



*Research article*

## Blow-up analysis of a nonlinear pseudo-parabolic equation with memory term

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**Abstract:** This paper deals with the blow-up phenomena for a nonlinear pseudo-parabolic equation with a memory term  $u_t - \Delta u - \Delta u_t + \int_0^t g(t - \tau)\Delta u(\tau)d\tau = |u|^p u$  in a bounded domain, with the initial and Dirichlet boundary conditions. We first obtain the finite time blow-up results for the solutions with initial data at non-positive energy level as well as arbitrary positive energy level, and give some upper bounds for the blow-up time  $T^*$  depending on the sign and size of initial energy  $E(0)$ . In addition, a lower bound for the life span  $T^*$  is derived by means of a differential inequality technique if blow-up does occur.

**Keywords:** pseudo-parabolic equation; memory term; blow up; upper bound; lower bound

**Mathematics Subject Classification:** 35B44, 35D40, 35K61, 35K70

### 1. Introduction

Studied here is an initial boundary value problem for the nonlinear pseudo-parabolic equation with memory term

$$u_t - \Delta u - \Delta u_t + \int_0^t g(t - \tau)\Delta u(\tau)d\tau = |u|^p u, \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \tag{1.3}$$

where the exponent  $p > 0$ , and  $\Omega$  with smooth boundary  $\partial\Omega$  is a bounded domain in  $\mathbb{R}^n$  ( $n \geq 1$ ). The relaxation function  $g$  represents the kernel of memory term, satisfying certain conditions to be

specified later. The reason why this type of equations is so attractive is that it has extensive physical background, which appears in the study of heat conduction and viscous flow in materials with memory, electric signals in telegraph line with nonlinear damping [1], vibration of nonlinear viscoelastic rod [2], bidirectional nonlinear shallow water waves [3], and the velocity evolution of ion-acoustic waves in a collision less plasma [4] and so on.

The Eq. (1.1) describes many important and famous mathematical models which have extensive theoretical connotation. For example, in the absence of the memory term (i.e.,  $g \equiv 0$ ), the Eq. (1.1) becomes a semilinear pseudo-parabolic equation

$$u_t - \Delta u - \Delta u_t = f(u). \quad (1.4)$$

As is known to all, the nonlinear pseudo-parabolic equations appear in the dynamics of the thermodynamics, hydrodynamics and filtration theory, etc (see [1, 5, 6]). There is a lot of literature on the study of such kinds of semilinear pseudo-parabolic equation, such as the existence and uniqueness of certain solutions in [1, 7, 8], blow-up results in [8–10], asymptotic behavior in [8, 11] and so on. In [8], Xu etc. derived the finite time blow-up results in  $H_0^1(\Omega)$ -norm. Luo [9] derived the upper bound and lower bound for the blow-up time under some relative assumptions. In recent years, the qualitative properties of parabolic and pseudo-parabolic equations (systems) with nonlinear gradient terms have been attracting some authors' attention and many interesting results have also been obtained. For example, Dong and Zhou [12] considered a class of parabolic system with different gradient coefficients and coupled nonlocal sources, of which the global existence and a series of finite time blow-up results were discussed by using comparison principle and asymptotic analysis methods. In [13], Marras etc. gave some estimates about the upper and lower bounds of blow-up time for the pseudo-parabolic equation (system) with nonlinear gradient terms under suitable conditions. More results on these types of equations (systems) can be seen in [1, 12, 13, 27] and the references therein.

In [14], Gripenberg investigated the initial boundary value problem for a volterra integro-differential parabolic equation

$$u_t = \int_0^t k(t-s)\sigma(u_x(x,s))_x ds + f(x,t). \quad (1.5)$$

He obtained the global existence of a strong solution under certain conditions.

In the absence of dispersion term  $\Delta u_t$ , the model (1.1) is reduced to the following nonlinear parabolic equation

$$u_t - \Delta u = \int_0^t b(t-\tau)\Delta u(\tau)d\tau + f(u). \quad (1.6)$$

Yin [15] discussed the initial boundary value problem of Eq. (1.6) and obtained the global existence of a classical solution under a one-sided growth condition. In [16], Messaoudi replaced the memory term  $b(t-\tau)$  by  $-g(t-\tau)$  in Eq. (1.6). He proved the finite time blow-up of the solutions with negative and vanishing initial energy.

As far as we know, there are few results of viscoelastic pseudo-parabolic equations. In [17, 18], Shang and Guo studied the initial boundary value problem and initial value problem for a class of integro-differential equations of pseudo-parabolic type. They proved the global existence, uniqueness,

asymptotic behavior of solutions for the problem, and gave the sufficient conditions for the nonexistence of global solutions in one dimension case. Ptashnyk [19] studied the initial boundary value problem of degenerate viscoelastic quasilinear pseudo-parabolic equations. For the initial value problem of a class of degenerate viscoelastic pseudo-parabolic equations, Carroll and Showalter [20] represented some background in physics and applied sciences, and introduced a number of computing techniques for the study of the pseudo-parabolic equations with different types of memory terms. Di and Shang [21] considered the initial boundary value problem of Eq. (1.1). They obtained the global existence and finite time blow-up of the solutions with the low initial energy  $E(0) < d_k$  by using the potential well method. Later, the results of [21] were improved by Sun, Liu and Wu [22], to certain solutions with the low initial energy  $J(u_0) < d_\infty$ . Here, the functionals  $E(t)$ ,  $J(u(t))$  and  $d_k$  denote the total energy, potential energy and potential well depth associated with problem (1.1)–(1.3), respectively. About the application of the potential well theory to the study of evolution equations, we also refer the reader to see [21–25] and references therein.

In this paper, we further extend and improve the blow-up results obtained in [21, 22], the idea of which are based on the lemmas of Li and Tsai [26] (up to a appropriate modification). It is worth mentioning that the method of this paper is not directly related to the potential well theory, we shall discuss the blow-up phenomena of the solutions for problem (1.1)–(1.3) under the initial energy  $E(0) < 0$ ,  $0 \leq E(0) < \frac{\eta}{p}E_1$  and  $\frac{\eta}{p}E_1 \leq E(0) < \frac{\|u_0\|_{H^1}^2}{\gamma}$ , respectively. Moreover, we also establish the corresponding upper bounds for blow-up time  $T^*$  at the three different initial energy levels, which depend on the sign and size of initial energy  $E(0)$ . Finally, a lower bound of blow-up time  $T^*$  is obtained by applying a differential inequality technique, if blow-up occurs to the initial boundary problem (1.1)–(1.3).

The outline of this paper is as follows: In Section 2, we introduce some notations, functionals and important lemmas to be used throughout this article. Section 3 is devoted to study the finite time blow-up results for the solutions with initial data at non-positive energy level as well as arbitrary positive energy level, and give some upper bounds for blow-up time  $T^*$  depending on the sign and size of initial energy  $E(0)$ . In Section 4, a lower bound for the life span  $T^*$  is derived by means of a differential inequality technique if blow-up does occur.

## 2. Preliminaries

Our attention of this section is to give some notations, functionals and important lemmas in order to state the main results of this paper.

Throughout the whole paper, the following abbreviations are used for precise statement:

$$L^p(\Omega) = L^p, \quad \|u\|_{L^p(\Omega)} = \|u\|_p = \left( \int_{\Omega} |u|^p dx \right)^{\frac{1}{p}},$$

$$H_0^1(\Omega) = W_0^{1,2}(\Omega) = H_0^1, \quad \|u\|_{H_0^1(\Omega)} = \|u\|_{H_0^1} = \left( \int_{\Omega} |u|^2 + |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

And  $(u, v) = \int_{\Omega} uv dx$  for  $L^2$ -inner product will also denotes the notation of duality pairing between dual spaces. The Sobolev space  $H_0^1(\Omega)$  will be defined with the inner product  $\langle u, v \rangle = \int_{\Omega} uv dx + \int_{\Omega} \nabla u \cdot \nabla v dx$ .

The following conditions are the essential hypotheses to obtain the main results of this paper.

- (i)  $0 < p < \infty$  if  $n = 1, 2$  and  $0 < p \leq \frac{4}{n-2}$  if  $n \geq 3$ .  
(ii) The relaxation function  $g: [0, \infty) \rightarrow (0, \infty)$  is a  $C^1$  function satisfying

$$g'(t) \leq 0, \quad 0 < l = 1 - \int_0^\infty g(\tau) d\tau \leq 1 - \int_0^t g(\tau) d\tau = k(t), \quad (2.1)$$

and

$$\int_0^\infty g(\tau) d\tau < \frac{2\delta}{1 + 2\delta}, \quad (2.2)$$

where  $0 < \delta \leq \frac{p}{2}$ .

We first start with the following existence and uniqueness of local solution for problem (1.1)–(1.3), which can be obtained by using Faedo-Galerkin methods and Contraction Mapping Principle as in [1, 27, 28]. Here, the proof is thus omitted.

**Theorem 2.1.** *Let the conditions (i), (2.1) hold and  $u_0 \in H_0^1(\Omega)$ . Then there exists a unique local solution  $u$  of problem (1.1)–(1.3) such that*

$$u \in C([0, T]; H_0^1(\Omega)), \quad u_t \in C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H_0^1(\Omega)), \quad (2.3)$$

for some  $T > 0$ .

To obtain the results of this paper, we define the energy functional for the solution  $u$  of problem (1.1)–(1.3) in the form

$$E(t) := E(u(t)) = \int_0^t \|u_\tau\|_{H^1}^2 d\tau + \frac{1}{2}(g \diamond \nabla u)(t) + \frac{1}{2}\left(1 - \int_0^t g(\tau) d\tau\right) \|\nabla u\|_2^2 - \frac{1}{p+2} \|u\|_{p+2}^{p+2}, \quad (2.4)$$

where  $(g \diamond \nabla u)(t) = \int_0^t g(t - \tau) \|\nabla u(t) - \nabla u(\tau)\|_2^2 d\tau$ . Furthermore, multiplying Eq. (1.1) by  $u_t$  and integrating it over  $\Omega$ , we easily deduce from (i) and (2.1) that

$$E'(t) = \frac{1}{2}(g' \diamond \nabla u)(t) - \frac{1}{2}g(t) \|\nabla u\|_2^2 \leq 0. \quad (2.5)$$

**Remark 2.1.** In view of (2.1), (2.4) and Sobolev inequality, we discover that

$$\begin{aligned} E(t) &\geq \frac{1}{2}(g \diamond \nabla u)(t) + \frac{1}{2}k(t) \|\nabla u\|_2^2 - \frac{1}{p+2} \|u\|_{p+2}^{p+2} \\ &\geq \frac{1}{2}(g \diamond \nabla u)(t) + \frac{1}{2}l \|\nabla u\|_2^2 - \frac{B^{p+2}}{p+2} \|\nabla u\|_2^{p+2} \\ &\geq \frac{1}{2} \left[ (g \diamond \nabla u)(t) + l \|\nabla u\|_2^2 \right] - \frac{B^{p+2}}{(p+2)l^{\frac{p+2}{2}}} \left[ l \|\nabla u\|_2^2 + (g \diamond \nabla u)(t) \right]^{\frac{p+2}{2}} \\ &= h \left( \left( l \|\nabla u\|_2^2 + (g \diamond \nabla u)(t) \right)^{\frac{1}{2}} \right), \end{aligned} \quad (2.6)$$

where the function  $h(\lambda) = \frac{1}{2}\lambda^2 - \frac{B^{p+2}}{(p+2)l^{\frac{p+2}{2}}}\lambda^{p+2}$ ,  $\lambda = \left( l \|\nabla u\|_2^2 + (g \diamond \nabla u)(t) \right)^{\frac{1}{2}} > 0$ ,  $B$  is the optimal constant satisfying the Sobolev embedding inequality  $\|u\|_{p+2} \leq B \|\nabla u\|_2$ . A direct computation gives that  $h(\lambda)$  is

increasing for  $0 < \lambda < \lambda_1$ , decreasing for  $\lambda > \lambda_1$  and  $\lambda_1 = \left(\frac{l}{B^2}\right)^{\frac{p+2}{2p}}$  is the absolute maximum point of  $h(\lambda)$  satisfying

$$E_1 = h(\lambda_1) = \frac{l^{\frac{p+2}{p}}}{2B^{\frac{2p+4}{p}}} - \frac{B^{p+2}}{(p+2)l^{\frac{p+2}{2}}} \left(\frac{l^{\frac{p+2}{2p}}}{B^{\frac{p+2}{p}}}\right)^{p+2} = \frac{p}{2(p+2)} \left(\frac{l}{B^2}\right)^{\frac{p+2}{p}}. \quad (2.7)$$

**Lemma 2.1.** Assume that the conditions (i), (2.1) hold and  $u$  is a solution of problem (1.1)–(1.3) with initial data  $E(0) < E_1$  and  $l^{\frac{1}{2}}\|\nabla u_0\|_2 > \lambda_1$ . Then there exists  $\lambda_2 > \lambda_1$  satisfying

$$l\|\nabla u\|_2^2 + (g \diamond \nabla u)(t) \geq \lambda_2^2, \quad (2.8)$$

for  $t > 0$ .

*Proof.* From Remark 2.1, we know that  $h(\lambda)$  is increasing for  $0 < \lambda < \lambda_1$ , decreasing for  $\lambda > \lambda_1$  and  $h(\lambda) \rightarrow -\infty$  as  $\lambda \rightarrow \infty$ . By  $E(0) < E_1$ , we have that there exist  $\lambda'_2$  and  $\lambda_2$  such that  $\lambda_1 \in (\lambda'_2, \lambda_2)$  and  $h(\lambda'_2) = h(\lambda_2) = E(0)$ . To prove (2.8), by contradiction we assume that there exists a time  $t_0 > 0$  such that

$$l\|\nabla u(t_0)\|_2^2 + (g \diamond \nabla u)(t_0) < \lambda_2^2. \quad (2.9)$$

(1) If  $\lambda'_2 < \left(l\|\nabla u(t_0)\|_2^2 + (g \diamond \nabla u)(t_0)\right)^{\frac{1}{2}} < \lambda_2$ , it is inferred that

$$h\left(\left[l\|\nabla u(t_0)\|_2^2 + (g \diamond \nabla u)(t_0)\right]^{\frac{1}{2}}\right) > E(0) \geq E(t_0),$$

which contradicts (2.6).

(2) Assume that  $\left(l\|\nabla u(t_0)\|_2^2 + (g \diamond \nabla u)(t_0)\right)^{\frac{1}{2}} \leq \lambda'_2$  holds. Considering  $l^{\frac{1}{2}}\|\nabla u_0\|_2 > \lambda_1$ , we get from (2.6) that  $h(l^{\frac{1}{2}}\|\nabla u_0\|_2) < E(0) = h(\lambda_2)$ , which implies  $l^{\frac{1}{2}}\|\nabla u_0\|_2 > \lambda_2$ . Thus, applying the continuity of  $\left(l\|\nabla u\|_2^2 + (g \diamond \nabla u)(t)\right)^{\frac{1}{2}}$ , we know that there exists a  $t_1 \in (0, t_0)$  such that  $\lambda'_2 < \left(l\|\nabla u(t_1)\|_2^2 + (g \diamond \nabla u)(t_1)\right)^{\frac{1}{2}} < \lambda_2$ . Hence, we have

$$h\left(\left[l\|\nabla u(t_1)\|_2^2 + (g \diamond \nabla u)(t_1)\right]^{\frac{1}{2}}\right) > E(0) \geq E(t_0),$$

which also contradicts (2.6).

Next, we will mention an important lemma which is similar to the lemmas of [26] with slight modification.

**Lemma 2.2.** Let  $J(t)$  be a nonincreasing function on  $[t_0, \infty)$ ,  $t_0 \geq 0$ , and satisfy the differential inequality

$$J'(t)^2 \geq \alpha + \beta J(t)^{2+\frac{1}{\delta}}, \quad \text{for } t \geq t_0, \quad (2.10)$$

where  $\alpha > 0$  and  $\beta < 0$ . Then there exists a finite positive time  $T^*$  such that

$$\lim_{t \rightarrow T^{*-}} J(t) = 0, \quad (2.11)$$

and the upper bound for  $T^*$  is estimated by:

$$T^* \leq t_0 + \frac{1}{\sqrt{-\beta}} \ln \frac{\sqrt{\frac{\alpha}{-\beta}}}{\sqrt{\frac{\alpha}{-\beta}} - J(t_0)}, \quad (2.12)$$

where  $J(t_0) < \min \left\{ 1, \sqrt{\frac{\alpha}{-\beta}} \right\}$ .

### 3. Upper bound for blow-up time

In this section, we will give some blow-up results for the solutions with initial energy  $E(0) < 0$ ,  $0 \leq E(0) < \frac{\eta}{p} E_1$  and  $\frac{\eta}{p} E_1 \leq E(0) < \frac{\|u_0\|_{H^1}^2}{\gamma}$ , respectively. Furthermore, some upper bounds for blow-up time  $T^*$  depending on the sign and size of initial energy  $E(0)$  are obtained for problem (1.1)–(1.3).

To obtain the blow-up results and upper bounds, we first define functionals

$$a(t) = \int_0^t \|u\|_{H^1}^2 d\tau, \quad t \in [0, \infty), \quad (3.1)$$

and

$$J(t) = \left( \int_0^t \|u\|_{H^1}^2 d\tau + (T_0 - t) \|u_0\|_{H^1}^2 \right)^{-\delta} = \left( a(t) + (T_0 - t) \|u_0\|_{H^1}^2 \right)^{-\delta}, \quad t \in [0, T_0], \quad (3.2)$$

where  $\delta \in (0, \frac{p}{2}]$ ,  $T_0$  is positive constant to be chosen later and then give the following lemma.

**Lemma 3.1.** *Assume that the conditions (i), (ii) hold,  $u_0 \in H_0^1(\Omega)$  and  $u$  is a solution of problem (1.1)–(1.3), then we have*

$$a''(t) - 4(1 + \delta) \int_0^t \|u_t\|_{H^1}^2 d\tau \geq G(t), \quad (3.3)$$

where  $G(t) = -4(1 + \delta)E(0) + \eta[l\|\nabla u\|_2^2 + (g \diamond \nabla u)(t)]$ ,  $\eta = 1 + 2\delta - \frac{1}{\gamma} > 0$ .

*Proof.* From (3.1), a direct computation yields

$$a'(t) = \|u\|_{H^1}^2 = 2 \int_0^t \int_{\Omega} uu_t dx d\tau + 2 \int_0^t \int_{\Omega} \nabla u \cdot \nabla u_t dx d\tau + \|u_0\|_{H^1}^2, \quad (3.4)$$

and

$$\begin{aligned} a''(t) &= 2 \int_{\Omega} uu_t dx + 2 \int_{\Omega} \nabla u \cdot \nabla u_t dx \\ &= 2 \int_{\Omega} u[\Delta u - \int_0^t g(t - \tau)\Delta u(\tau) d\tau + |u|^p u] dx \\ &= -2\|\nabla u\|_2^2 + 2\|u\|_{p+2}^{p+2} + 2 \int_0^t \int_{\Omega} g(t - s)\nabla u(\tau) \cdot \nabla u(t) dx d\tau. \end{aligned} \quad (3.5)$$

By (2.4), (2.5) and (3.5), it is found that

$$\begin{aligned} & a''(t) - 4(1 + \delta) \int_0^t \|u_\tau\|_{H^1}^2 d\tau \\ & \geq -4(1 + \delta)E(0) + 2\delta\|\nabla u\|_2^2 + (2 + 2\delta)(g \diamond \nabla u)(t) + \left[2 - \frac{4 + 4\delta}{p + 2}\right] \|u\|_{p+2}^{p+2} \\ & \quad - (2 + 2\delta) \int_0^t g(\tau) d\tau \|\nabla u\|_2^2 + 2 \int_0^t \int_\Omega g(t - s) \nabla u(\tau) \cdot \nabla u(t) dx d\tau. \end{aligned} \quad (3.6)$$

From the Young inequality, it is inferred that

$$\begin{aligned} & \int_0^t \int_\Omega g(t - s) \nabla u(\tau) \cdot \nabla u(t) dx d\tau \\ & = \int_0^t \int_\Omega g(t - \tau) \nabla u(t) \cdot [\nabla u(\tau) - \nabla u(t)] dx d\tau + \int_0^t g(t - \tau) d\tau \|\nabla u(t)\|_2^2 \\ & \geq -\frac{1}{2}(g \diamond \nabla u)(t) + \frac{1}{2} \int_0^t g(\tau) d\tau \|\nabla u\|_2^2. \end{aligned} \quad (3.7)$$

Hence, combining (2.2), (3.6) and (3.7), it follows that

$$\begin{aligned} & a''(t) - 4(1 + \delta) \int_0^t \|u_\tau\|_{H^1}^2 d\tau \\ & \geq -4(1 + \delta)E(0) + (1 + 2\delta)(g \diamond \nabla u)(t) + 2\delta\|\nabla u\|_2^2 - (1 + 2\delta) \int_0^t g(\tau) d\tau \|\nabla u\|_2^2 \\ & \geq -4(1 + \delta)E(0) + (1 + 2\delta - \frac{1}{l})l\|\nabla u\|_2^2 + (1 + 2\delta - \frac{1}{l})(g \diamond \nabla u)(t) \\ & = -4(1 + \delta)E(0) + \eta[l\|\nabla u\|_2^2 + (g \diamond \nabla u)(t)], \end{aligned} \quad (3.8)$$

where  $\eta = 1 + 2\delta - \frac{1}{l} > 0$ . The proof is completed.

Next, we shall state and prove the finite time blow-up results on the solutions of problem (1.1)–(1.3).

**Theorem 3.1.** Assume that the conditions (i), (ii) hold and  $\|u_0\|_{H^1}^2 < \frac{1}{T_0}$ . Furthermore assume that either one of the following three conditions is satisfied:

- (1)  $E(0) < 0$ ;
- (2)  $0 \leq E(0) < \frac{\eta}{p}E_1$ , and  $\lambda_1 < l^{\frac{1}{2}}\|\nabla u_0\|_2$ ;
- (3)  $\frac{\eta}{p}E_1 \leq E(0) < \frac{\|u_0\|_{H^1}^2}{\gamma}$ .

Then, the solutions  $u$  of problem (1.1)–(1.3) blow up in finite time, which means that the maximum existence time  $T^*$  of  $u$  is finite and

$$\lim_{t \rightarrow T^{*-}} \int_0^t \|u\|_{H^1}^2 d\tau = +\infty. \quad (3.9)$$

Moreover, the upper bounds for blow-up time  $T^*$  can be estimated according to the sign and size of energy  $E(0)$ :

Case (1): If  $E(0) < 0$ , then an upper bound of blow-up time  $T^*$  is given by

$$T^* \leq \sqrt{\frac{-(2\delta + 1)}{8\delta^2(\delta + 1)E(0)}} \ln \frac{1}{1 - \sqrt{T_0}\|u_0\|_{H^1}};$$

Case (2): If  $0 \leq E(0) < \frac{\eta}{p}E_1$ , and  $\lambda_1 < l^{\frac{1}{2}}\|\nabla u_0\|_2$ , then an upper bound of blow-up time  $T^*$  is given by

$$T^* \leq \sqrt{\frac{(2\delta + 1)}{8\delta^2(\delta + 1)[\frac{\eta}{p}E_1 - E(0)]}} \ln \frac{1}{1 - \sqrt{T_0}\|u_0\|_{H^1}};$$

Case (3):  $\frac{\eta}{p}E_1 \leq E(0) < \frac{\|u_0\|_{H^1}^2}{\gamma}$ , then an upper bound of blow-up time  $T^*$  is given by

$$T^* \leq \sqrt{\frac{(2\delta + 1)}{2\delta^2 D Z(0)}} \ln \frac{1}{1 - \sqrt{T_0}\|u_0\|_{H^1}},$$

where  $Z(0) = \|u_0\|_{H^1}^2 - \gamma E(0)$ ,  $\gamma = \frac{4(1+\delta)}{D}$  and  $D = \frac{\eta\mu_1}{1+\mu_1}$ ;  $\mu_1$  is the first eigenvalue of operate  $-\Delta$  subject to the homogeneous Dirichlet boundary condition.

*Proof.* We will consider the different cases depending on the sign and size of initial energy  $E(0)$ .

**Case (1):** If  $E(0) \leq 0$ , then it follows from (3.6) that

$$a''(t) \geq -4(1 + \delta)E(0) + \eta[l\|\nabla u\|_2^2 + (g \diamond \nabla u)(t)] + 4(1 + \delta) \int_0^t \|u_t\|_{H^1}^2 d\tau > 0, \quad (3.10)$$

for  $t \geq 0$ . Thus, we get  $a'(t) > a'(0) = \|u_0\|_{H^1}^2 \geq 0$ , for  $t > 0$ . Obviously, by the direct computation, we have from (3.2) that

$$J'(t) = -\delta J(t)^{1+\frac{1}{\delta}} \left( a'(t) - \|u_0\|_{H^1}^2 \right), \quad (3.11)$$

and

$$\begin{aligned} J''(t) &= -\delta J(t)^{1+\frac{2}{\delta}} \left\{ a''(t)[a(t) + (T_0 - t)\|u_0\|_{H^1}^2] \right. \\ &\quad \left. - (1 + \delta)[a'(t) - \|u_0\|_{H^1}^2]^2 \right\} = -\delta J(t)^{1+\frac{2}{\delta}} V(t), \end{aligned} \quad (3.12)$$

where

$$V(t) = a''(t)[a(t) + (T_0 - t)\|u_0\|_{H^1}^2] - (1 + \delta)[a'(t) - \|u_0\|_{H^1}^2]^2. \quad (3.13)$$

Then, by Lemma 3.1, we obtain that

$$\begin{aligned} &a''(t)[a(t) + (T_0 - t)\|u_0\|_{H^1}^2] \\ &\geq [G(t) + 4(1 + \delta) \int_0^t \|u_t\|_{H^1}^2 d\tau] \left[ \int_0^t \|u\|_{H^1}^2 d\tau + (T_0 - t)\|u_0\|_{H^1}^2 \right] \\ &\geq G(t)J(t)^{-\frac{1}{\delta}} + 4(1 + \delta) \int_0^t \|u_t\|_{H^1}^2 d\tau \int_0^t \|u\|_{H^1}^2 d\tau. \end{aligned} \quad (3.14)$$

On the other hand,

$$[a'(t) - \|u_0\|_{H^1}^2]^2 = 4 \left( \int_0^t \int_{\Omega} uu_t dx d\tau \right)^2 + 4 \left( \int_0^t \int_{\Omega} \nabla u \cdot \nabla u_t dx d\tau \right)^2$$



$$+ 8 \int_0^t \int_{\Omega} uu_t dx d\tau \int_0^t \int_{\Omega} \nabla u \cdot \nabla u_t dx d\tau. \quad (3.15)$$

Applying the Hölder and Cauchy inequalities, we discover that

$$4 \left( \int_0^t \int_{\Omega} uu_t dx d\tau \right)^2 \leq 4 \int_0^t \|u\|_2^2 d\tau \int_0^t \|u_t\|_2^2 d\tau, \quad (3.16)$$

$$4 \left( \int_0^t \int_{\Omega} \nabla u \cdot \nabla u_t dx d\tau \right)^2 \leq 4 \int_0^t \|\nabla u\|_2^2 d\tau \int_0^t \|\nabla u_t\|_2^2 d\tau, \quad (3.17)$$

and

$$\begin{aligned} & 8 \int_0^t \int_{\Omega} uu_t dx d\tau \int_0^t \int_{\Omega} \nabla u \cdot \nabla u_t dx d\tau \\ & \leq 8 \left( \int_0^t \|u\|_2^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^t \|u_t\|_2^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^t \|\nabla u\|_2^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^t \|\nabla u_t\|_2^2 d\tau \right)^{\frac{1}{2}} \\ & \leq 4 \int_0^t \|u\|_2^2 d\tau \int_0^t \|\nabla u_t\|_2^2 d\tau + 4 \int_0^t \|\nabla u\|_2^2 d\tau \int_0^t \|u_t\|_2^2 d\tau. \end{aligned} \quad (3.18)$$

Inserting (3.14)–(3.18) into (3.13), it follows that

$$V(t) = a''(t)[a(t) + (T_0 - t)\|u_0\|_{H^1}^2] - (1 + \delta)[a'(t) - \|u_0\|_{H^1}^2]^2 \geq G(t)J(t)^{-\frac{1}{\delta}}. \quad (3.19)$$

Thus, by (3.12), (3.19) and the definition of  $G(t)$ , there appears the relation

$$J''(t) \leq -\delta G(t)J(t)^{1+\frac{1}{\delta}} \leq 4\delta(1 + \delta)E(0)J(t)^{1+\frac{1}{\delta}}, \quad \text{for } t \geq 0. \quad (3.20)$$

Note that by  $a'(t) > a'(0) = \|u_0\|_{H^1}^2$  for  $t > 0$  and (3.11), we deduce that  $J'(t) < 0$  for  $t > 0$  and  $J'(0) = 0$ . Multiplying (3.20) with  $J'(t)$  and integrating it from 0 to  $t$ , then we conclude that

$$J'(t)^2 \geq -\frac{8\delta^2(\delta + 1)}{2\delta + 1}E(0)J(0)^{2+\frac{1}{\delta}} + \frac{8\delta^2(\delta + 1)}{2\delta + 1}E(0)J(t)^{2+\frac{1}{\delta}} = \alpha + \beta J(t)^{2+\frac{1}{\delta}}, \quad (3.21)$$

where

$$\beta = \frac{8\delta^2(\delta + 1)}{2\delta + 1}E(0) < 0, \quad (3.22)$$

$$\alpha = -\frac{8\delta^2(\delta + 1)}{2\delta + 1}E(0)J(0)^{2+\frac{1}{\delta}} > 0, \quad (3.23)$$

where  $J(0) = (T_0\|u_0\|_{H^1}^2)^{-\delta} > 1$ .

Then, by the combination of (3.21)–(3.23) and Lemma 2.2, we can obtain that there exists a finite time  $T^*$  satisfying  $\lim_{t \rightarrow T^*-} J(t) = 0$ , which also implies that

$$\lim_{t \rightarrow T^*-} \int_0^t \|u\|_{H^1}^2 d\tau = +\infty.$$

Furthermore, from Lemma 2.2, it follows that an upper bound of blow-up time  $T^*$  is given by

$$T^* \leq \sqrt{\frac{-(2\delta + 1)}{8\delta^2(\delta + 1)E(0)}} \ln \frac{1}{1 - \sqrt{T_0}\|u_0\|_{H^1}}. \quad (3.24)$$

**Case (2):** If  $0 \leq E(0) < \frac{\eta}{p}E_1$ , and  $l^{\frac{1}{2}}\|\nabla u_0\|_2 > \lambda_1$ , then utilizing Lemma 2.1, definition of  $\lambda_1$  and (2.7), we have

$$\begin{aligned} G(t) &= -4(1 + \delta)E(0) + \eta[l\|\nabla u\|_2^2 + (g \diamond \nabla u)(t)] \\ &\geq -4(1 + \delta)E(0) + \eta\lambda_2^2 \\ &> -4(1 + \delta)E(0) + \frac{2\eta(p + 2)}{p}E_1 \\ &\geq 4(1 + \delta)\left[\frac{\eta}{p}E_1 - E(0)\right] > 0, \end{aligned} \quad (3.25)$$

where in the last inequality we have used  $0 < \delta \leq \frac{p}{2}$ . Inserting (3.25) into (3.8), it follows that

$$a''(t) \geq G(t) + 4(1 + \delta) \int_0^t \|u_\tau\|_{H^1}^2 d\tau > 4(1 + \delta)\left[\frac{\eta}{p}E_1 - E(0)\right] + 4(1 + \delta) \int_0^t \|u_\tau\|_{H^1}^2 d\tau > 0. \quad (3.26)$$

Hence, we also have  $a'(t) > a'(0) = \|u_0\|_{H^1}^2 \geq 0$ , for  $t > 0$ .

Then, using the similar arguments to the case (1), we obtain that

$$J''(t) = -\delta J(t)^{1+\frac{2}{\delta}} V(t), \quad \text{and} \quad V(t) \geq G(t)J(t)^{-\frac{1}{\delta}}. \quad (3.27)$$

Hence, from (3.26) and (3.27), it follows that

$$J''(t) \leq -\delta G(t)J(t)^{1+\frac{1}{\delta}} \leq -4\delta(1 + \delta)\left[\frac{\eta}{p}E_1 - E(0)\right]J(t)^{1+\frac{1}{\delta}}, \quad \text{for } t \geq 0. \quad (3.28)$$

Applying the same discussion as in case (1), we also have  $J'(t) < 0$  for  $t > 0$  and  $J'(0) = 0$ . Multiplying (3.28) with  $J'(t)$  and integrating it from 0 to  $t$ , then we discover that

$$J'(t)^2 \geq \frac{8\delta^2(\delta + 1)}{2\delta + 1} \left[\frac{\eta}{p}E_1 - E(0)\right] \left(J(0)^{2+\frac{1}{\delta}} - J(t)^{2+\frac{1}{\delta}}\right) = \alpha_1 + \beta_1 J(t)^{2+\frac{1}{\delta}}, \quad (3.29)$$

where

$$\beta_1 = -\frac{8\delta^2(\delta + 1)}{2\delta + 1} \left[\frac{\eta}{p}E_1 - E(0)\right] < 0, \quad (3.30)$$

$$\alpha_1 = \frac{8\delta^2(\delta + 1)}{2\delta + 1} \left[\frac{\eta}{p}E_1 - E(0)\right] J(0)^{2+\frac{1}{\delta}} > 0. \quad (3.31)$$

Therefore, we can obtain from Lemma 2.2 and (3.29)–(3.31) that there exists a finite time  $T^*$  such that  $\lim_{t \rightarrow T^*-} J(t) = 0$ , which means that

$$\lim_{t \rightarrow T^*-} \int_0^t \|u\|_{H^1}^2 d\tau = +\infty.$$

Furthermore, from Lemma 2.2, we get that an upper bound of blow-up time  $T^*$  is given by

$$T^* \leq \sqrt{\frac{(2\delta + 1)}{8\delta^2(\delta + 1)[\frac{\eta}{p}E_1 - E(0)]}} \ln \frac{1}{1 - \sqrt{T_0}\|u_0\|_{H^1}}. \quad (3.32)$$

**Case (3):** When  $\frac{\eta}{p}E_1 \leq E(0) < \frac{\|u_0\|_{H^1}^2}{\gamma}$ , we define the functional

$$Z(t) = \|u\|_{H^1}^2 - \gamma E(0) = a'(t) - \gamma E(0), \quad (3.33)$$

where  $\gamma = \frac{4(1+\delta)}{D}$  and  $D = \frac{\eta\mu_1}{1+\mu_1} > 0$ ;  $\mu_1$  is the first eigenvalue of operate  $-\Delta$  subject to the homogeneous Dirichlet boundary condition such that  $\|\nabla u\|_2^2 \geq \mu_1\|u\|_2^2$ . Using (3.33) and Poincaré inequality, we get that

$$\begin{aligned} \frac{d}{dt}Z(t) &= a''(t) \geq -4(1 + \delta)E(0) + \eta[l\|\nabla u\|_2^2 + (g \diamond \nabla u)(t)] + 4(1 + \delta) \int_0^t \|u_\tau\|_{H^1}^2 d\tau \\ &\geq \frac{\eta\mu_1}{1 + \mu_1} \left[ \|u\|_{H^1}^2 - \frac{4(1 + \delta)(1 + \mu_1)}{\eta\mu_1} E(0) \right] + 4(1 + \delta) \int_0^t \|u_\tau\|_{H^1}^2 d\tau \\ &= D \left[ \|u\|_{H^1}^2 - \gamma E(0) \right] + 4(1 + \delta) \int_0^t \|u_\tau\|_{H^1}^2 d\tau \\ &= DZ(t) + 4(1 + \delta) \int_0^t \|u_\tau\|_{H^1}^2 d\tau. \end{aligned} \quad (3.34)$$

Then, from

$$\|u_0\|_{H^1}^2 - \gamma E(0) = a'(0) - \gamma E(0) = Z(0) > 0, \quad (3.35)$$

and (3.34), it is deduced that

$$Z(t) = a'(t) - \gamma E(0) \geq Z(0)e^{Dt} \geq Z(0) > 0, \quad \text{for } t \geq 0. \quad (3.36)$$

By (3.34) and (3.36), there appear the relation

$$\frac{d}{dt}Z(t) = a''(t) \geq DZ(t) \geq DZ(0) > 0. \quad (3.37)$$

Hence, we also have  $a'(t) > a'(0) = \|u_0\|_{H^1}^2 \geq 0$ , for  $t > 0$ .

Next, using the same discussion as in case (1), it follows that

$$J''(t) \leq -\delta G(t)J(t)^{1+\frac{1}{\delta}}, \quad \text{for } t \geq 0. \quad (3.38)$$

By (3.34) and (3.36) again, we also get that

$$G(t) \geq DZ(t) \geq DZ(0).$$

Hence, it follows that

$$J''(t) \leq -\delta G(t)J(t)^{1+\frac{1}{\delta}} \leq -\delta DZ(0)J(t)^{1+\frac{1}{\delta}}, \quad \text{for } t \geq 0. \quad (3.39)$$

Multiplying (3.39) with  $J'(t)$  and integrating it from 0 to  $t$ , then there appears the relation

$$J'(t)^2 \geq \frac{2\delta^2 DZ(0)}{2\delta + 1} \left( J(0)^{2+\frac{1}{\delta}} - J(t)^{2+\frac{1}{\delta}} \right) = \alpha_2 + \beta_2 J(t)^{2+\frac{1}{\delta}}, \quad (3.40)$$

where

$$\beta_2 = -\frac{2\delta^2 DZ(0)}{2\delta + 1} < 0, \quad (3.41)$$

$$\alpha_2 = \frac{2\delta^2 DZ(0)}{2\delta + 1} J(0)^{2+\frac{1}{\delta}} > 0, \quad t \geq 0, \quad (3.42)$$

Similarly, we can obtain from Lemma 2.2 and (3.40)–(3.42) that there exists a finite time  $T^*$  such that

$$\lim_{t \rightarrow T^{*-}} \int_0^t \|u\|_{H^1}^2 d\tau = +\infty.$$

Similarly, from Lemma 2.2, we conclude that an upper bound of blow-up time  $T^*$  is given by

$$T^* \leq \sqrt{\frac{(2\delta + 1)}{2\delta^2 DZ(0)}} \ln \frac{1}{1 - \sqrt{T_0} \|u_0\|_{H^1}}. \quad (3.43)$$

This completes the proof of Theorem 3.1.

#### 4. Lower bound for blow-up time

Our goal of this section is turned to determine a lower bound for blow-up time  $T^*$  when blow up occurs to the initial boundary value problem (1.1)–(1.3).

**Theorem 4.1.** *Assume that the conditions (i), (ii) hold. Let  $u_0 \in H_0^1(\Omega)$  and  $u$  be a blow-up solution of problem (1.1)–(1.3), then a lower bound for blow-up time  $T^*$  can be estimated in the form*

$$T^* \geq \int_{F(0)}^{\infty} \frac{1}{\eta + \frac{2p+6}{p+2} B^{p+2} \eta^{\frac{p+2}{2}} + 2E(0)} d\eta,$$

where  $B$  is the optimal constant satisfying the Sobolev embedding inequality  $\|u\|_{p+2} \leq B \|\nabla u\|_2$  and  $F(0) = \|u_0\|_{H^1}^2$ .

*Proof.* Let us define the auxiliary function

$$F(t) = \|u\|_2^2 + \|\nabla u\|_2^2, \quad \text{for } t \geq 0. \quad (4.1)$$

Differentiating (4.1) with respect to  $t$  and integration by parts, then we discover that

$$\begin{aligned} F'(t) &= 2 \int_{\Omega} uu_t dx - 2 \int_{\Omega} u \Delta u_t dx \\ &= 2 \int_{\Omega} u \left[ \Delta u - \int_0^t g(t-\tau) \Delta u(\tau) d\tau + |u|^p u \right] dx \end{aligned}$$

$$= -2\|\nabla u\|_2^2 + 2 \int_0^t g(t-\tau) \nabla u(\tau) \cdot \nabla u(t) d\tau + 2\|u\|_{p+2}^{p+2}. \quad (4.2)$$

Making use of the Young inequality, we have

$$\begin{aligned} & 2 \int_0^t g(t-\tau) \nabla u(\tau) \cdot \nabla u(t) d\tau \\ &= 2 \int_0^t g(t-\tau) [\nabla u(\tau) - \nabla u(t)] \cdot \nabla u(t) d\tau + 2 \int_0^t g(\tau) d\tau \|\nabla u(t)\|_2^2 \\ &\leq 3 \int_0^t g(\tau) d\tau \|\nabla u(t)\|_2^2 + \int_0^t g(t-\tau) [\nabla u(\tau) - \nabla u(t)]^2 d\tau. \end{aligned} \quad (4.3)$$

Inserting (4.3) into (4.2), it follows that

$$F'(t) \leq 3 \left( \int_0^t g(\tau) d\tau - 1 \right) \|\nabla u\|_2^2 + \|\nabla u\|_2^2 + (g \diamond u)(t) + 2\|u\|_{p+2}^{p+2} \quad (4.4)$$

From (2.1), (2.5), (4.4) and the definitions of  $E(t)$ ,  $F(t)$ , we have

$$\begin{aligned} F'(t) &\leq \|\nabla u\|_2^2 + 2E(0) + \left[ 2 + \frac{2}{p+2} \right] \|u\|_{p+2}^{p+2} \\ &\leq \|\nabla u\|_2^2 + \frac{2p+6}{p+2} B^{p+2} \|\nabla u\|_2^{p+2} + 2E(0) \\ &\leq F(t) + \frac{2p+6}{p+2} B^{p+2} F(t)^{\frac{p+2}{2}} + 2E(0), \end{aligned} \quad (4.5)$$

where

$$B = \sup_{u \in H_0^1(\Omega)} \frac{\|u\|_{p+2}}{\|\nabla u\|_2}. \quad (4.6)$$

Integrating the inequality (4.5) from 0 to  $t$ , we deduce that

$$\int_{F(0)}^{F(t)} \frac{1}{\eta + \frac{2p+6}{p+2} B^{p+2} \eta^{\frac{p+2}{2}} + 2E(0)} d\eta \leq t. \quad (4.7)$$

If  $u$  blows up in  $H_0^1(\Omega)$ -norm, then we establish a lower bound for  $T^*$  by the form

$$T^* \geq \int_{F(0)}^{\infty} \frac{1}{\eta + \frac{2p+6}{p+2} B^{p+2} \eta^{\frac{p+2}{2}} + 2E(0)} d\eta, \quad (4.8)$$

which thereby completes the proof of Theorem 4.1.

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## Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

## References

1. A. B. Al'shin, M. O. Korpusov, A. G. Siveshnikov, *Blow up in nonlinear Sobolev type equations*, De Gruyter Series in Nonlinear Analysis and Applications, Walter de Gruyter, 2011.
2. A. Y. Kolesov, E. F. Mishchenko, N. K. Rozov, *Asymptotic methods of investigation of periodic solutions of nonlinear hyperbolic equations*, Trudy Matematicheskogo Instituta im. V. A. Steklova RAN, **222** (1998), 3–191.
3. B. K. Shivamoggi, *A symmetric regularized long wave equation for shallow water waves*, Phys. Fluids, **29** (1986), 890–891.
4. P. Rosenau, *Evolution and breaking of the ion-acoustic waves*, Phys. Fluids, **31** (1988), 1317–1319.
5. E. S. Dzekhtser, *Generalization of the equations of motion of ground waters with free surface*, Dokl. Akad. Nauk. SSSR., **202** (1972), 1031–1033.
6. M. O. Korpusov, A. G. Sveshnikov, *Three-dimensional nonlinear evolution equations of pseudoparabolic type in problems of mathematical physics*, Comp. Math. Math. Phys., **43** (2003), 1765–1797.
7. R. E. Showalter, *Existence and representation theorem for a semilinear Sobolev equation in Banach space*, SIAM. J. Math. Anal., **3** (1972), 527–543.
8. R. Z. Xu, J. Su, *Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations*, J. Funct. Anal., **264** (2013), 2732–2763.
9. P. Luo, *Blow-up phenomena for a pseudo-parabolic equation*, Math. Method. Appl. Sci., **38** (2015), 2636–2641.
10. H. F. Di, Y. D. Shang, X. M. Peng, *Blow-up phenomena for a pseudo-parabolic equation with variable exponents*, Appl. Math. Lett., **64** (2017), 67–73.
11. Y. Liu, W. S. Jiang, F. L. Huang, *Asymptotic behaviour of solutions to some pseudo-parabolic equations*, Appl. Math. Lett., **25** (2012), 111–114.
12. Z. Dong, J. Zhou, *Blow-up of solutions to a parabolic system with nonlocal source*, Appl. Anal., **97** (2018), 825–841.
13. M. Marras, S. Vernier-Piro, G. Viglialoro, *Blow-up phenomena for nonlinear pseudo-parabolic equations with gradient term*, Discrete Cont. Dyn-B., **22** (2017), 2291–2300.
14. G. Gripenberg, *Global existence of solutions of volterra integro-differential equations of parabolic type*, J. Differ. Equations, **102** (1993), 382–390.
15. H. M. Yin, *Weak and classical solutions of some nonlinear volterra intergro-differential equations*, Commun. Part. Diff. Eq., **17** (1992), 1369–1385.
16. S. A. Messaoudi, *Blow-up of solutions of a semilinear heat equation with a visco-elastic term*, Nonlinear Elliptic and Parabolic Problems, Birkhäuser Basel, 2005.

17. Y. D. Shang, B. L. Guo, *On the problem of the existence of global solutions for a class of nonlinear convolutional integro-differential equations of pseudoparabolic type*, Acta Math. Appl. Sin., **26** (2003), 511–524.
18. Y. D. Shang, B. L. Guo, *Initial-boundary value problems and initial value problems for nonlinear pseudoparabolic integro-differential equations*, Math. Appl., **15** (2002), 40–45.
19. M. Ptashnyk, *Degenerate quasilinear pseudoparabolic equations with memory terms and variational inequalities*, Nonlinear Analysis: Theory, Methods & Applications, **66** (2007), 2653–2675.
20. R. W. Carroll, R. E. Showalter, *Singular and degenerate Cauchy problems*, Academic Press, 1976.
21. H. F. Di, Y. D. Shang, *Global existence and nonexistence of solutions for the nonlinear pseudo-parabolic equation with a memory term*, Math. Method. Appl. Sci., **38** (2015), 3923–3936.
22. F. L. Sun, L. S. Liu, Y. H. Wu, *Global existence and finite time blow-up of solutions for the semilinear pseudo-parabolic equation with a memory term*, Appl Anal., **98** (2019), 735–755.
23. H. A. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equation of the form  $Pu_t = -Au + F(u)$* , Arch. Ration. Mech. An., **51** (1973), 371–386.
24. H. F. Di, Y. D. Shang, X. X. Zheng, *Global well-posedness for a fourth order pseudo-parabolic equation with memory and source terms*, Discrete Cont. Dyn-B., **21** (2017), 781–801.
25. R. Z. Xu, L. Wei, Y. Niu, *Global well-posedness of coupled parabolic systems*, Sci. China Math., **62** (2020), 321–356.
26. M. R. Li, L. Y. Tsai, *Existence and nonexistence of global solutions of some system of semilinear wave equations*, Nonlinear Analysis: Theory, Methods & Applications, **54** (2003), 1397–1415.
27. J. L. Lions, *Quelques méthodes de résolutions des problèmes aux limites non linéaires*, Paris: Dunod, 1969.
28. M. Escobedo, M. A. Herrero, *A semilinear parabolic system in bounded domain*, Ann. Mat. Pur. Appl., **165** (1993), 315–336.



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