x [-f_3(x)] = 4$. □

Remark 3.1. So far, we have the results about the completely monotonic integer degrees of such functions, that is, $\deg_{cmi}^x [R_1(x)] = 1$ and $\deg_{cmi}^x [R_n(x)] = 2(n-1)$ for $n = 2, 3$, and find that the existing conclusions support the conjecture (1.5).

4. Complete monotonicity of the functions $-R_n'(x)$ ($1 \le n \le 3$) and their completely monotonic integer degrees

In this section, we shall calculate the completely monotonic degrees of the functions $(-1)^m R_n^{(m)}(x)$, where $m = 1$ and $1 \le n \le 3$.

Theorem 4.1 The function $-x^2 R_1'(x) = x^2 f_1'(x)$ is completely monotonic on $(0, \infty)$, and

$$\deg_{cmi}^x \left[(-1)^1 R_1'(x)\right] = 2.$$

Proof. By the integral representation (2.3) we obtain

$$x^2 f_1'(x) = \frac{1}{4} \int_0^\infty \left[-t p_1\left(\frac{t}{2}\right)\right]'' e^{-xt} dt.$$

So we complete the proof of result that $x^2 f_1'(x)$ is completely monotonic on $(0, \infty)$ when proving

$$\left[-t p_1\left(\frac{t}{2}\right)\right]'' > 0 \Longleftrightarrow \left[t p_1\left(\frac{t}{2}\right)\right]'' < 0 \Longleftrightarrow [t p_1(t)]'' < 0.$$

In fact,

$$t p_1(t) \;=\; t\left(\frac{\coth t}{t} - \frac{1}{t^2} - \frac{1}{3}\right) = \frac{\cosh t}{\sinh t} - \frac{1}{3}t - \frac{1}{t},$$

$$
\begin{aligned}
[tp_1(t)]' &= \frac{1}{t^2} - \frac{\cosh^2 t}{\sinh^2 t} + \frac{2}{3}, \\
[tp_1(t)]'' &= \frac{2}{\sinh^3 t}\left[\cosh t - \left(\frac{\sinh t}{t}\right)^3\right] < 0.
\end{aligned}
$$

Then we have $\deg_{cmi}^{x}\left[(-1)^1 R_1'(x)\right] \geq 2$.

Here $-x^3 R_1'(x) = x^3 f_1'(x)$ is not completely monotonic on $(0,\infty)$. By (2.2) and (2.3) we have

$$
x^3 f_1'(x) = \frac{1}{4}\int_0^\infty \left[-tp_1\left(\frac{t}{2}\right)\right]''' e^{-xt}dt,
$$

and

$$
[-tp_1(t)]''' = 2\frac{3t^4\cosh^2 t - t^4\sinh^2 t - 3\sinh^4 t}{t^4\sinh^4 t}
$$

with $[-tp_1(t)]'''|_{t=2} \approx -3.6237 \times 10^{-2} < 0$. □

Theorem 4.2. The function $-x^3 R_2'(x)$ is completely monotonic on $(0,\infty)$, and

$$
\deg_{cmi}^{x}[-R_2'(x)] = \deg_{cmi}^{x}[-f_2'(x)] = 3.
$$

Proof. First, we can prove that the function $-x^3 R_2'(x)$ is completely monotonic on $(0,\infty)$. Using the integral representation (2.3) we obtain

$$
-x^3 f_2'(x) = \frac{1}{4}\int_0^\infty \left[tp_2\left(\frac{t}{2}\right)\right]^{(3)} e^{-xt}dt,
$$

and complete the proof of the staged argument when proving

$$
\left[tp_2\left(\frac{t}{2}\right)\right]^{(3)} > 0 \Longleftrightarrow [tp_2(t)]^{(3)} > 0.
$$

In fact,

$$
\begin{aligned}
p_2(t) &= \frac{\coth t}{t} + \frac{1}{45}t^2 - \frac{1}{t^2} - \frac{1}{3}, \\
L(t) &:= tp_2(t) = \frac{\cosh t}{\sinh t} - \frac{1}{3}t - \frac{1}{t} + \frac{1}{45}t^3,
\end{aligned}
$$

$$
\begin{aligned}
L'''(t) &= \frac{-180\cosh 2t + 45\cosh 4t - 124t^4\cosh 2t + t^4\cosh 4t - 237t^4 + 135}{60t^4\sinh^4 t} \\
&= \frac{1}{60t^4\sinh^4 t}\left[\sum_{n=3}^{\infty}\frac{2^{2n+2}b_n}{(2n+4)!}t^{2n+4}\right] > 0,
\end{aligned}
$$

where

$$
b_n = 2^{2n}\left(4n^4 + 20n^3 + 35n^2 + 25n + 2886\right)
$$

$$
\begin{aligned}
&-4\left(775n+1085n^2+620n^3+124n^4+366\right)\\
>\ & 0
\end{aligned}
$$

for all $n \geq 3$.

On the other hand, by (2.3) we obtain

$$x^4(-1)^1 R_2'(x) = -x^4 f_2'(x) = \frac{1}{4}\int_0^\infty \left[tp_2\left(\frac{t}{2}\right)\right]^{(4)} e^{-xt}dt,$$

and

$$L^{(4)}(t) = [tp_2(t)]^{(4)} = 16\frac{\cosh t}{\sinh t} - 40\frac{\cosh^3 t}{\sinh^3 t} + 24\frac{\cosh^5 t}{\sinh^5 t} - \frac{24}{t^5}$$

is not positive on $(0,\infty)$ due to $L^{(4)}(10) \approx -2.399\,3\times 10^{-4} < 0$, we have that $-x^4R_2'(x)$ is not completely monotonic on $(0,\infty)$. □

Theorem 4.3. The function $-x^5R_3'(x)$ is completely monotonic on $(0,\infty)$, and

$$\deg_{cmi}^x[-R_3'(x)] = \deg_{cmi}^x[f_3'(x)] = 5.$$

Proof. We shall prove that $-x^5R_3'(x) = x^5f_3'(x)$ is completely monotonic on $(0,\infty)$ and $-x^6R_3'(x) = x^6f_3(x)$ is not. By (2.2) and (2.3) we obtain

$$x^r f_3'(x) = \frac{1}{4}\int_0^\infty \left[-tp_3\left(\frac{t}{2}\right)\right]^{(r)} e^{-xt}dt,\ r \geq 0.$$

and

$$
\begin{aligned}
p_3(t) &= \frac{\coth t}{t} - \frac{1}{t^2} + \frac{1}{45}t^2 - \frac{2}{945}t^4 - \frac{1}{3},\\
M(t) &:= tp_3(t) = \frac{\cosh t}{\sinh t} - \frac{1}{3}t - \frac{1}{t} + \frac{1}{45}t^3 - \frac{2}{945}t^5,\\
M^{(5)}(t) &= -\frac{1}{252}\frac{p(t)}{t^6\sinh^6 t},
\end{aligned}
$$

we have

$$
\begin{aligned}
[-M(t)]^{(5)} &= \frac{1}{252}\frac{p(t)}{t^6\sinh^6 t},\\
[-M(t)]^{(6)} &= \frac{1}{4}\frac{q(t)}{t^7\sinh^7 t},
\end{aligned}
$$

where

$$
\begin{aligned}
p(t) &= -14\,175\cosh 2t + 5670\cosh 4t - 945\cosh 6t + 13\,134t^6\cosh 2t\\
&\quad +492t^6\cosh 4t + 2t^6\cosh 6t + 16\,612t^6 + 9450,\\
q(t) &= 945\sinh 3t - 315\sinh 5t + 45\sinh 7t - 1575\sinh t - 456t^7\cosh 3t\\
&\quad -8t^7\cosh 5t - 2416t^7\cosh t.
\end{aligned}
$$

Since

$$\begin{aligned} p(t) &= \sum_{n=4}^{\infty} \frac{2 \cdot 6^{2n} + 492 \cdot 4^{2n} + 13\,134 \cdot 2^{2n}}{(2n)!} t^{2n+6} \\ &\quad - \sum_{n=4}^{\infty} \frac{945 \cdot 6^{2n+6} - 5670 \cdot 4^{2n+6} + 14\,175 \cdot 2^{2n+6}}{(2n+6)!} t^{2n+6} \\ &> 0, \\ q(0.1) &\approx -2.9625 \times 10^{-5} < 0, \end{aligned}$$

we obtain the expected conclusions. □

Remark 4.1. The experimental results show that the conjecture (1.6) may be true.

5. Complete monotonicity of the functions $R_n''(x)$ ($1 \le n \le 3$) and their completely monotonic integer degrees

Theorem 5.1. The function $x^3 R_1''(x)$ is completely monotonic on $(0, \infty)$, and

$$\deg_{cmi}^{x} [R_1''(x)] = \deg_{cmi}^{x} [-f_1''(x)] = 3. \tag{5.1}$$

Proof. By (2.2) and (2.3) we obtain

$$x^3 R_1''(x) = -x^3 f_1''(x) = \frac{1}{4} \int_0^{\infty} \left[-t^2 p_1\left(\frac{t}{2}\right)\right]''' e^{-xt} dt,$$

and

$$t^2 p_1(t) = t \frac{\cosh t}{\sinh t} - \frac{1}{3} t^2 - 1,$$

$$\left[-t^2 p_1(t)\right]''' = \frac{2}{3 \sinh^4 t} \left[\sum_{n=2}^{\infty} \frac{3(n-1) 2^{2n+1}}{(2n+1)!} t^{2n+1}\right] > 0.$$

So $x^3 R_1''(x)$ is completely monotonic on $(0, \infty)$.

But $x^4 R_1''(x)$ is not completely monotonic on $(0, \infty)$ due to

$$x^4 R_1''(x) = \frac{1}{4} \int_0^{\infty} \left[-t^2 p_1\left(\frac{t}{2}\right)\right]^{(4)} e^{-xt} dt,$$

and

$$\left[-t^2 p_1(t)\right]^{(4)} = \frac{1}{\sinh^5 t} (4 \sinh 3t + 12 \sinh t - 22t \cosh t - 2t \cosh 3t)$$

with $\left[-t^2 p_1(t)\right]^{(4)} |_{t=10} \approx -5.276\,6 \times 10^{-7} < 0.$

So

$$\deg_{cmi}^{x} [R_1''(x)] = \deg_{cmi}^{x} [-f_1''(x)] = 3.$$

□

Remark 5.1. Here, we actually give a negative answer to the second paragraph of conjecture (1.7).

Theorem 5.2. The function $x^4 R_2''(x)$ is completely monotonic on $(0,\infty)$, and

$$\deg_{cmi}^{x}[R_2''(x)] = \deg_{cmi}^{x}[f_2''(x)] = 4.$$

Proof. By (2.2) and (2.3) we

$$x^4 f_2''(x) = \frac{1}{4}\int_0^\infty \left[t^2 p_2\left(\frac{t}{2}\right)\right]^{(4)} e^{-xt}dt,$$

and

$$\begin{aligned} t^2 p_2(t) &= \frac{1}{45}t^4 - \frac{1}{3}t^2 + t\frac{\cosh t}{\sinh t} - 1, \\ \left[t^2 p_2(t)\right]^{(4)} &= \frac{1}{30}\frac{(-125\sinh 3t + \sinh 5t - 350\sinh t + 660t\cosh t + 60t\cosh 3t)}{\sinh^5 t} \\ &= \frac{1}{30\sinh^5 t}\left[\sum_{n=3}^{\infty}\frac{5\left(5^{2n} + (24n-63)3^{2n} + 264n + 62\right)}{(2n+1)!}t^{2n+1}\right] \\ &> 0. \end{aligned}$$

Since

$$x^5 f_2''(x) = \frac{1}{4}\int_0^\infty \left[t^2 p_2\left(\frac{t}{2}\right)\right]^{(5)} e^{-xt}dt,$$

and

$$\left[t^2 p_2(t)\right]^{(5)} = \frac{50\sinh 2t - 66t + 5\sinh 4t - 52t\cosh 2t - 2t\cosh 4t}{\sinh^6 t}$$

with $\left[t^2 p_2(t)\right]^{(5)}|_{t=10} \approx -9.8935\times 10^{-7} < 0$, we have that $x^5 f_2''(x)$ is not completely monotonic on $(0,\infty)$. So

$$\deg_{cmi}^{x}[R_2''(x)] = 4.$$

□

Theorem 5.3. The function $x^6 R_3''(x)$ is completely monotonic on $(0,\infty)$, and

$$\deg_{cmi}^{x}[R_3''(x)] = \deg_{cmi}^{x}[-f_3''(x)] = 6.$$

Proof. By the integral representation (2.3) we obtain

$$\begin{aligned} -x^6 f_3''(x) &= \frac{1}{4}\int_0^\infty \left[-t^2 p_3\left(\frac{t}{2}\right)\right]^{(6)} e^{-xt}dt, \\ -x^7 f_3''(x) &= \frac{1}{4}\int_0^\infty \left[-t^2 p_3\left(\frac{t}{2}\right)\right]^{(7)} e^{-xt}dt. \end{aligned}$$

It follows from (2.2) that

$$p_3(t) = \frac{\coth t}{t} - \frac{1}{t^2} + \frac{1}{45}t^2 - \frac{2}{945}t^4 - \frac{1}{3},$$

$$
\begin{aligned}
N(t) &: = t^2 p_3(t) = \frac{1}{45}t^4 - \frac{1}{3}t^2 - \frac{2}{945}t^6 + t\frac{\cosh t}{\sinh t} - 1, \\
-N^{(6)}(t) &= \frac{1}{42}\frac{r(t)}{\sinh^7 t}, \\
-N^{(7)}(t) &= \frac{\begin{pmatrix} 2416t - 1715\sinh 2t - 392\sinh 4t - 7\sinh 6t \\ +2382t\cosh 2t + 240t\cosh 4t + 2t\cosh 6t \end{pmatrix}}{\sinh^8 t},
\end{aligned}
$$

where

$$
\begin{aligned}
r(t) &= 6321\sinh 3t + 245\sinh 5t + \sinh 7t + 10\,045\sinh t - 25\,368t\cosh t \\
&\quad -4788t\cosh 3t - 84t\cosh 5t \\
&= \sum_{n=4}^{\infty} \frac{c_n}{(2n+1)!} t^{2n+1}
\end{aligned}
$$

with

$$
c_n = 7 \cdot 7^{2n} - (168n - 1141)\,5^{2n} - (9576n - 14\,175)\,3^{2n} - (50\,736n + 15\,323)\,.
$$

Since $c_i > 0$ for $i = 4, 5, 6, 7$, and

$$
\begin{aligned}
c_{n+1} - 49c_n &= (4032n - 31\,584)\,5^{2n} + (383\,040n - 653\,184)\,3^{2n} \\
&\quad +2435\,328n + 684\,768 > 0
\end{aligned}
$$

for all $n \geq 8$. So $c_n > 0$ for all $n \geq 4$. Then $r(t) > 0$ and $-N^{(6)}(t) > 0$ for all $t > 0$. So $x^6 R_3''(x)$ is completely monotonic on $(0, \infty)$.

In view of $-N^{(7)}(1.5) \approx -0.57982 < 0$, we get $x^7 R_3''(x)$ is not completely monotonic on $(0, \infty)$. The proof of this theorem is complete. □

Remark 5.2. The experimental results show that the conjecture (1.7) may be true for $n, m \geq 2$.

6. Conjectures for the completely monotonic integer degrees of the functions $(-1)^m R_n^{(m)}(x)$

In this way, the first two paragraphs for conjectures (1.5) and (1.6) have been confirmed, leaving the following conjectures to be confirmed:

$$
\deg_{cmi}^{x}[R_n(x)] = 2(n-1),\ n \geq 4; \tag{6.1}
$$

$$
\deg_{cmi}^{x}[-R_n'(x)] = 2n - 1,\ n \geq 4; \tag{6.2}
$$

and for $m \geq 1$,

$$
\deg_{cmi}^{x}\left[(-1)^m R_n^{(m)}(x)\right] = \begin{cases} m, & \text{if } n = 0 \\ m+1, & \text{if } n = 1 \\ m + 2(n-1), & \text{if } n \geq 2 \end{cases}, \tag{6.3}
$$

where the first formula and second formula in (6.3) are two new conjectures which are different from the original ones.

By the relationship (2.3) we propose the following operational conjectures.

Conjecture 6.1. Let $n \geq 4$, and $p_n(t)$ be defined as (2.2). Then

$$[(-1)^n p_n(t)]^{(2n-2)} > 0 \tag{6.4}$$

holds for all $t \in (0, \infty)$ and

$$[(-1)^n p_n(t)]^{(2n-1)} > 0 \tag{6.5}$$

is not true for all $t \in (0, \infty)$.

Conjecture 6.2. Let $n \geq 4$, and $p_n(t)$ be defined as (2.2). Then

$$(-1)^n [t p_n(t)]^{(2n-1)} > 0 \tag{6.6}$$

holds for all $t \in (0, \infty)$ and

$$(-1)^n [t p_n(t)]^{(2n)} > 0 \tag{6.7}$$

is not true for all $t \in (0, \infty)$.

Conjecture 6.3. Let $m \geq 1$, and $p_n(t)$ be defined as (2.2). Then

$$[t^m p_0(t)]^{(m)} > 0, \tag{6.8}$$

$$[-t^m p_1(t)]^{(m+1)} > 0 \tag{6.9}$$

hold for all $t \in (0, \infty)$, and

$$[t^m p_0(t)]^{(m+1)} > 0, \tag{6.10}$$

$$[-t^m p_1(t)]^{(m+2)} > 0 \tag{6.11}$$

are not true for all $t \in (0, \infty)$.

Conjecture 6.4. Let $m \geq 1$, $n \geq 2$, and $p_n(t)$ be defined as (2.2). Then

$$(-1)^n [t^m p_n(t)]^{(m+2n-2)} > 0 \tag{6.12}$$

holds for all $t \in (0, \infty)$ and

$$(-1)^n [t^m p_n(t)]^{(m+2n-1)} > 0 \tag{6.13}$$

is not true for all $t \in (0, \infty)$.

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Conflict of interest

The author declares no conflict of interest in this paper.

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