Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(4): 3378–3390.
DOI: 10.3934/math.2020218
Received: 06 February 2020
Accepted: 19 March 2020
Published: 01 April 2020

Research article

Asymptotic behavior for a class of population dynamics

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Abstract: This paper investigates the asymptotic behavior for a class of n-dimensional population dynamics systems described by delay differential equations. With the help of technique of differential inequality, we show that each solution of the addressed systems tends to a constant vector as $t \to \infty$, which includes many generalizations of Bernfeld-Haddock conjecture. By the way, our results extend some existing literatures.

Keywords: population dynamics; time-varying delay; asymptotic behavior; Bernfeld-Haddock conjecture
Mathematics Subject Classification: 34C25, 34K13, 34K25

1. Introduction

Recently, many differential equations models arising from population dynamics, biological mathematics and engineer have attracted more and more attention [1–6], and the hot topics include asymptotical stability [7–12], limit cycles [13, 14], bifurcation [15–17], and periodic solutions [18–20]. At the international conference on nonlinear systems and their applications in 1976, the following conjecture:
Conjecture. *Each solution of the scalar differential equation*

$$x'(t) = -x^{\frac{1}{3}}(t) + x^{\frac{1}{3}}(t-r), r > 0 \tag{1.1}$$

tends to a constant as $t \to \infty$.
was proposed by Bernfeld and Haddock [21]. Model (1.1) arises from population dynamics of a single species, where $x^{\frac{1}{3}}(t)$ and $x^{\frac{1}{3}}(t-r)$ always describe the instantaneous mortality rate and the feedback controls depending on the values of the stable variable with respective delay r, respectively.

Since the seminal works obtained by Jehu [22] and Ding [23] on the above conjecture, variants of the conjecture have been extensively investigated (see, [24–34]). A simple generalization of the conjecture is described by the following autonomous population dynamics model

$$x'(t) = -F(x(t)) + G(x(t-r)), \tag{1.2}$$

where $F, G \in C(\mathbb{R}^1, \mathbb{R}^1)$ describe the instantaneous mortality rate and the feedback controls depending on the values of the stable variable with respective delays r, respectively. The higher dimensional generalizations for compartmental systems were also presented in [24–30]. Yet the time-varying delays are more realistic than constant delays in population and ecology models, so Liu [35,36] generalized the conjecture as the following non-autonomous population dynamics models with time-varying delays:

$$x'(t) = p(t)[-x^{\frac{1}{3}}(t) + x^{\frac{1}{3}}(t - r(t))], \tag{1.3}$$

and Xiao [37] established the following generalized version

$$\begin{cases} x_1'(t) = \gamma_1(t)[-F_1(x_1(t)) + G_1(x_2(t-\tau_2(t)))], \\ x_2'(t) = \gamma_2(t)[-F_2(x_2(t)) + G_2(x_1(t-\tau_1(t)))]. \end{cases} \tag{1.4}$$

Here, $F_i, G_i \in C(\mathbb{R}, \mathbb{R})$, F_i is increasing on $\mathbb{R}$, $p, \gamma_1, \gamma_2 \in C(\mathbb{R}, (0, +\infty))$, $i = 1, 2$.

In addition, the results in above references indicate that the solution of the systems which is bounded tends to a constant solution as $t \to +\infty$. There are two main methods to prove them, one is the analysis method of the monotone dynamical system [24, 25, 29, 33, 34], the other is the differential inequality analysis technique [23, 27, 32, 35–37]. In particular, assume that,

$$F_1, F_2 \in \Omega = \Big\{F \in C(\mathbb{R}, \mathbb{R}) \mid F(0) = 0\Big\},$$

where F is strictly increasing on $\mathbb{R}$, and continuous differentiable on $\mathbb{R} \setminus \{0\}$, the author in [37] established the convergence of system (1.1) and generalized the Bernfeld-Haddock conjecture to two-dimensional system. Unfortunately, if $F_i \not\equiv G_i (i = 1, 2)$, there is a simple but serious error in proving Theorem 3.1 in [37]. For the convenience of reading, we'll point out the details of this error in Remark 3.1. Furthermore, in order to correct the above error, we take the following n-dimensional non-autonomous population dynamics model with time-varying delays into consideration:

$$\begin{cases} x_1'(t) = \gamma_1(t)[-F_1(x_1(t)) + F_1(x_2(t-\tau_2(t)))], \\ x_2'(t) = \gamma_2(t)[-F_2(x_2(t)) + F_2(x_3(t-\tau_3(t)))], \\ \cdots \\ x_i'(t) = \gamma_i(t)[-F_i(x_i(t)) + F_i(x_{i+1}(t-\tau_{i+1}(t)))], \\ \cdots \\ x_{n-1}'(t) = \gamma_{n-1}(t)[-F_{n-1}(x_{n-1}(t)) + F_{n-1}(x_n(t-\tau_n(t)))], \\ x_n'(t) = \gamma_n(t)[-F_n(x_n(t)) + F_n(x_1(t-\tau_1(t)))], \end{cases} \tag{1.5}$$

and $F_i \in \Omega, \gamma_i, \tau_i \in C(\mathbb{R}, \mathbb{R}_+), i \in J = \{1, 2, \cdots, n\}$. Evidently, for $F_i \equiv G_i (i \in J)$, Eqs 1.1, 1.3, 1.4 and the systems in [35] are special models of system (1.5).

Since the non-autonomous delay systems usually do not produce a semiflow, the approach in [23–25, 27, 29, 32, 33] can not be used to prove the asymptotic behavior of (1.5). In particular, as far as we know, there are no references about the asymptotic behavior of solutions to the n-dimensional system (1.5) involving time-varying delays. On the basis of the above discussions, we expect to propose a novel proof to show a similar conclusion to Bernfeld-Haddock conjecture that every solution of (1.5) tends to a constant vector as $t \to +\infty$.

The paper is organized in following way. We present the initial condition and some preliminary results in Section 2. The boundness and asymptotic behavior of solutions are investigated in Section 3, which are our main results. In the next section, some examples with numerical simulations are carried out to illustrate the validity of the obtained results.

2. Materials and method

In this section, we give the initial condition and present the relevant results which will be used in Section 3.

Denote

$$f^+ = \sup_{t\in\mathbb{R}} f(t) \qquad \text{and} \qquad f^- = \inf_{t\in\mathbb{R}} f(t),$$

where f is and continuous bounded function on $\mathbb{R}$. We suppose that

$$\tau_{max} = \max\{\tau_i^+ : i \in J\},\ \tau_{min} = \min\{\tau_i^- : i \in J\} > 0.$$

Define $C = \prod_{i=1}^n C([-\tau_i^+, 0], \mathbb{R})$ as the Banach space equipped with the supremum norm. Moreover, we assume the initial condition

$$x_i(t_0 + \theta) = \varphi_i(\theta),\ \theta \in [-\tau_i^+, 0],\ t_0 \in \mathbb{R},\ \varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \in C,\ i \in J. \tag{2.1}$$

Let $x(t; t_0, \varphi) = (x_1(t; t_0, \varphi), x_2(t; t_0, \varphi), \cdots, x_n(t; t_0, \varphi))$ be the solution of (1.5) with the initial value condition (2.1). And $[t_0, \eta(\varphi))$ is the maximal right-interval of existence of $x(t; t_0, \varphi)$.

Now, it is assumed that $G \in \Omega$, and we recall the following Lemmas and Propositions.

Lemma 2.1. (see [34]) For any constant $c \neq 0, t_0$ and x_0, the system

$$\begin{cases} x'(t) = -G(x(t)) + G(c), \\ x(t_0) = x_0 \end{cases} \tag{2.2}$$

has a unique left-hand solution $x(t; t_0, x_0)$.

According to Proposition 4* and Proposition 5* in [35], the following results can be achieved:

Proposition 2.1. Consider the initial value problem

$$\begin{cases} x'(t) = -G(u) + G(a + \varepsilon), \\ u(t_0) = u_0\ (u_0 < a) \end{cases} \tag{2.3}$$

where $a \neq 0$ and $0 \leq \varepsilon \leq \frac{|a|}{2}$. Then, we suppose that $u = u(t; t_0, u_0)$ is the solution of (2.3), and the constant $\beta > 0$. There must exist $\sigma > 0$ independent of t_0 and ε such that

$$(a + \varepsilon) - u(t; t_0, u_0) \geq \sigma > 0 \ \text{ for } \ t \in [t_0, t_0 + \beta].$$

Proposition 2.2. Consider the initial value problem

$$\begin{cases} x'(t) = -G(u) + G(a - \varepsilon), \\ u(t_0) = u_0\ (u_0 > a) \end{cases} \tag{2.4}$$

where $a \neq 0$ and $0 \leq \varepsilon \leq \frac{|a|}{2}$. Then, we suppose that $u = u(t; t_0, u_0)$ is the solution of (2.3), and the constant $\beta > 0$. There must exist $\gamma > 0$ independent of t_0 and ε such that

$$u(t; t_0, u_0) - (a - \varepsilon) \geq \gamma > 0 \quad \text{for } t \in [t_0, t_0 + \beta].$$

Lemma 2.2. (see [37]) Let $t_0, x_0 \in \mathbb{R}$, $\alpha > 0$, $H(t, x) \in C([t_0, t_0 + \alpha] \times \mathbb{R}, \mathbb{R})$, and $H(t, x)$ is non-increasing with respect to the x. Then the following differential equation

$$\begin{cases} \frac{dx}{dt} = H(t, x), \\ x(t_0) = x_0 \end{cases}$$

has a unique solution $x = x(t)$ on $[t_0, t_0 + \alpha]$.
Lemma 2.3. Consider the system (1.5), assume that $\varphi \in C$, there is a unique solution $x(t; t_0, \varphi)$ on $[t_0, +\infty)$.
Proof. Let $x(t) = x(t; t_0, \varphi)$, then, for all $i \in J$, we shall prove that there is a unique solution $x(t)$ on $[t_0, t_0 + \tau_{min}]$. Let

$$g_i(t) = F_i(x_{\bar{i}}(t - \tau_{\bar{i}}(t))) = F_i(\varphi_{\bar{i}}(t - \tau_{\bar{i}}(t) - t_0)),$$

where

$$\bar{i} = \begin{cases} i + 1, & i \neq n, \\ 1, & i = n, \end{cases} \tag{2.5}$$

for any $t \in [t_0, t_0 + \tau_{min}]$. Consider the following differential equation

$$\begin{cases} x_i'(t) &= \gamma_i(t)[-F_i(x_i(t)) + g_i(t)], \\ x_i(t_0) &= \varphi_i(0). \end{cases}$$

According to Lemma 2.2, there is a unique solution $x_i(t)$ on $[t_0, t_0 + \tau_{min}]$ for all $i \in J$, and $x(t)$ exists and is unique on $[t_0, t_0 + \tau_{min}]$. Hence, it is obvious that there is a unique solution $x(t)$ on $[t_0 + \tau_{min}, t_0 + 2\tau_{min}], [t_0 + 2\tau_{min}, t_0 + 3\tau_{min}], \cdots$. We now complete the proof of Lemma 2.3.

3. Results

Theorem 3.1. $x(t) = x(t; t_0, \varphi)$ is bounded and tends to a constant vector as $t \to +\infty$.

Proof. For convenience, we label

$$v_i(t) = \max_{t-\tau_{max} \leq s \leq t} x_i(s), u_i(t) = \min_{t-\tau_{max} \leq s \leq t} x_i(s), \forall\ t \geq t_0 + \tau_{max},\ i \in J,$$

$$v(t) = \max\{v_i(t) : i \in J\},\ M = \{t | t \in [t_0 + \tau_{max}, +\infty), v(t) = x_i(t) \text{ for some } i \in J\},$$

and

$$u(t) = \min\{u_i(t) : i \in J\},\ T = \{t | t \in [t_0 + \tau_{max}, +\infty), u(t) = x_i(t) \text{ for some } i \in J\}.$$

Then, in order to get the boundness of $x(t)$, we first prove $D^+v(t) \leq 0$ for all $t \geq t_0 + \tau_{max}$. The proof of this part is divided into the following two cases:
Case 1. If $t \in [t_0 + \tau_{max}, +\infty) \setminus M$, then there exist $i_0 \in J$ and $t^* \in [t - \tau_{max}, t)$ satisfing that

$$v(t) = v_{i_0}(t) = \max_{t-\tau_{max} \leq s \leq t} x_{i_0}(s) = x_{i_0}(t^*) > \max\{x_i(t) : i \in J\}.$$

Since $x_i(t)$ are continuous, we choose a constant $0 < \delta < \tau_{max}$ to satisfy that

$$x_i(s) < x_{i_0}(t^*), \quad \forall s \in [t, t+\delta], i \in J,$$

which follows that

$$x_i(s) \le x_{i_0}(t^*) = \max_{t-\tau_{max} \le s \le t} x_{i_0}(s) = v_{i_0}(t) = v(t), \quad \forall s \in [t-\tau_{max}, t+\delta], i \in J,$$

and hence, for all $h \in (0, \delta)$,

$$\begin{aligned} v(t+h) &= \max\{\max_{t+h-\tau_{max} \le s \le t+h} x_i(s) : i \in J\} \\ &\le \max\{\max_{t-\tau_{max} \le s \le t+\delta} x_i(s) : i \in J\} \\ &\le \max_{t-\tau_{max} \le s \le t} x_{i_0}(s) = v_{i_0}(t) = v(t). \end{aligned}$$

Then, the following results can be obtained:

$$D^+ v(t) = \limsup_{h \to 0^+} \frac{v(t+h) - v(t)}{h} \le 0.$$

Case 2. If $t \in M$, one can pick $i_0 \in J$ such that

$$v(t) = v_{i_0}(t) = x_{i_0}(t) = \max_{t-\tau_{max} \le s \le t} x_{i_0}(s). \tag{3.1}$$

Then, (1.5) and (2.7) give us

$$\begin{aligned} 0 &\le x'_{i_0}(t) \\ &= \gamma_{i_0}(t)[-F_{i_0}(x_{i_0}(t)) + F_{i_0}(x_{\bar{i}_0}(t - \tau_{\bar{i}_0}(t)))] \\ &\le \gamma_{i_0}(t)[-F_{i_0}(x_{i_0}(t)) + F_{i_0}(x_{i_0}(t))] \\ &= 0. \end{aligned}$$

Let $\rho = \frac{1}{2}\tau_{min}$. When $v(s) = x_{i_0}(s)$ for all $s \in (t, t+\rho]$, we have

$$\begin{aligned} D^+ v(t) &= \limsup_{h \to 0^+} \frac{v(t+h) - v(t)}{h} \\ &= \limsup_{h \to 0^+} \frac{v(t+h) - x_{i_0}(t)}{h} \\ &= \limsup_{h \to 0^+} \frac{x_{i_0}(t+h) - x_{i_0}(t)}{h} \\ &= x'_{i_0}(t) \\ &= 0, \end{aligned}$$

where $0 < h < \rho$.

On the other hand, if there exists $s_1 \in (t, t+\rho]$ such that $v(s_1) > x_{i_0}(s_1)$, it suffices to deal with the following two categories (**i**) and (**ii**).

(**i**) If $v(s_1) = v_{i_0}(s_1) = \max\limits_{s_1-\tau_{max}\le s\le s_1} x_{i_0}(s)$, then, we can choose a constant $\tilde{t} \in [s_1 - \tau_{max}, s_1)$ such that

$$v(s_1) = x_{i_0}(\tilde{t}) = \max_{s_1-\tau_{max}\le s\le s_1} x_{i_0}(s).$$

Noting that $t - \tau_{max} < s_1 - \tau_{max} \le t + \rho - \tau_{max} < t < s_1$, we gain

$$x_{i_0}(\tilde{t}) \ge x_{i_0}(t) = v(t) = v_{i_0}(t) = \max_{t-\tau_{max}\le s\le t} x_{i_0}(s).$$

We claim

$$x_{i_0}(\tilde{t}) = x_{i_0}(t). \tag{3.2}$$

Otherwise, $x_{i_0}(\tilde{t}) > x_{i_0}(t)$. Then it is not hard to obtain that $t < \tilde{t} < s_1$ and

$$0 \le x'_{i_0}(\tilde{t}) \;=\; \gamma_{i_0}(\tilde{t})[-F_{i_0}(x_{i_0}(\tilde{t})) + F_{i_0}(x_{\bar{i}_0}(\tilde{t} - \tau_{\bar{i}_0}(\tilde{t})))],$$

which follows

$$x_{\bar{i}_0}(\tilde{t} - \tau_{\bar{i}_0}(\tilde{t})) \ge x_{i_0}(\tilde{t}).$$

In combination with $t - \tau_{max} \le t - \tau_{\bar{i}_0}(\tilde{t}) < \tilde{t} - \tau_{\bar{i}_0}(\tilde{t}) < \tilde{t} - \rho < t < s_1$, we obtain

$$x_{i_0}(\tilde{t}) \le x_{\bar{i}_0}(\tilde{t} - \tau_{\bar{i}_0}(\tilde{t})) \le \max_{t-\tau_{max}\le s\le t} x_{\bar{i}_0}(s) \le v(t) = x_{i_0}(t),$$

which leads to a contradiction and suggests that the above claim is true. Then

$$\max_{t-\tau_{max}\le s\le s_1} x_{i_0}(s) = x_{i_0}(t),$$

which together with $t - \tau_{max} < s_1 - \tau_{max} \le t + \rho - \tau_{max} < t < s_1$, and

$$v_i(t) \le v_{i_0}(t),\ v_i(s_1) \le v_{i_0}(s_1),\ \text{for all } i \in J,$$

hereafter, we obtain

$$\max_{t-\tau_{max}\le s\le s_1} x_{i_0}(s) = x_{i_0}(t) = v(t),\ v(t+h) = x_{i_0}(t),\ \ \forall\, 0 < h < s_1 - t,$$

and hence

$$\begin{aligned} D^+v(t) &= \limsup_{h\to 0^+} \frac{v(t+h) - v(t)}{h} \\ &= \limsup_{h\to 0^+} \frac{x_{i_0}(t) - x_{i_0}(t)}{h} \\ &= 0. \end{aligned}$$

(**ii**) If there exists $\tilde{i} \in J,\ \tilde{i} \ne i_0$ such that

$$v(s_1) = v_{\tilde{i}}(s_1) = \max_{s_1-\tau_{max}\le s\le s_1} x_{\tilde{i}}(s) > v_{i_0}(s_1),$$

then, we can find a constant $t_1 \in [s_1 - \tau_{max}, s_1]$ such that

$$v(s_1) = x_{\tilde{i}}(t_1) = \max_{s_1-\tau_{max}\le s\le s_1} x_{\tilde{i}}(s) > v_{i_0}(s_1) \ge x_{i_0}(t). \tag{3.3}$$

If $s_1 - \tau_{max} \leq t_1 \leq t$, and from (3.1), we have

$$v(t) = v_{i_0}(t) = \max_{t-\tau_{max} \leq s \leq t} x_{i_0}(s) \geq x_{\tilde{i}}(t_1),$$

which contradicts with (3.3). Therefore, $t < t_1 \leq s_1$ as well as

$$0 \leq x'_{\tilde{i}}(t_1) \quad = \quad \gamma_{\tilde{i}}(t_1)[-F_{\tilde{i}}(x_{\tilde{i}}(t_1)) + F_{\tilde{i}}(x_{\tilde{i}+1}(t_1 - \tau_{\tilde{i}+1}(t_1)))],$$

yields

$$x_{\tilde{i}+1}(t_1 - \tau_{\tilde{i}+1}(t_1)) \geq x_{\tilde{i}}(t_1) > x_{i_0}(t). \tag{3.4}$$

From (3.1) and the fact that $t - \tau_{max} < t_1 - \tau_{\tilde{i}+1} \leq t_1 - \tau_{min} \leq t + \rho - \tau_{min} < t$, one can see that

$$x_{i_0}(t) \geq x_{\tilde{i}+1}(t_1 - \tau_{\tilde{i}+1}(t_1)), \tag{3.5}$$

which contradicts with (3.4). Therefore, category (**ii**) does not hold, and we can draw that $D^+v(t) \leq 0$ for all $t \geq t_0 + \tau_{max}$ from the proof of the above two cases.

Accordingly, from the definitions of $u(t)$ and T, by using a similar argument as that adopted above, one can evidence that $D_+u(t) \geq 0, \forall t \geq t_0 + \tau_{max}$. Overall, we know that u is non-decreasing and v is non-increasing on $[t_0 + \tau_{max}, +\infty)$. Now, the boundness of the $x(t; t_0, \varphi)$ is proved.

Next, we prove the convergence of $x(t)$. Let $l_i = \liminf_{t\to+\infty} x_i(t; t_0, \varphi), L_i = \limsup_{t\to+\infty} x_i(t; t_0, \varphi), i \in J$.

From the boundedness of x, we can obtain

$$\lim_{t\to+\infty} v(t) = P, \lim_{t\to+\infty} u(t) = D, \text{ and } P \geq L_i \geq l_i \geq D, i \in J.$$

Clearly, it is need to show $L_i = l_i$ for all $i \in J$. Combined with the above discussions, we just need to prove that $L_i > l_i$ for all $i \in J$ does not hold. We just consider the case $L_1 > l_1$, and the remainder of the argument is analogous for $i \in J \setminus \{1\}$. Suppose that, on the contrary, $L_1 > l_1$. It follows that $D < P$, one of P and D is not equal to 0. Thus, we suppose that $P \neq 0$ and another case is similar. For $\bar{H} \in (l_1, L_1) \subset (D, P)$, we can choose $t_0^* > t_0 + \tau_{max}$ and $\{\sigma_m\}_{m=1}^{+\infty} \subset [t_0^* + \tau_{max}, +\infty)$ such that

$$x_1(\sigma_m) = \bar{H}, \quad \lim_{m\to+\infty} \sigma_m = +\infty, \quad \text{and} \quad x_i(t) \leq P + \frac{|P|}{2} \text{ for all } t \in [t_0^*, +\infty), i \in J.$$

From the fact that $v(t)$ is a monotone function and $\varepsilon_m = v(\sigma_m) - P \to 0$ (as $m \to +\infty$), we can presume that, for any positive integer m,

$$F_1(P) \leq F_1(v(\sigma_m)) = F_1(P + \varepsilon_m), 0 \leq \varepsilon_m \leq \frac{|P|}{2}.$$

Since

$$v(\sigma_m) \geq v(t) \geq x_i(t), \quad \forall t \in [\sigma_m, \sigma_m + (n+1)\tau_{max}], i \in J,$$

and

$$-F_1(x_1(t)) + F_1(v(\sigma_m)) \geq 0, \quad \forall t \in [\sigma_m, \sigma_m + (n+1)\tau_{max}],$$

one can find that, for all $t \in [\sigma_m, \sigma_m + (n+1)\tau_{max}]$,

$$\begin{aligned} x_1'(t) &= \gamma_1(t)[-F_1(x_1(t)) + F_1(x_2(t-\tau_2(t)))] \\ &\leq \gamma_1(t)[-F_1(x_1(t)) + F_1(v(\sigma_m))] \\ &\leq \gamma_1^+[-F_1(x_1(t)) + F_1(P+\varepsilon_m)]. \end{aligned} \tag{3.6}$$

Let $z(t) = z(t; \sigma_m, \varepsilon_m)$ be the solution of following system

$$z'(t) = \gamma^+[-F_1(z(t)) + F_1(P+\varepsilon_m)],\ z(\sigma_m) = \bar{H}. \tag{3.7}$$

Note that $\bar{H} < P$, on the basis of Proposition 2.1, we get

$$P + \varepsilon_m - z(t; \sigma_m, \varepsilon_m) \geq \mu > 0,\ t \in [\sigma_m, \sigma_m + (n+1)\tau_{max}],$$

where μ is unconcerned with σ_m and ε_m. With the help of (3.6) and (3.7), we can take a constant $\alpha \in (0, \mu)$ such that

$$x_1(t) \leq z(t) < P + \varepsilon_m - \alpha,\ t \in [\sigma_m, \sigma_m + (n+1)\tau_{max}], \tag{3.8}$$

$$v_1(s) = \max_{s-\tau_{max} \leq t \leq s} x_1(t) < P + \varepsilon_m - \alpha,\ s \in [\sigma_m + \tau_{max}, \sigma_m + (n+1)\tau_{max}],$$

and

$$v_1(\sigma_m + 2\tau_{max}) \leq v_1(\sigma_m + \tau_{max}) < P + \varepsilon_m - \alpha. \tag{3.9}$$

For $s \in [\sigma_m + 2\tau_{max}, \sigma_m + (n+1)\tau_{max}]$, according to the fact that

$$v_n(s) = \max_{s-\tau_{max} \leq t \leq s} x_n(t), \tag{3.10}$$

it follows that there exists $t_n \in [s - \tau_{max}, s] \subseteq [\sigma_m + \tau_{max}, \sigma_m + (n+1)\tau_{max}]$ such that

$$v_n(s) = x_n(t_n) = \max_{s-\tau_{max} \leq t \leq s} x_n(t)$$

and

$$0 \leq x_n'(t_n) = \gamma_n(t_n)[-F_n(x_n(t_n)) + F_n(x_1(t_n - \tau_1(t_n)))],$$

therefore

$$x_n(t_n) \leq x_1(t_n - \tau_1(t_n)) < P + \varepsilon_m - \alpha,$$

and

$$v_n(\sigma_m + k\tau_{max}) < P + \varepsilon_m - \alpha, \text{for all } k = 2, 3, \cdots, (n+1).$$

For $s \in [\sigma_m + 3\tau_{max}, \sigma_m + (n+1)\tau_{max}]$, in view of the fact that

$$v_{n-1}(s) = \max_{s-\tau_{max} \leq t \leq s} x_{n-1}(t), \tag{3.11}$$

one can choose $t_{n-1} \in [s - \tau_{max}, s] \subseteq [\sigma_m + 2\tau_{max}, \sigma_m + (n+1)\tau_{max}]$ satisfying

$$v_{n-1}(s) = x_{n-1}(t_{n-1}) = \max_{s-\tau_{max} \leq t \leq s} x_{n-1}(t)$$

and

$$0 \leq x'_{n-1}(t_{n-1}) = \gamma_{n-1}(t_{n-1})[-F_{n-1}(x_{n-1}(t_{n-1})) + F_n(x_1(t_{n-1} - \tau_{n-1}(t_{n-1})))],$$

which implies that

$$x_{n-1}(t_n) \leq x_n(t_{n-1} - \tau_{n-1}(t_{n-1})) < P + \varepsilon_m - \alpha,$$

and

$$v_{n-1}(\sigma_m + k\tau_{max}) < P + \varepsilon_m - \alpha, \text{for all } k = 3, 4, \cdots, (n+1).$$

Similarly,

$$v_j(\sigma_m + k\tau_{max}) < P + \varepsilon_m - \alpha, \text{for all } k = n - j + 2, \cdots, (n+1), \text{and } j = 2, 3, \cdots, n-2.$$

Consequently,

$$v(\sigma_m + n\tau_{max}) = \max\{v_i(\sigma_m + n\tau_{max}) : i \in J\} < P + \varepsilon_m - \alpha.$$

However, we know that $\lim_{m\to+\infty} v(\sigma_m+\tau_{max}) = \lim_{t\to+\infty} v(t) = P$, which leads to a contradiction. Therefore $L_1 = l_1$.

Finally, based on the above analysis, we obtain $L_i = l_i, i \in J$ and complete the proof of Theorem 3.1.

Remark 3.1. As mentioned before, there are some mistakes in the proof of Theorem 3.1 in [37]. The author briefly explained the proof of $D_+u(t) \geq 0$, but under the premise of $G_i \leq F_i$, it is impossible to get the conclusion of $D_+u(t) \geq 0$ in the same way as proving $D^+v(t) \leq 0$. And for system (1.5), proving $D^+v(t) \leq 0$ is similar to $D_+u(t) \geq 0$.

In fact, if $\gamma_i(t) = 1$, $\tau_i = \tau$, for different values of n, many generalizations of Bernfeld-Haddock conjecture are special cases of (1.5). Thus, this paper is a more extensive generalization.

Remark 3.2. To some extent, for different dimensions and delays, many systems mentioned above are special cases of n-dimensional non-autonomous differential equations with time-varying delays, which means that this article not only points out the errors in previous results, but also generalizes it.

4. Numerical simulations

In this section, we give two examples of satisfying population dynamics system (1.5), for different n, we have

$$\begin{cases} x'_1(t) &= (1 + 2\cos^2 t)[-x_1^{\frac{1}{3}}(t) + x_2^{\frac{1}{3}}(t - (1 + |\sin t|))], \\ x'_2(t) &= (1 + 3\cos^4 t)[-x_2^{\frac{3}{5}}(t) + x_1^{\frac{3}{5}}(t - (1 + |\cos t|))], \\ x_{t_0} &= \varphi \in C([-2, 0], \mathbb{R}) \times C([-2, 0], \mathbb{R}), \end{cases} \tag{4.1}$$

and

$$\begin{cases} x'_1(t) &= (1 + 5\cos^2 t)[-x_1^{\frac{1}{3}}(t) + x_2^{\frac{1}{3}}(t - (1 + |\sin t|))], \\ x'_2(t) &= (1 + 2\cos^4 t)[-x_2^{\frac{3}{5}}(t) + x_3^{\frac{3}{5}}(t - (1 + |\cos t|))], \\ x'_3(t) &= (1 + 2\cos^2 t)[-x_3^{\frac{1}{3}}(t) + x_4^{\frac{1}{3}}(t - (1 + |\sin t|))], \\ x'_4(t) &= (1 + \cos^4 t)[-x_4^{\frac{3}{5}}(t) + x_1^{\frac{3}{5}}(t - (1 + |\cos t|))], \\ x_{t_0} &= \varphi \in C([-2, 0], \mathbb{R}) \times C([-2, 0], \mathbb{R}) \times C([-2, 0], \mathbb{R}) \times C([-2, 0], \mathbb{R}). \end{cases} \tag{4.2}$$

As Theorem 3.1 and Remark 3.1 mentioned, one can obtain that every solution of the Eqs 4.1 and 4.2 tends to a constant vector as $t \to \infty$. The curves in Figures 1 and 2 make it easy to see that our conclusion is correct.

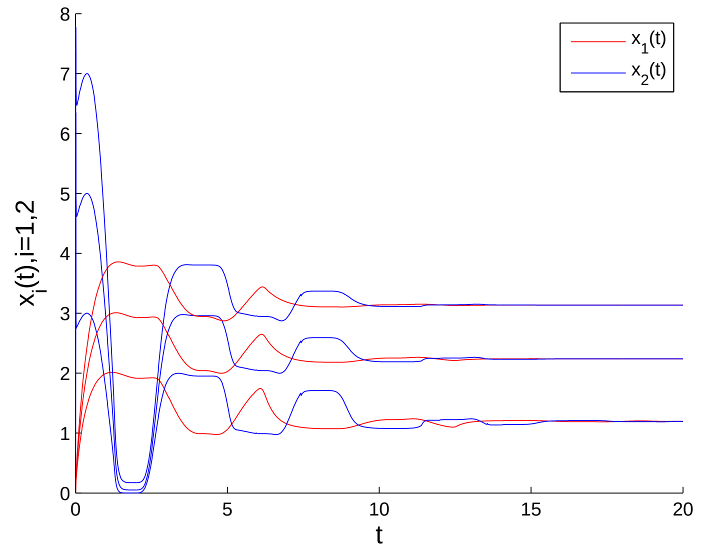

Figure 1. $\varphi(s) = (3\sin s, 3\sin s), (5\sin s, 5\sin s), (7\sin s, 7\sin s)$, $s \in [-2, 0]$, the solutions $x(t)$ of (4.1).

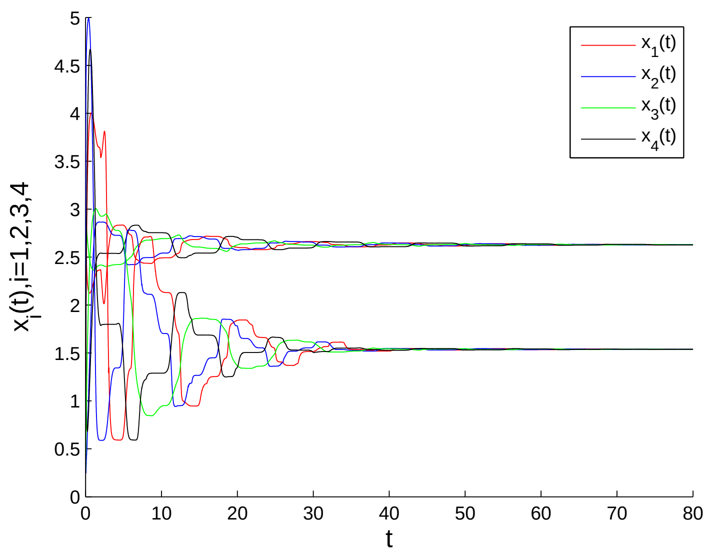

Figure 2. $\varphi(s) = (1 + 2\cos s, 1 + 2\cos s), (-5\sin s, -5\sin s)$, $s \in [-2, 0]$, the solutions $x(t)$ of (4.2).

5. Conclusions

This paper investigates the asymptotic behavior for a class of n-dimensional population dynamics systems described by delay differential equations. With the help of technique of differential inequality, we show that each solution of the addressed systems tends to a constant vector as $t \to \infty$, which includes many generalizations of Bernfeld-Haddock conjecture. Our results extend some existing literatures. However, if we generalize the conjecture to a differential equation with impulses, can we have a similar conclusion? It is an interesting and meaningful work, we leave it as an open problem.

Acknowledgments

The authors would like to thank the anonymous referees and the editor for very helpful suggestions and comments which led to improvements of our original paper. This work was supported by the National Natural Science Foundation of China (Nos. 11971076, 51839002)

Conflict of interests

We confirm that we have no conflict of interest.

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