Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(4): 3423–3433.
DOI:10.3934/math.2020221
Received: 20 February 2020
Accepted: 30 March 2020
Published: 02 April 2020

Research article

Stein's lemma for truncated generalized skew-elliptical random vectors

Baishuai Zuo and Chuancun Yin*

School of Statistics, Qufu Normal University, Qufu 273165, Shandong, China

* **Correspondence:** Email: ccyin@qfnu.edu.cn.

Abstract: Inspired by Shushi [1] and Adcock et al. [2], we consider Stein's lemma for truncated generalized skew-elliptical random vectors. We provide two Stein's lemmas. One is Stein's lemma for truncated generalized skew-elliptical random vectors, the other is a special form of Stein's lemma for truncated generalized skew-elliptical random vectors. Finally, the conditional tail expectation allocation, the lower-orthant conditional tail expectation at probability level q, the upper-orthant conditional tail expectation at probability level q, the truncated version of Wang's premium, the multivariate tail conditional expectation and the multivariate tail covariance matrix as applications are given.

Keywords: generalized skew-elliptical distributions; generalized skew-normal distributions; truncated random vectors; Stein's lemma
Mathematics Subject Classification: 62E10, 62H05

1. Introduction and motivation

Stein [3] provide an expression $E[h(X)(X-\mu)]$ for normal random variable X, where $h(x)$ is an almost differentiable function. Then a number of scholars have generalized the formula. For examples, Landsman [4], Landsman and Ne*š*lehov*á* [5], Landsman et al. [6] derive Stein's lemma for multivariate elliptical distributions. Adcock and Shutes [7], Adcock [8], Adcock et al. [2] derive Stein's lemma for multivariate skew distributions. Liu [9] use Stein's lemma derive the Siegel's formula, and Li [10], Landsman et al. [11] apply this lemma to study risk measures.

Recently, Shushi [1] provide Stein's lemma for truncated elliptical random vectors, and inspired by this, we shall generalize Stein's lemma for truncated generalized skew-elliptical distributions. As applications, we consider the conditional tail expectation (CTE) allocation, the lower-orthant CTE at probability level q, the upper-orthant CTE at probability level q, the truncated version of Wang's premium, the multivariate tail conditional expectation and the multivariate tail covariance matrix measures of Stein's lemma for generalized skew-elliptical random vectors.

The rest of the paper is organized as follows. Section 2 reviews the definitions and properties of the generalized skew-elliptical distributions. In Section 3, We provide two Stein's lemmas, one is Stein's lemma for truncated generalized skew-elliptical random vectors, the other is a special form of Stein's lemma for truncated generalized skew-elliptical random vectors. Several measures as applications in risk theory are given in Section 4. Conclusions are summarized in Section 5.

2. Generalized skew-elliptical distributions

Let $\mathbf{Y}$ be an n-dimensional generalized skew-elliptical random vector, and denoted by $\mathbf{Y} \sim GSE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n, \pi(\cdot))$. If it's probability density function exists, the form will be (see Adcock et al. [2])

$$f_Y(\boldsymbol{y}) = \frac{2}{\sqrt{|\boldsymbol{\Sigma}|}} g_n \left\{ \frac{1}{2} (\boldsymbol{y} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{y} - \boldsymbol{\mu}) \right\} \pi \left(\Sigma^{-\frac{1}{2}} (\boldsymbol{y} - \boldsymbol{\mu}) \right), \ \boldsymbol{y} \in \mathbb{R}^n, \tag{2.1}$$

where

$$f_X(\boldsymbol{x}) := \frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} g_n \left\{ \frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \right\}, \ \boldsymbol{x} \in \mathbb{R}^n, \tag{2.2}$$

is the density of n-dimensional elliptical random vector $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$. Here $\boldsymbol{\mu}$ is an $n \times 1$ location vector, $\boldsymbol{\Sigma}$ is an $n \times n$ scale matrix, and $g_n(u)$, $u \geq 0$, is the density generator of $\mathbf{X}$. $\pi(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^n$, is called the skewing function satisfying $\pi(-\boldsymbol{x}) = 1 - \pi(\boldsymbol{x})$ and $0 \leq \pi(\boldsymbol{x}) \leq 1$. The characteristic function of $\mathbf{X}$ takes the form $\varphi_X(\boldsymbol{t}) = \exp\left\{ i\boldsymbol{t}^T \boldsymbol{\mu} \right\} \psi\left(\frac{1}{2} \boldsymbol{t}^T \boldsymbol{\Sigma} \boldsymbol{t} \right)$, $\boldsymbol{t} \in \mathbb{R}^n$, with function $\psi(t) : [0, \infty) \to \mathbb{R}$, called the characteristic generator (see Fang et al. [12]). We definite a cumulative generator $\overline{G}_n(u)$. It takes the form (see Landsman et al. [11], Part 4 or Landsman [13])

$$\overline{G}_n(u) = \int_u^\infty g_n(v) \mathrm{d}v. \tag{2.3}$$

3. Main results

In this section, consider a random vector $\mathbf{Y} \sim GSE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n, \pi(\cdot))$ with finite vector $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n)^T$, positive defined matrix $\boldsymbol{\Sigma} = (\sigma_{ij})_{i,j=1}^n$ and probability density function $f_Y(\boldsymbol{y})$.

Let $\varpi : \mathbb{R}^m \to \mathbb{R}$, $1 \leq m \leq n$, be an almost differentiable function, and we write

$$\nabla \varpi(\boldsymbol{y}_{(1)}) = \left(\frac{\partial \varpi(\boldsymbol{y}_{(1)})}{\partial y_1}, \frac{\partial \varpi(\boldsymbol{y}_{(1)})}{\partial y_2}, \cdots, \frac{\partial \varpi(\boldsymbol{y}_{(1)})}{\partial y_n} \right)^T.$$

Let $\mathbf{X}^* \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \overline{G}_n)$ be an elliptical random vector with generator $\overline{G}_n(u)$, whose the density function (if it exists)

$$f_{X^*}(\boldsymbol{x}) = \frac{-1}{\psi'(0) \sqrt{|\boldsymbol{\Sigma}|}} \overline{G}_n \left\{ \frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \right\}, \ \boldsymbol{x} \in \mathbb{R}^n. \tag{3.1}$$

Let $\mathbf{Y}^* \sim GSE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \overline{G}_n, \pi(\cdot))$ be a generalized skew-elliptical random vector.

Let's derive Stein's lemma for truncated generalized skew-elliptical random vectors below. Firstly, we define a subset $\mathbb{D} \subseteq \mathbb{R}^n$, which is a subset of all possible outcomes of $\mathbf{Y} \in \mathbb{R}^n$, and

$$E^{\mathbb{D}}[\varpi(\mathbf{Y})(\mathbf{Y}-\boldsymbol{\mu})] := E\left[\varpi(\mathbf{Y})(\mathbf{Y}-\boldsymbol{\mu})|\mathbf{Y} \in \mathbb{D}\right].$$

Then partition $\mathbf{Y} = (\mathbf{Y_{(1)}}^T,\ \mathbf{Y_{(2)}}^T)^T$, where $\mathbf{Y_{(1)}} = (Y_1,\ Y_2, \cdots, Y_m)^T$ and $\mathbf{Y_{(2)}} = (Y_{m+1},\ Y_{m+2}, \cdots, Y_n)^T$. Furthermore, $\mathbf{X} = (\mathbf{X_{(1)}}^T,\ \mathbf{X_{(2)}}^T)^T$ and $\boldsymbol{\mu} = (\boldsymbol{\mu_{(1)}}^T,\ \boldsymbol{\mu_{(2)}}^T)^T$ are similar partitions.

Theorem 3.1. Let $\mathbf{Y} \sim GSE_n(\boldsymbol{\mu},\ \mathbf{\Sigma},\ g_n,\ \pi(\cdot))$ be an n-dimensional generalized skew-elliptical random vector with probability density function (2.1). The function ϖ satisfies $E^{\mathbb{D}}[\|\ \nabla\varpi(\mathbf{Y_{(1)}}^*)\ \|] < \infty$, $E\left[\|\ \varpi(\mathbf{Y_{(1)}}^*)\nabla \mathbf{1}_{\boldsymbol{Y}^*\in\mathbb{D}}\ \|\right] < \infty$ and $E^{\mathbb{D}}\left[\|\ \varpi(\mathbf{X_{(1)}}^*)\nabla\pi\left(\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^*-\boldsymbol{\mu})\right)\ \|\right] < \infty$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. Furthermore, we suppose

$$\lim_{|y_k|\to\infty} \varpi\left(\mathbf{A}_{1,1}\boldsymbol{y_{(1)}} + \mathbf{A}_{1,2}\boldsymbol{y_{(2)}} + \boldsymbol{\mu_{(1)}}\right)\pi(\boldsymbol{y})\mathbf{1}_{\boldsymbol{y}\in\mathbb{D}}\overline{G}_n\left(\frac{1}{2}\boldsymbol{y}^T\boldsymbol{y}\right) = 0. \tag{3.2}$$

Then

$$\begin{aligned}
&E^{\mathbb{D}}[\varpi(\mathbf{Y_{(1)}})(\mathbf{Y}-\boldsymbol{\mu})]\\
=&\frac{-\psi'(0)}{\Pr(\boldsymbol{Y}\in\mathbb{D})}\Big\{\Pr(\boldsymbol{Y}^*\in\mathbb{D})\mathbf{\Sigma}E^{\mathbb{D}}[\nabla\varpi(\mathbf{Y_{(1)}}^*)] + \mathbf{\Sigma}^{\frac{1}{2}}E[\varpi(\mathbf{Y_{(1)}}^*)\nabla\mathbf{1}_{\boldsymbol{Y}^*\in\mathbb{D}}]\\
&+2\Pr(\boldsymbol{X}^*\in\mathbb{D})\mathbf{\Sigma}E^{\mathbb{D}}\left[\varpi(\mathbf{X_{(1)}}^*)\nabla\pi\left(\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^*-\boldsymbol{\mu})\right)\right]\Big\},
\end{aligned} \tag{3.3}$$

where $\nabla\mathbf{1}_{\boldsymbol{Y}^*\in\mathbb{D}} = \left(\frac{\partial\mathbf{1}_{\boldsymbol{Y}^*\in\mathbb{D}}}{\partial Y_1^*}, \frac{\partial\mathbf{1}_{\boldsymbol{Y}^*\in\mathbb{D}}}{\partial Y_2^*}, \cdots, \frac{\partial\mathbf{1}_{\boldsymbol{Y}^*\in\mathbb{D}}}{\partial Y_n^*}\right)^T$, and $\mathbf{1}$ is the indicator function. In addition, $\nabla\pi\left(\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^*-\boldsymbol{\mu})\right) = \frac{\mathrm{d}}{\mathrm{d}\mathbf{X}^*}\pi\left(\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^*-\boldsymbol{\mu})\right)$ and

$$\mathbf{\Sigma}^{\frac{1}{2}} = \begin{pmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{pmatrix}.$$

Proof. Using definition, we obtain

$$\begin{aligned}
&E^{\mathbb{D}}[\varpi(\mathbf{Y_{(1)}})(\mathbf{Y}-\boldsymbol{\mu})]\\
=&\frac{2|\mathbf{\Sigma}|^{-\frac{1}{2}}}{\Pr(\boldsymbol{Y}\in\mathbb{D})}\int_{\mathbb{D}}\varpi(\boldsymbol{y_{(1)}})(\boldsymbol{y}-\boldsymbol{\mu})g_n\left\{\frac{1}{2}(\boldsymbol{y}-\boldsymbol{\mu})^T\mathbf{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\right\}\pi\left(\mathbf{\Sigma}^{-\frac{1}{2}}(\boldsymbol{y}-\boldsymbol{\mu})\right)\mathrm{d}\boldsymbol{y}.
\end{aligned}$$

Setting $\boldsymbol{z} = \mathbf{\Sigma}^{-\frac{1}{2}}(\boldsymbol{y}-\boldsymbol{\mu})$, we have

$$\begin{aligned}
&E^{\mathbb{D}}[\varpi(\mathbf{Y})(\mathbf{Y}-\boldsymbol{\mu})]\\
=&\frac{2\mathbf{\Sigma}^{\frac{1}{2}}}{\Pr(\boldsymbol{Y}\in\mathbb{D})}\int_{\mathbb{R}^n}\varpi\left(\mathbf{A}_{1,1}\boldsymbol{z_{(1)}}+\mathbf{A}_{1,2}\boldsymbol{z_{(2)}}+\boldsymbol{\mu_{(1)}}\right)\pi(\boldsymbol{z})\mathbf{1}_{\boldsymbol{z}\in\mathbb{D}_z}\boldsymbol{z}g_n\left\{\frac{1}{2}\boldsymbol{z}^T\boldsymbol{z}\right\}\mathrm{d}\boldsymbol{z}\\
=&\frac{-2\mathbf{\Sigma}^{\frac{1}{2}}}{\Pr(\boldsymbol{Y}\in\mathbb{D})}\int_{\mathbb{R}^n}\varpi\left(\mathbf{A}_{1,1}\boldsymbol{z_{(1)}}+\mathbf{A}_{1,2}\boldsymbol{z_{(2)}}+\boldsymbol{\mu_{(1)}}\right)\pi(\boldsymbol{z})\mathbf{1}_{\boldsymbol{z}\in\mathbb{D}_z}\mathrm{d}\overline{G}_n\left\{\frac{1}{2}\boldsymbol{z}^T\boldsymbol{z}\right\}\\
=&\frac{2\mathbf{\Sigma}^{\frac{1}{2}}}{\Pr(\boldsymbol{Y}\in\mathbb{D})}\int_{\mathbb{R}^n}\nabla\left[\varpi\left(\mathbf{A}_{1,1}\boldsymbol{z_{(1)}}+\mathbf{A}_{1,2}\boldsymbol{z_{(2)}}+\boldsymbol{\mu_{(1)}}\right)\pi(\boldsymbol{z})\mathbf{1}_{\boldsymbol{z}\in\mathbb{D}_z}\right]\overline{G}_n\left\{\frac{1}{2}\boldsymbol{z}^T\boldsymbol{z}\right\}\mathrm{d}\boldsymbol{z},
\end{aligned}$$

where $\Pr(\boldsymbol{Z} \in \mathbb{D}_{\boldsymbol{Z}}) = \Pr(\boldsymbol{Y} \in \mathbb{D})$, and in the third equality we have used (3.2).
While

$$\begin{aligned}
&\nabla\left[\varpi\left(\mathbf{A}_{1,1}z_{(1)}+\mathbf{A}_{1,2}z_{(2)}+\boldsymbol{\mu}_{(1)}\right)\pi(z)\mathbf{1}_{z\in\mathbb{D}_z}\right]\\
&=\pi(z)\mathbf{1}_{z\in\mathbb{D}_z}\nabla\varpi\left(\mathbf{A}_{1,1}z_{(1)}+\mathbf{A}_{1,2}z_{(2)}+\boldsymbol{\mu}_{(1)}\right)\\
&\quad+\varpi\left(\mathbf{A}_{1,1}z_{(1)}+\mathbf{A}_{1,2}z_{(2)}+\boldsymbol{\mu}_{(1)}\right)\mathbf{1}_{z\in\mathbb{D}_z}\nabla\pi(z)\\
&\quad+\varpi\left(\mathbf{A}_{1,1}z_{(1)}+\mathbf{A}_{1,2}z_{(2)}+\boldsymbol{\mu}_{(1)}\right)\pi(z)\nabla\mathbf{1}_{z\in\mathbb{D}_z},
\end{aligned}$$

so that

$$\begin{aligned}
&E^{\mathbb{D}}[\varpi(\mathbf{Y}_{(1)})(\mathbf{Y}-\boldsymbol{\mu})]\\
=&\frac{2\boldsymbol{\Sigma}^{\frac{1}{2}}}{\Pr(\boldsymbol{Y}\in\mathbb{D})}\int_{\mathbb{R}^n}\Big[\pi(z)\mathbf{1}_{z\in\mathbb{D}_z}\nabla\varpi\left(\mathbf{A}_{1,1}z_{(1)}+\mathbf{A}_{1,2}z_{(2)}+\boldsymbol{\mu}_{(1)}\right)\\
&+\varpi\left(\mathbf{A}_{1,1}z_{(1)}+\mathbf{A}_{1,2}z_{(2)}+\boldsymbol{\mu}_{(1)}\right)\mathbf{1}_{z\in\mathbb{D}_z}\nabla\pi(z)\\
&+\varpi\left(\mathbf{A}_{1,1}z_{(1)}+\mathbf{A}_{1,2}z_{(2)}+\boldsymbol{\mu}_{(1)}\right)\pi(z)\nabla\mathbf{1}_{z\in\mathbb{D}_z}\Big]\overline{G}_n\left\{\frac{1}{2}z^Tz\right\}\mathrm{d}z\\
=&\frac{2\boldsymbol{\Sigma}^{\frac{1}{2}}}{\Pr(\boldsymbol{Y}\in\mathbb{D})\sqrt{|\boldsymbol{\Sigma}|}}\int_{\mathbb{R}^n}\Big[\pi\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{y}-\boldsymbol{\mu})\right)\mathbf{1}_{\boldsymbol{y}\in\mathbb{D}}\boldsymbol{\Sigma}^{\frac{1}{2}}\nabla\varpi(\boldsymbol{y}_{(1)})\\
&+\varpi(\boldsymbol{y}_{(1)})\mathbf{1}_{\boldsymbol{y}\in\mathbb{D}}\boldsymbol{\Sigma}^{\frac{1}{2}}\nabla\pi\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{y}-\boldsymbol{\mu})\right)\\
&+\varpi(\boldsymbol{y}_{(1)})\pi\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{y}-\boldsymbol{\mu})\right)\nabla\mathbf{1}_{\boldsymbol{y}\in\mathbb{D}}\Big]\overline{G}_n\left\{\frac{1}{2}(\boldsymbol{y}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\right\}\mathrm{d}\boldsymbol{y},
\end{aligned}$$

therefore we obtain (3.3), which completes the proof of Theorem 3.1.

As a special case, Stein's lemma for n-dimensional truncated generalized skew-normal distribution is shown as follows.

Corollary 3.1. Let $\mathbf{Y} \sim GSN_n(\boldsymbol{\mu}, \Sigma, \pi(\cdot))$ be an n-dimensional generalized skew-normal random vector with probability density function

$$f_Y(\boldsymbol{y}) = \frac{2}{\sqrt{|\boldsymbol{\Sigma}|}(2\pi)^{\frac{n}{2}}}\exp\left\{-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\right\}\pi\left(\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{y}-\boldsymbol{\mu})\right),\ \boldsymbol{y}\in\mathbb{R}^n,$$

where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_n)^T$ and function $\pi(\cdot) : \mathbb{R} \to \mathbb{R}$. Then

$$\begin{aligned}
E^{\mathbb{D}}[\varpi(\mathbf{Y}_{(1)})(\mathbf{Y}-\boldsymbol{\mu})] = \boldsymbol{\Sigma}E^{\mathbb{D}}[\nabla\varpi(\mathbf{Y}_{(1)})] &+ \frac{\boldsymbol{\Sigma}^{\frac{1}{2}}}{\Pr(\mathbf{Y}\in\mathbb{D})}E[\varpi(\mathbf{Y}_{(1)})\nabla\mathbf{1}_{\mathbf{Y}\in\mathbb{D}}]\\
&+ 2\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\gamma}E^{\mathbb{D}}\left[\varpi(\mathbf{X}_{(1)})\pi'\left(\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{X}-\boldsymbol{\mu})\right)\right],
\end{aligned}\tag{3.4}$$

where $\mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and $\pi'(\cdot)$ is the derivative of $\pi(\cdot)$.

Proof. Let $g_n(u) = (2\pi)^{-\frac{n}{2}}\exp\{-u\}$ and $\pi\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{y}-\boldsymbol{\mu})\right) = \pi\left(\boldsymbol{\gamma}^T\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{y}-\boldsymbol{\mu})\right)$ in Theorem 3.1. Due to $g_n(u) = \overline{G}_n(u) = (2\pi)^{-\frac{n}{2}}\exp\{-u\}$, so that we obtain (3.4). This completes the proof of Corollary 3.1.

Remark 3.1. Let $\pi(\cdot) = \Phi(\cdot)$(the cdf of a standard normal distribution) in Corollary 3.1, we obtain Stein's lemma for n-dimensional truncated skew-normal distribution as follows.

$$E^{\mathbb{D}}[\varpi(\mathbf{Y}_{(1)})(\mathbf{Y}-\boldsymbol{\mu})] = \boldsymbol{\Sigma}E^{\mathbb{D}}[\nabla\varpi(\mathbf{Y}_{(1)})] + \frac{\boldsymbol{\Sigma}^{\frac{1}{2}}}{\Pr(\mathbf{Y}\in\mathbb{D})}E\left[\varpi(\mathbf{Y}_{(1)})\nabla\mathbf{1}_{\mathbf{Y}\in\mathbb{D}}\right]$$

$$+\sqrt{\frac{2}{\pi}}\mathbf{\Sigma}^{\frac{1}{2}}\boldsymbol{\gamma}E^{\mathbb{D}}\left[\varpi(\mathbf{X}_{(1)})\exp\left\{-\frac{1}{2}\left(\boldsymbol{\gamma}^{T}\mathbf{\Sigma}^{-\frac{1}{2}}(\boldsymbol{X}-\boldsymbol{\mu})\right)^{2}\right\}\right].$$

Remark 3.2. In Loperfido [14], a trivariate distribution with the following pdf was introduced:

$$f(y_1,\ y_2,\ y_3) = 2\phi(y_1)\phi(y_2)\phi(y_3)\Phi(ay_1y_2y_3),$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are pdf and the cdf of the standard normal distribution, respectively. Moreover, a is a nonnull real value. The conditional distribution of Y_1 given that $Y_2 = y_2$ and $Y_3 = y_3$ is skew-normal with pdf

$$f(y_1|Y_2 = y_2,\ Y_3 = y_3) = 2\phi(y_1)\Phi(cy_1),$$

where $c = ay_2y_3$ is a real value. Stein's lemma for this distribution can be obtained from Remark 3.1.

The following theorem gives a special form of Stein's lemma for truncated generalized skew-elliptical random vectors.

Theorem 3.2. Let $\mathbf{Y} \sim GSE_n(\boldsymbol{\mu},\ \mathbf{\Sigma},\ g_n,\ \pi(\cdot))$ be an n-dimensional generalized skew-elliptical random vector with probability density function (2.1). The function ϖ satisfies $E^{\mathbb{D}}[\|\ \nabla\varpi(\mathbf{Y}^*)\ \|] < \infty$, $E\,[\|\ \varpi(\mathbf{Y}^*)\nabla\mathbf{1}_{\boldsymbol{Y}^*\in\mathbb{D}}\ \|] < \infty$ and $E^{\mathbb{D}}\left[\|\ \varpi(\mathbf{X}^*)\nabla\pi\left(\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^*-\boldsymbol{\mu})\right)\|\right] < \infty$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. Furthermore, we suppose

$$\lim_{|y_k|\to\infty}\varpi\left(\boldsymbol{\mu}+\mathbf{\Sigma}^{\frac{1}{2}}\boldsymbol{y}\right)\pi(\boldsymbol{y})\mathbf{1}_{\boldsymbol{y}\in\mathbb{D}}\overline{G}_n\left(\frac{1}{2}\boldsymbol{y}^T\boldsymbol{y}\right) = 0. \tag{3.5}$$

Then

$$\begin{aligned}&E^{\mathbb{D}}[\varpi(\mathbf{Y})(\mathbf{Y}-\boldsymbol{\mu})]\\ =&\frac{-\psi'(0)}{\Pr(\boldsymbol{Y}\in\mathbb{D})}\Big\{\Pr(\boldsymbol{Y}^*\in\mathbb{D})\mathbf{\Sigma}E^{\mathbb{D}}[\nabla\varpi(\mathbf{Y}^*)]+\mathbf{\Sigma}^{\frac{1}{2}}E[\varpi(\mathbf{Y}^*)\nabla\mathbf{1}_{\boldsymbol{Y}^*\in\mathbb{D}}]\\ &+2\Pr(\boldsymbol{X}^*\in\mathbb{D})\mathbf{\Sigma}E^{\mathbb{D}}\left[\varpi(\mathbf{X}^*)\nabla\pi\left(\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^*-\boldsymbol{\mu})\right)\right]\Big\}.\end{aligned} \tag{3.6}$$

Proof. Let $\varpi(\boldsymbol{y}_{(1)}) = \varpi(\boldsymbol{y})$ in Theorem 3.1, we obtain (3.6), which completes the proof of Theorem 3.2.

Remark 3.3. Let $\pi(\cdot) = \frac{1}{2}$ in Theorem 3.2, we obtain

$$\begin{aligned}&E^{\mathbb{D}}[\varpi(\mathbf{Y})(\mathbf{Y}-\boldsymbol{\mu})]\\ =&\frac{-\psi'(0)}{\Pr(\boldsymbol{Y}\in\mathbb{D})}\left\{\Pr(\boldsymbol{Y}^*\in\mathbb{D})\mathbf{\Sigma}E^{\mathbb{D}}[\nabla\varpi(\mathbf{Y}^*)]+\mathbf{\Sigma}^{\frac{1}{2}}E[\varpi(\mathbf{Y}^*)\nabla\mathbf{1}_{\boldsymbol{Y}^*\in\mathbb{D}}]\right\},\end{aligned}$$

which is an equivalent form of Theorem 1 in Shushi [1].

4. Applications in risk theory

Let $S_{\mathbf{X}}$ denote the aggregate or portfolio risk $S_{\mathbf{X}} = X_1 + X_2 + \cdots + X_n$, the risk allocation for the conditional tail expectation (CTE) is a rule that decomposes the CTE of $S_{\mathbf{X}}$ to each X_i such that $E[S_{\mathbf{X}}|S_{\mathbf{X}} > s_p] = \sum_{j=1}^{n}\rho(X_j|\Omega)$, where $\rho(X_j|\Omega)$ stands for the allocated risk to the jth line, and $\Omega =$

$\{X_1,\ X_2,\ X_n\}$ represents the whole portfolio. Since $\rho(X_j|\Omega) = E[X_j|S_{\mathbf{X}} > s_p]$, so that (see Kim et al. [15])

$$E[S_{\mathbf{X}}|S_{\mathbf{X}} > s_p] = \sum_{j=1}^{n} E[X_j|S_{\mathbf{X}} > s_p].$$

We now derive $E[S_{\mathbf{Y}}|S_{\mathbf{Y}} > s_p]$ for generalized skew-elliptical random vectors.

Theorem 4.1. Consider $\mathbf{Y} \sim GSE_n(\boldsymbol{\mu},\ \boldsymbol{\Sigma},\ g_n,\ \pi(\cdot))$ is an n-dimensional generalized skew-elliptical random vector. Then the CTE allocation for $S_{\mathbf{Y}}$ is given by

$$\begin{aligned} E[S_{\mathbf{Y}}|S_{\mathbf{Y}} > s_p] &= \boldsymbol{e}^T\boldsymbol{\mu} - \frac{\psi'(0)}{\Pr(S_{\mathbf{Y}} > s_p)}\Big\{\boldsymbol{e}^T\boldsymbol{\Sigma}^{\frac{1}{2}}E\left[\nabla\mathbf{1}_{S_{\mathbf{Y}^*}>s_p}\right] \\ &+ 2\Pr(S_{\mathbf{X}^*} > s_p)\boldsymbol{e}^T\boldsymbol{\Sigma}E\left[\nabla\pi\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^* - \boldsymbol{\mu})\right)|S_{\mathbf{X}^*} > s_p\right]\Big\}, \end{aligned} \tag{4.1}$$

where $S_{\mathbf{Y}} = Y_1 + Y_2 + \cdots + Y_n$, and $\boldsymbol{e} = (1,\ 1,\ \cdots,\ 1)^T$ is an $n \times 1$ vector whose elements are all equal to 1.

Proof. Let $\varpi(\mathbf{Y}) = 1$, and $\mathbf{Y} \in \mathbb{D}$ subject to $S_{\mathbf{Y}^*} > s_p$ in Theorem 3.2, we can obtain

$$\begin{aligned} E[\mathbf{Y}|S_{\mathbf{Y}} > s_p] &= \boldsymbol{\mu} - \frac{\psi'(0)}{\Pr(S_{\mathbf{Y}} > s_p)}\Big\{\boldsymbol{\Sigma}^{\frac{1}{2}}E\left[\nabla\mathbf{1}_{S_{\mathbf{Y}^*}>s_p}\right] \\ &+ 2\Pr(S_{\mathbf{X}^*} > s_p)\boldsymbol{\Sigma}E\left[\nabla\pi\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^* - \boldsymbol{\mu})\right)|S_{\mathbf{X}^*} > s_p\right]\Big\}. \end{aligned} \tag{4.2}$$

Since $E[S_{\mathbf{Y}}|S_{\mathbf{Y}} > s_p] = \boldsymbol{e}^T E[\mathbf{Y}|S_{\mathbf{Y}} > s_p]$, so we can get formula (4.1). This completes the proof of Theorem 4.1.

In addition, we introduce two CTE measure as follows (see Cousin and Bernardino [16] or Shushi [1]).

The lower-orthant CTE at probability level q

$$\underline{MCTE}_q(\mathbf{X}) = E[\mathbf{X}|F(\mathbf{X}) \geq q],\ q \in (0,\ 1).$$

The upper-orthant CTE at probability level q

$$\overline{MCTE}_q(\mathbf{X}) = E\left[\mathbf{X}|\overline{F}(\mathbf{X}) \leq 1 - q\right],\ q \in (0,\ 1).$$

Here F is distribution function of $\mathbf{X}$, and $\overline{F}$ is survival function of $\mathbf{X}$.

We now give lower-orthant CTE and upper-orthant CTE for generalized skew-elliptical random vectors.

Theorem 4.2. Consider $\mathbf{Y} \sim GSE_n(\boldsymbol{\mu},\ \boldsymbol{\Sigma},\ g_n,\ \pi(\cdot))$ is an n-dimensional generalized skew-elliptical random vector. Then

$$\begin{aligned} \underline{MCTE}_q(\mathbf{Y}) &= \boldsymbol{\mu} - \frac{\psi'(0)}{\Pr(F(\mathbf{Y}) \geq q)}\Big\{\boldsymbol{\Sigma}^{\frac{1}{2}}E\left[\nabla\mathbf{1}_{F(\mathbf{Y}^*)\geq q}\right] \\ &+ 2\Pr(F(\mathbf{X}^*) \geq q)\boldsymbol{\Sigma}E\left[\nabla\pi\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^* - \boldsymbol{\mu})\right)|F(\mathbf{X}^*) \geq q\right]\Big\}, \end{aligned} \tag{4.3}$$

$$\overline{MCTE}_q(\mathbf{Y}) = \boldsymbol{\mu} - \frac{\psi'(0)}{\Pr(\overline{F}(\mathbf{Y}) \leq 1 - q)}\Big\{\boldsymbol{\Sigma}^{\frac{1}{2}}E\left[\nabla\mathbf{1}_{\overline{F}(\mathbf{Y}^*)\leq 1-q}\right]$$

$$+2\Pr(\overline{F}(\mathbf{X}^*)\leq 1-q)\mathbf{\Sigma}E\left[\nabla\pi\left(\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^*-\boldsymbol{\mu})\right)|\overline{F}(\mathbf{X}^*)\leq 1-q\right]\bigg\}. \tag{4.4}$$

Proof. Let $\varpi(\mathbf{Y})=1$, and $\mathbf{Y}\in\mathbb{D}$ subject to $F(\mathbf{Y})\geq q$ in Theorem 3.2, we directly obtain (4.3). Let $\varpi(\mathbf{Y})=1$, and $\mathbf{Y}\in\mathbb{D}$ submit to $\overline{F}(\mathbf{Y})\leq 1-q$ in Theorem 3.2, we get (4.4). This completes the proof of Theorem 4.2.

The truncated version of Wang's premium can be defined, as follows (see Shushi [1]):

$$\pi_{q,\,\lambda}(X_i,\ \mathbf{X})=\frac{E\left[X_i\exp\left\{\boldsymbol{\lambda}^T\mathbf{X}\right\}|F(\mathbf{X})\geq q\right]}{E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{X}\right\}|F(\mathbf{X})\geq q\right]},\ \lambda_i\geq 0,$$

with the tuned exponential tilting $\exp\left\{\boldsymbol{\lambda}^T\mathbf{X}\right\}$ and $\boldsymbol{\lambda}=(\lambda_1,\ \lambda_2,\cdots,\lambda_n)^T\in\mathbb{R}^n$, $q\in(0,\ 1)$, where moment generating function is exists, i.e.,

$$E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{X}\right\}\right]<\infty. \tag{4.5}$$

We now derive $\pi_{q,\,\lambda}(\mathbf{Y})$ for generalized skew-elliptical random vectors.

Theorem 4.3. Consider $\mathbf{Y}\sim GSE_n(\boldsymbol{\mu},\ \mathbf{\Sigma},\ g_n,\ \pi(\cdot))$ is an n-dimensional generalized skew-elliptical random vector, and satisfying formula (4.5). Then

$$\begin{aligned}&\pi_{q,\,\lambda}(\mathbf{Y})\\ =&\boldsymbol{\mu}-\frac{\psi'(0)}{\Pr(F(\mathbf{Y})\geq q)}\left\{\Pr(F(\boldsymbol{Y}^*)\geq q)\mathbf{\Sigma}\zeta+\mathbf{\Sigma}^{\frac{1}{2}}\eta+2\Pr(F(\mathbf{X}^*)\geq q)\mathbf{\Sigma}\xi\right\},\end{aligned} \tag{4.6}$$

where

$$\zeta=\frac{E\left[\boldsymbol{\lambda}\exp\left\{\boldsymbol{\lambda}^T\mathbf{Y}^*\right\}|F(\mathbf{Y}^*)\geq q\right]}{E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{Y}\right\}|F(\mathbf{Y})\geq q\right]},\ \eta=\frac{E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{Y}^*\right\}\nabla\mathbf{1}_{F(\mathbf{Y}^*)\geq q}\right]}{E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{Y}\right\}|F(\mathbf{Y})\geq q\right]},$$
$$\textit{and }\xi=\frac{E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{X}^*\right\}\nabla\pi\left(\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^*-\boldsymbol{\mu})\right)|F(\mathbf{X}^*)\geq q\right]}{E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{Y}\right\}|F(\mathbf{Y})\geq q\right]}.$$

Proof. Substituting $\varpi(\mathbf{Y})=\exp\left\{\boldsymbol{\lambda}^T\mathbf{Y}\right\}$ into Theorem 3.2, we obtain

$$\begin{aligned}E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{Y}\right\}(\mathbf{Y}-\boldsymbol{\mu})|F(\mathbf{Y})\geq q\right]&=\frac{-\psi'(0)}{\Pr(F(\mathbf{Y})\geq q)}\bigg\{\Pr(F(\boldsymbol{Y}^*)\geq q)\\ &\cdot\mathbf{\Sigma}E\left[\boldsymbol{\lambda}\exp\left\{\boldsymbol{\lambda}^T\mathbf{Y}^*\right\}|F(\mathbf{Y}^*)\geq q\right]+\mathbf{\Sigma}^{\frac{1}{2}}E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{Y}^*\right\}\nabla\mathbf{1}_{F(\mathbf{Y}^*)\geq q}\right]\\ &+2\Pr(F(\mathbf{X}^*)\geq q)\mathbf{\Sigma}E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{X}^*\right\}\nabla\pi\left(\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^*-\boldsymbol{\mu})\right)|F(\mathbf{X}^*)\geq q\right]\bigg\},\end{aligned}$$

so that

$$\begin{aligned}E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{Y}\right\}\mathbf{Y}|F(\mathbf{Y})\geq q\right]&=\frac{-\psi'(0)}{\Pr(F(\mathbf{Y})\geq q)}\bigg\{\Pr(F(\boldsymbol{Y}^*)\geq q)\\ &\cdot\mathbf{\Sigma}E\left[\boldsymbol{\lambda}\exp\left\{\boldsymbol{\lambda}^T\mathbf{Y}^*\right\}|F(\mathbf{Y}^*)\geq q\right]+\mathbf{\Sigma}^{\frac{1}{2}}E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{Y}^*\right\}\nabla\mathbf{1}_{F(\mathbf{Y}^*)\geq q}\right]\end{aligned}$$

$$+2\Pr(F(\mathbf{X}^*)\geq q)\boldsymbol{\Sigma}E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{X}^*\right\}\nabla\pi\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^*-\boldsymbol{\mu})\right)|F(\mathbf{X}^*)\geq q\right]\Big\}$$
$$+\boldsymbol{\mu}E\left[\exp\left\{\boldsymbol{\lambda}^T\mathbf{Y}\right\}|F(\mathbf{Y})\geq q\right].$$

Therefore we obtain (4.6), which completes the proof of Theorem 4.3.

Landsman et al. [11] defined the multivariate tail conditional expectation (MTCE) measure

$$MTCE_{\boldsymbol{q}}(\mathbf{Y})=E[\mathbf{Y}|\mathbf{Y}>VaR_{\boldsymbol{q}}(\mathbf{Y})]=E[\mathbf{Y}|Y_1>VaR_{q_1}(Y_1),\cdots,Y_n>VaR_{q_n}(Y_n)],$$

where $VaR_{\boldsymbol{q}}(\mathbf{Y})=(VaR_{q_1}(Y_1),\ VaR_{q_2}(Y_2),\cdots,VaR_{q_n}(Y_n))^T$, $VaR_{q_i}(Y_i)=y_{q_i}$ is the value at risk of Y_i under the q_i-th quantile, $q_i\in[0,\ 1)$, $i=1,\ 2,\cdots,n$, and $\boldsymbol{q}=(q_1,\ q_2,\cdots,q_n)$. In addition, we define multivariate tail covariance matrix

$$MTCov_{\boldsymbol{q}}(\mathbf{Y})=E\left[(\mathbf{Y}-MTCE_{\boldsymbol{q}}(\mathbf{Y}))(\mathbf{Y}-MTCE_{\boldsymbol{q}}(\mathbf{Y}))^T|\mathbf{Y}>VaR_{\boldsymbol{q}}(\mathbf{Y})\right].$$

The following two theorems give $MTCE_{\boldsymbol{q}}(\mathbf{Y})$ and $MTCov_{\boldsymbol{q}}(\mathbf{Y})$, respectively.

Theorem 4.4. Consider $\mathbf{Y}\sim GSE_n(\boldsymbol{\mu},\ \boldsymbol{\Sigma},\ g_n,\ \pi(\cdot))$ is an n-dimensional generalized skew-elliptical random vector. Then

$$MCTE_{\boldsymbol{q}}(\mathbf{Y})=\boldsymbol{\mu}-\frac{\psi'(0)}{\Pr(\mathbf{Y}>VaR_{\boldsymbol{q}}(\mathbf{Y}))}\Big\{\boldsymbol{\Sigma}^{\frac{1}{2}}E\left[\nabla\mathbf{1}_{\mathbf{Y}^*>VaR_{\boldsymbol{q}}(\mathbf{Y}^*)}\right]$$
$$+2\Pr(\mathbf{X}^*>VaR_{\boldsymbol{q}}(\mathbf{X}^*))\boldsymbol{\Sigma}E\left[\nabla\pi\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^*-\boldsymbol{\mu})\right)|\mathbf{X}^*>VaR_{\boldsymbol{q}}(\mathbf{X}^*)\right]\Big\}.\tag{4.7}$$

Proof. Let $\varpi(\mathbf{Y})=1$, and $\mathbf{Y}\in\mathbb{D}$ subject to $\mathbf{Y}>VaR_{\boldsymbol{q}}(\mathbf{Y})$ in Theorem 3.2, we can obtain (4.7). This is completes proof of Theorem 4.4.

Theorem 4.5. Consider $\mathbf{Y}\sim GSE_n(\boldsymbol{\mu},\ \boldsymbol{\Sigma},\ g_n,\ \pi(\cdot))$ is an n-dimensional generalized skew-elliptical random vector. Then

$$MTCov_{\boldsymbol{q}}(\mathbf{Y})=(a_{i,j})_{i,j=1}^n,\tag{4.8}$$

where

$$a_{i,j}=\left[\mu_j-\boldsymbol{e}_j^TMCTE_{\boldsymbol{q}}(\mathbf{Y})\right]\boldsymbol{e}_i^TMCTE_{\boldsymbol{q}}(\mathbf{Y})$$
$$-\frac{\psi'(0)}{\Pr(\mathbf{Y}>VaR_{\boldsymbol{q}}(\mathbf{Y}))}\Big\{\Pr(\mathbf{Y}^*>VaR_{\boldsymbol{q}}(\mathbf{Y}^*))\sigma_{i,j}+\boldsymbol{e}_j^T\boldsymbol{\Sigma}^{\frac{1}{2}}E\left[Y_i^*\nabla\mathbf{1}_{\mathbf{Y}^*>VaR_{\boldsymbol{q}}(\mathbf{Y}^*)}\right]$$
$$+2\Pr(\mathbf{X}^*>VaR_{\boldsymbol{q}}(\mathbf{X}^*))\boldsymbol{e}_j^T\boldsymbol{\Sigma}E\left[X_i^*\nabla\pi\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^*-\boldsymbol{\mu})\right)|\mathbf{X}^*>VaR_{\boldsymbol{q}}(\mathbf{X}^*)\right]\Big\},$$

and $\boldsymbol{e}_i=(0,\cdots,0,\ 1,\ 0,\cdots,0)^T$ is an $n\times1$ vector whose elements are all equal to zero except i-th element, which is equal to 1.

Proof. Let $\varpi(\mathbf{Y})=Y_i$, and $\mathbf{Y}\in\mathbb{D}$ subject to $\mathbf{Y}>VaR_{\boldsymbol{q}}(\mathbf{Y})$ in Theorem 3.2, we have

$$E[Y_i(\mathbf{Y}-\boldsymbol{\mu})|\mathbf{Y}>VaR_{\boldsymbol{q}}(\mathbf{Y})]$$
$$=\frac{-\psi'(0)}{\Pr(\mathbf{Y}>VaR_{\boldsymbol{q}}(\mathbf{Y}))}\Big\{\Pr(\mathbf{Y}^*>VaR_{\boldsymbol{q}}(\mathbf{Y}^*))\boldsymbol{\Sigma}\boldsymbol{e}_i+\boldsymbol{\Sigma}^{\frac{1}{2}}E\left[Y_i^*\nabla\mathbf{1}_{\mathbf{Y}^*>VaR_{\boldsymbol{q}}(\mathbf{Y}^*)}\right]$$

$$+ 2\Pr(\mathbf{X}^* > VaR_{\boldsymbol{q}}(\mathbf{X}^*))\boldsymbol{\Sigma} E\left[X_i^* \nabla \pi \left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^* - \boldsymbol{\mu})\right) | \mathbf{X}^* > VaR_{\boldsymbol{q}}(\mathbf{X}^*)\right]\Big\}.$$

Multiplying $E[Y_i(\mathbf{Y} - \boldsymbol{\mu})|\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y})]$ by $\boldsymbol{e}_j^T$ from the left, we get

$$\begin{aligned}
&E[Y_i(Y_j - \mu_j)|\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y})] \\
=&\frac{-\psi'(0)}{\Pr(\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y}))}\Big\{ \Pr(\mathbf{Y}^* > VaR_{\boldsymbol{q}}(\mathbf{Y}^*))\sigma_{i,j} + \boldsymbol{e}_j^T \boldsymbol{\Sigma}^{\frac{1}{2}} E\left[Y_i^* \nabla \mathbf{1}_{\mathbf{Y}^* > VaR_{\boldsymbol{q}}(\mathbf{Y}^*)}\right] \\
&+ 2\Pr(\mathbf{X}^* > VaR_{\boldsymbol{q}}(\mathbf{X}^*))\boldsymbol{e}_j^T \boldsymbol{\Sigma} E\left[X_i^* \nabla \pi \left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^* - \boldsymbol{\mu})\right) | \mathbf{X}^* > VaR_{\boldsymbol{q}}(\mathbf{X}^*)\right]\Big\}.
\end{aligned}$$

So that

$$\begin{aligned}
&E[Y_i Y_j|\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y})] = \mu_j E[Y_i|\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y})] \\
&- \frac{\psi'(0)}{\Pr(\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y}))}\Big\{ \Pr(\mathbf{Y}^* > VaR_{\boldsymbol{q}}(\mathbf{Y}^*))\sigma_{i,j} + \boldsymbol{e}_j^T \boldsymbol{\Sigma}^{\frac{1}{2}} E\left[Y_i^* \nabla \mathbf{1}_{\mathbf{Y}^* > VaR_{\boldsymbol{q}}(\mathbf{Y}^*)}\right] \\
&+ 2\Pr(\mathbf{X}^* > VaR_{\boldsymbol{q}}(\mathbf{X}^*))\boldsymbol{e}_j^T \boldsymbol{\Sigma} E\left[X_i^* \nabla \pi \left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^* - \boldsymbol{\mu})\right) | \mathbf{X}^* > VaR_{\boldsymbol{q}}(\mathbf{X}^*)\right]\Big\} \\
&= \mu_j \boldsymbol{e}_i^T MCTE_{\boldsymbol{q}}(\mathbf{Y}) \\
&- \frac{\psi'(0)}{\Pr(\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y}))}\Big\{ \Pr(\mathbf{Y}^* > VaR_{\boldsymbol{q}}(\mathbf{Y}^*))\sigma_{i,j} + \boldsymbol{e}_j^T \boldsymbol{\Sigma}^{\frac{1}{2}} E\left[Y_i^* \nabla \mathbf{1}_{\mathbf{Y}^* > VaR_{\boldsymbol{q}}(\mathbf{Y}^*)}\right] \\
&+ 2\Pr(\mathbf{X}^* > VaR_{\boldsymbol{q}}(\mathbf{X}^*))\boldsymbol{e}_j^T \boldsymbol{\Sigma} E\left[X_i^* \nabla \pi \left(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{X}^* - \boldsymbol{\mu})\right) | \mathbf{X}^* > VaR_{\boldsymbol{q}}(\mathbf{X}^*)\right]\Big\},
\end{aligned} \tag{4.9}$$

where in the last line we have used the following relation

$$E[Y_i|\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y})] = \boldsymbol{e}_i^T MCTE_{\boldsymbol{q}}(\mathbf{Y}),\ i = 1,\ 2, \cdots, n. \tag{4.10}$$

While

$$\begin{aligned}
a_{i,j} &= E[(Y_i - p_i)(Y_j - p_j)|\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y})] \\
&= E\left[\left(Y_i - \boldsymbol{e}_i^T MCTE_{\boldsymbol{q}}(\mathbf{Y})\right)\left(Y_j - \boldsymbol{e}_j^T MCTE_{\boldsymbol{q}}(\mathbf{Y})\right)|\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y})\right] \\
&= E\left[Y_i Y_j|\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y})\right] - \boldsymbol{e}_j^T MCTE_{\boldsymbol{q}}(\mathbf{Y}) E[Y_i|\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y})] \\
&\quad - \boldsymbol{e}_i^T MCTE_{\boldsymbol{q}}(\mathbf{Y}) E[Y_j|\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y})] + \boldsymbol{e}_i^T MCTE_{\boldsymbol{q}}(\mathbf{Y}) \boldsymbol{e}_j^T MCTE_{\boldsymbol{q}}(\mathbf{Y}) \\
&= E[Y_i Y_j|\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y})] - \boldsymbol{e}_j^T MCTE_{\boldsymbol{q}}(\mathbf{Y}) \boldsymbol{e}_i^T MCTE_{\boldsymbol{q}}(\mathbf{Y}) \\
&\quad - \boldsymbol{e}_i^T MCTE_{\boldsymbol{q}}(\mathbf{Y}) \boldsymbol{e}_j^T MCTE_{\boldsymbol{q}}(\mathbf{Y}) + \boldsymbol{e}_i^T MCTE_{\boldsymbol{q}}(\mathbf{Y}) \boldsymbol{e}_j^T MCTE_{\boldsymbol{q}}(\mathbf{Y}) \\
&= E[Y_i Y_j|\mathbf{Y} > VaR_{\boldsymbol{q}}(\mathbf{Y})] - \boldsymbol{e}_j^T MCTE_{\boldsymbol{q}}(\mathbf{Y}) \boldsymbol{e}_i^T MCTE_{\boldsymbol{q}}(\mathbf{Y}),
\end{aligned} \tag{4.11}$$

where $MCTE_{\boldsymbol{q}}(\mathbf{Y}) = \boldsymbol{p} = (p_1,\ p_2, \cdots, p_n)^T$, $p_i = \boldsymbol{e}_i^T MCTE_{\boldsymbol{q}}(\mathbf{Y})$, $i = 1,\ 2, \cdots, n$, and in the fourth equality we have used (4.10).
Using relations (4.9) and (4.11), we obtain (4.8). This is completes proof of Theorem 4.5.

5. Conclusions

In this paper, we extended Stein's lemma in Shushi [1], deriving two Stein's lemmas for truncated generalized skew-elliptical random vectors. Moreover, as applications in risk theory, we obtained the expressions for the conditional tail expectation (CTE) allocation, the lower-orthant CTE at probability level q, the upper-orthant CTE at probability level q, the truncated version of Wang's premium, the multivariate tail conditional expectation and the multivariate tail covariance matrix measures. Our results also possibly apply to model of financial returns, for example, the multivariate SGARCH model proposed by De Luca, Genton and Loperfido [17] and further studied by De Luca and Loperfido [18]. We hope that these important problems can be addressed in future research.

Acknowledgments

The authors would like to thank two anomymous referees for their very useful comments which greatly helped in improving the quality of the present paper. The research was supported by the National Natural Science Foundation of China (No. 11171179, 11571198, 11701319).

Conflicts of interest

The authors declare that they have no conflicts of interest.

References

1. T. Shushi, *Stein's lemma for truncated elliptical random vectors*, Stat. Probabil. Lett., **137** (2018), 297–303.
2. C. Adcock, Z. Landsman, T. Shushi, *Stein's lemma for generalized skew-elliptical random vectors*, Commun. Stat-Theor. M., 2019.
3. C. M. Stein, *Estimation of the mean of a multivariate normal distribution*, The Annals of Statistics, **9** (1981), 1135–1151.
4. Z. Landsman, *On the generalization of Stein's lemma for elliptical class of distributions*, Stat. Probabil. Lett., **76** (2006), 1012–1016.
5. Z. Landsman, J. Nešlehová, *Stein's lemma for elliptical random vectors*, J. Multivariate Anal., **99** (2008), 912–927.
6. Z. Landsman, S. Vanduffel, J. Yao, *A note on Stein's lemma for multivariate elliptical distributions*, J. Stat. Plan. Infer., **143** (2013), 2016–2022.
7. C. J. Adcock, K. Shutes, *On the multivariate extended skew-normal, normal-exponential, and normal-gamma distributions*, Journal of Statistical Theory and Practice, **6** (2012), 636–664.
8. C. J. Adcock, *Mean-variance-skewness efficient surfaces, Stein's lemma and the multivariate extended skew-Student distribution*, Eur. J. Oper. Res., **234** (2014), 392–401.
9. J. S. Liu, *Siegel's formula via Stein's identities*, Stat. Probabil. Lett., **21** (1994), 247–251.

10. K. C. Li, *On principal Hessian directions for data visualization and dimension reduction: Another application of Stein's lemma*, J. Am. Stat. Assoc., **87** (1992), 1025–1039.

11. Z. Landsman, U. Makov, T. Shushi, *A multivariate tail covariance measure for elliptical distributions*, Insurance: Mathematics and Economics, **81** (2018), 27–35.

12. K. T. Fang, S. Kotz, K. W. Ng, *Symmetric Multivariate and Related Distributions*, CRC Press, New York, 1990.

13. Z. M. Landsman, E. A. Valdez, *Tail conditional expectations for elliptical distributions*, North American Actuarial Journal, **7** (2003), 55–71.

14. N. Loperfido, *Skewness-based projection pursuit: A computational approach*, Comput. Stat. Data An., **120** (2018), 42–57.

15. J. H. T. Kim, S. Y. Kim, *Tail risk measures and risk allocation for the class of multivariate normal mean-variance mixture distributions*, Insurance: Mathematics and Economics, **86** (2019), 145–157.

16. A. Cousin, E. D. Bernardino, *On multivariate extensions of conditional-tail-expectation*, Insurance: Mathematics and Economics, **55** (2014), 272–282.

17. G. De Luca, M. Genton, N. Loperfido, *A multivariate skew-GARCH model*, In: *Econometric Analysis of Financial and Economic Time Series*, Emerald Group Publishing Limited, Bingley, 2006, 33–57.

18. G. De Luca, N. Loperfido, *Modelling multivariate skewness in financial returns: a SGARCH approach*, The European Journal of Finance, **21** (2015), 1113–1131.

AIMS Press