AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(4): 3365–3377.
DOI:10.3934/math.2020217
Received: 31 December 2019
Accepted: 26 March 2020
Published: 31 March 2020

Research article

Regions of variability for a subclass of analytic functions

Mohsan Raza[1,*], **Wasim Ul Haq**[2], **Jin-Lin Liu**[3] **and Saddaf Noreen**[1]

[1] Department of Mathematics, Government College University, Faisalabad, Pakistan

[2] Department of Mathematics, Abbottabad University of Science and Technology Abbottabad, Pakistan

[3] Department of Mathematics, Yangzhou University, Yangzhou 225002, China

* **Correspondence:** Email: mohsan976@yahoo.com.

Abstract: Let $A \in \mathbb{C}$, $B \in [-1, 0)$ and $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $C_\alpha[A, B]$ denotes the class of analytic functions f in the open unit disc with $f(0) = 0 = f'(0) - 1$ such that

$$e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \cos\alpha p(z) + i\sin\alpha,$$

with

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)},$$

where $w(0) = 0$ and $|w(z)| < 1$. Region of variability problems provides accurate information about a class of univalent functions than classical growth distortion and rotation theorems. In this article we find the regions of variability $V_\lambda(z_0, A, B)$ for $\log f'(z_0)$ when f ranges over the class $C_\alpha[\lambda, A, B]$ defined as

$$C_\alpha[\lambda, A, B] = \left\{f \in C_\alpha[A, B] : f''(0) = (A - B)e^{-i\alpha}\cos\alpha\right\}$$

for any fixed $z_0 \in E$ and $\lambda \in \overline{E}$. As a consequence, the regions of variability are also illustrated graphically for different sets of parameters.

Keywords: robertson functions; spirallike functions; Janowski functions; Schwarz lemma; region of variability
Mathematics Subject Classification: 30C45, 30C55, 30C80

1. Introduction

Let A be the class of functions f analytic in the open unit disc $E = \{z : |z| < 1,\ z \in \mathbb{C}\}$ with the usual normalization $f(0) = f'(0) - 1 = 0$ and S be the subclass of A consisting of functions which are univalent in E. Consider that A as a topological vector space endowed with the topology of uniform convergence over a compact subsets of E. Also let $\mathcal{B}$ denote the class of analytic functions w in E such that $|w(z)| < 1$ and $w(0) = 0$. A function f is said to be subordinate to a function g written as $f \prec g$, if there exists a Schwarz function $w \in \mathcal{B}$ such that $f(z) = g(w(z))$. In particular if g is univalent in E, then $f(0) = g(0)$ and $f(E) \subset g(E)$.

A set D in the complex plane is said to be starlike with respect to a point w_0, an interior point of D if the line segment with initial point w_0 lies entirely in D. If a function f maps E onto a domain that is starlike with respect to w_0, then we say that f is starlike with respect to w_0. In the special case that f is starlike function with respect to origin. The class of univalent starlike functions with respect to origin is denoted by S^*. The class of starlike functions with respect to the origin have been studied extensively, for some details [5, 6].

In 1933 Spacek [19] extended the idea of starlike functions by using the logarithmic spirals instead of line segment. Let $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The curve $\gamma_\alpha : t \to \exp\left(te^{i\alpha}\right)$, $t \in \mathbb{R}$ and its rotation $e^{i\theta}\gamma_\alpha(t)$, $\theta \in \mathbb{R}$ are called α-spirals. A domain $D \subset \mathbb{C}$ is said to be α-spirallike with respect to the origin if the spiral with initial point 0 to every point in D lies in D. A function $f \in A$ is spirallike if it maps E onto a domain which is spirallike with respect to 0. The class of spirallike functions is denoted by S_α. In otherwise

$$S_\alpha = \left\{ f \in A : Re\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > 0,\ z \in E \right\}.$$

Kim and Sugawa [10] introduced the notion of α-argument. Consider that $\theta = \arg_\alpha w$ with $w \in e^{i\theta}\gamma_\alpha(\mathbb{R})$. By using the α-argument, it is clear that $f \in S_\alpha$ if and only if

$$\frac{\partial}{\partial\theta}\left(arg_\alpha f(re^{i\theta})\right) > 0 \ (\theta \in \mathbb{R}, 0 < r < 1).$$

For some details about the spirallike functions [1, 5, 6]. A function f with $f(0) = f'(0) - 1 = 0$ is said to be Robertson functions if

$$Re\left\{e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0,\ z \in E.$$

The class of Robertson functions is denoted by C_α and defined by Robertson [17]. It is clear that $C_0 = C$, usual class of convex functions. In the view of above discussions about the spirallike functions and Robertson functions, we have the following relation. Let $f \in A$. Then $f \in C_\alpha$ if and only if $zf' \in S_\alpha$. By using the concept of Janowski [8], Sen et al. [18] studied the class $C_\alpha[A, B]$. Let $A \in \mathbb{C}$, $B \in [-1, 0)$ and $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $C_\alpha[A, B]$ denotes the class of analytic functions f in the open unit disc with $f(0) = 0 = f'(0) - 1$ such that

$$e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \cos\alpha p(z) + i\sin\alpha,$$

with

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}.$$

The single-valued branch of logarithms for $f'(z)$ when $f \in C_\alpha[A,B]$ is denoted by $\log f'(z)$ with $\log f'(0) = 0$. Yanagihara [21,22] determined the region of variability for the class of convex functions. Ponnusamy et al. [13] found the region of variability for some subclasses of univalent functions. For some work on region of variability, [7, 11–16] and references therein.

Using the Herglotz representation for Janowski functions, it can be seen that for $f \in C_\alpha[A,B]$ there exists a unique positive unit measure μ in $(-\pi,\pi]$ such that

$$1 + \frac{zf''(z)}{f'(z)} = e^{-i\alpha}\left[\cos\alpha \int_{-\pi}^{\pi} \frac{1 + Aze^{-it}}{1 + Bze^{-it}} d\mu(t) + i\sin\alpha\right]. \tag{1.1}$$

This shows that

$$\log f'(z) = \frac{A-B}{B} e^{-i\alpha}\cos\alpha \int_{-\pi}^{\pi} \frac{Be^{-it}}{1 + Bze^{-it}} d\mu(t). \tag{1.2}$$

It follows from (1.2) that for each fixed $z_0 \in E$, the region of variability is the set $V_\lambda[z_0,A,B]$ given as

$$\left\{\frac{A-B}{B} e^{-i\alpha}\cos\alpha \log(1 + Bz) : |z| \le |z_0|\right\}.$$

Let

$$p_f(z) = e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

Then there exists a Schwarz function $w \in \mathcal{B}$, such that

$$w(z) = \frac{p_f(z) - e^{i\alpha}}{A\cos\alpha + iB\sin\alpha - Bp_f(z)}. \tag{1.3}$$

Let $f \in C_\alpha[A,B]$. Then by applying Schwarz lemma, that is $\left|w_p'(0)\right| \le 1$ [4], we get

$$|f''(0)| = (A-B)\cos\alpha\left|\omega_f'(0)\right| \le (A-B)\cos\alpha,$$

for some $\lambda \in \overline{E}$, Now for $\lambda \in \overline{E} = \{z \in \mathbb{C} : |z| \le 1\}$ and $z_0 \in E$, we introduce

$$C_\alpha[\lambda,A,B] = \left\{f \in C_\alpha[A,B] : f''(0) = \lambda(A-B)e^{-i\alpha}\cos\alpha\right\}$$

and

$$V_\lambda(z_0,A,B) = \{\log f'(z_0) : f \in C_\alpha[\lambda,A,B]\}.$$

The aim of this article is to investigate the region of variability $V_\lambda(z_0,A,B)$ for the class $f \in C_\alpha[\lambda,A,B]$.

2. Basic properties of $V_\lambda(z_0,A,B)$

We start our investigations by studying certain general properties of the set $V_\lambda(z_0,A,B)$ such as compactness and convexity.

Proposition 2.1. *(i) $V_\lambda(z_0, A, B)$ is a compact subset of $\mathbb{C}$.*

(ii) $V_\lambda(z_0, A, B)$ is a convex subset of $\mathbb{C}$.

(iii) If $|\lambda| = 1$ or $z_0 = 0$, then $V_\lambda(z_0, A, B) = \left\{\frac{A-B}{B} e^{-i\alpha} \cos\alpha \log(1 + B\lambda z_0)\right\}$ and if $|\lambda| < 1$ and $z_0 \neq 0$, then the set $V_\lambda(z_0, A, B)$ has $e^{-i\alpha} \cos\alpha \log(1 + B\lambda z_0)$ an interior point.

Proof. **(i)** Since $C_\alpha[\lambda, A, B]$ is a compact subset of $\mathbb{C}$, therefore $V_\lambda(z_0, A, B)$ is also compact.

(ii) Let $f_1, f_2 \in C_\alpha[\lambda, A, B]$. Then for $0 \leq t \leq 1$, the function

$$f(z) = \int_0^z (f_1(\varsigma))^{1-t} (f_2(\varsigma))^t \, d\varsigma$$

is also in $C_\alpha[\lambda, A, B]$, therefore $V_\lambda(z_0, A, B)$ is convex because $\log f'(z) = (1-t)\log f_1'(z) + t \log f_2'(z)$, $t \in [0, 1]$.

(iii) Since $|\lambda| = |w_f'(0)| = 1$, then from Schwarz lemma, we obtain $w_f(z) = \lambda z$, which yields $p(z) = \frac{1+\lambda A z}{1+\lambda B z}$. This implies that

$$\log f'(z) = \frac{A-B}{B} e^{-i\alpha} \cos\alpha \log(1 + B\lambda z_0).$$

Therefore

$$V_\lambda(z_0, A, B) = \left\{\frac{A-B}{B} e^{-i\alpha} \cos\alpha \log(1 + B\lambda z_0)\right\}.$$

This also trivially holds true when $z_0 = 0$. For $\lambda \in E$ and $\alpha \in \overline{E}$, set

$$\delta(z, \lambda) = \frac{z + \lambda}{1 + \overline{\lambda} z},$$

$$F_{a,\lambda}(z) = \int_0^z \left(\exp \int_0^{\varsigma_2} \frac{e^{-i\alpha} \cos\alpha\, (A - B)\, \delta(a\varsigma_1, \lambda)}{1 + B\varsigma_1 \delta(a\varsigma_1, \lambda))} d\varsigma_1 \right) d\varsigma_2, \; z \in E. \tag{2.1}$$

Then $F_{a,\lambda}(z)$ is in $C_\alpha[\lambda, A, B]$ and $w_f(z) = z\delta(az, \lambda)$. For fixed $\lambda \in E$ and $z_0 \in E\backslash\{0\}$ the function

$$E \ni a \mapsto \log F'_{a,\lambda}(z) = \int_0^{z_0} \frac{e^{-i\alpha} \cos\alpha\, (A - B)\, (a\varsigma + \lambda)}{1 + \left(a\overline{\lambda} + B\lambda\right)\varsigma + aB\varsigma^2} d\varsigma$$

is a non-constant analytic function of $a \in E$, and therefore is an open mapping. Hence $\log F'_{0,\lambda}(z) = \left\{\frac{A-B}{B} e^{-i\alpha} \cos\alpha \log(1 + B\lambda z_0)\right\}$ is an interior point of $\left\{\log F'_{a,\lambda}(z) : a \in E\right\} \subset V_\lambda(z_0, A, B)$. □

Keeping in view the above proposition, it is sufficient to find $V_\lambda(z_0, A, B)$ for $0 \leq \lambda < 1$ and $z_0 \in E\backslash\{0\}$.

3. Main results

In this section, we state and prove some results which are needed in the proof of our main theorem.

In the following proposition, we prove that for $f \in C_\alpha[\lambda, A, B]$, the ratio $f''(z)/f'(z)$ is contained in a closed disc with center $q(z, \lambda)$ and radius $r(z, \lambda)$.

Proposition 3.1. *For $C_\alpha[\lambda, A, B]$, we have*

$$\left|\frac{f''(z)}{f'(z)} - q(z,\lambda)\right| \le r(z,\lambda), \tag{3.1}$$

where

$$\begin{aligned} q(z,\lambda) &= \frac{D(z,\lambda) + |z|^2|\tau(z,\lambda)|^2 E(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2}, \\ r(z,\lambda) &= \frac{|z|\,|\tau(z,\lambda)|\,|D(z,\lambda)+E(z,\lambda)|}{1-|z|^2|\tau(z,\lambda)|^2}. \end{aligned} \tag{3.2}$$

Proof. Since $f \in C_\alpha[\lambda, A, B]$. Then by using Schwarz lemma for $w_p \in \mathcal{B}$ with $w_p'(0) = \lambda$ such that

$$\left|\frac{\frac{w_f(z)}{z} - \lambda}{1 - \bar{\lambda}\frac{w_f(z)}{z}}\right| \le |z|. \tag{3.3}$$

Now from (1.3) this can be written equivalently as

$$\left|\frac{\frac{f''(z)}{f'(z)} - D(z,\lambda)}{\frac{f''(z)}{f'(z)} + E(z,\lambda)}\right| \le |z|\,|\tau(z,\lambda)|, \tag{3.4}$$

where

$$\begin{aligned} D(z,\lambda) &= \frac{\lambda e^{-i\alpha}(A-B)\cos\alpha}{1+B\lambda z}, \quad E(z,\lambda) = \frac{(B-A)e^{-i\alpha}\cos\alpha}{\bar{\lambda}+Bz}, \\ \tau(z,\lambda) &= \frac{-\bar{\lambda}-Bz}{1+B\lambda z}. \end{aligned} \tag{3.5}$$

This is equivalent to

$$\left|\frac{f''(z)}{f'(z)} - \frac{D(z,\lambda)+|z|^2|\tau(z,\lambda)|^2E(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2}\right| \le \frac{|z|\,|\tau(z,\lambda)|\,|A(z,\lambda)+B(z,\lambda)|}{1-|z|^2|\tau(z,\lambda)|^2}. \tag{3.6}$$

Now after simple calculations, we have

$$1-|z|^2|\tau(z,\lambda)|^2 = \frac{1-B^2|z|^4 + 2B\left(1-|z|^2\right)Re\lambda z + |\lambda|^2|z|^2\left(B^2-1\right)}{|1+B\lambda z|^2}.$$

Also

$$D(z,\lambda)+E(z,\lambda) = \frac{e^{-i\alpha}\cos\alpha(A-B)\left(|\lambda|^2-1\right)}{(1+B\lambda z)\left(\bar{\lambda}+Bz\right)}$$

and

$$D(z,\lambda)+|z|^2|\tau(z,\lambda)|^2E(z,\lambda) = \frac{e^{-i\alpha}(A-B)\cos\alpha\left\{\lambda\left(1-|z|^2\right)+B\bar{z}\left(|\lambda|^2-|z|^2\right)\right\}}{|1+B\lambda z|^2}.$$

By setting

$$\begin{aligned} q(z,\lambda) &= \frac{D(z,\lambda)+|z|^2|\tau(z,\lambda)|^2 E(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2}, \\ r(z,\lambda) &= \frac{|z||\tau(z,\lambda)||D(z,\lambda)+E(z,\lambda)|}{1-|z|^2|\tau(z,\lambda)|^2}. \end{aligned}$$

The relation (3.1) occurs from (3.6) and the above relations. Equality is attained in (3.1) when $f = F_{e^{i\theta},\lambda}(z)$, for some $z \in E$. Conversely if equality occurs in (3.1) for some $z \in E\backslash\{0\}$, then equality must hold in (3.3). Thus by Schwarz lemma there exists $\theta \in \mathbb{R}$ such that $w_f(z) = z\delta(e^{i\theta}z,\lambda)$ for all $z \in E$. This implies $f = F_{e^{i\theta},\lambda}$. □

Geometrically the above proposition means that the functional $\log f'$ lies in the closed disk centred at $q(z,\lambda)$ with radius $r(z,\lambda)$.

For $\lambda = 0$, we have the following special result which gives us bounds on pre-Schwarzian norm of locally univalent functions.

Corollary 3.2. *Let $f \in C_\alpha[0,A,B]$. Then*

$$\left|\frac{f''(z)}{f'(z)} - \frac{-e^{-i\alpha}(A-B)\cos\alpha\, B\bar{z}|z|^2}{1-B^2|z|^4}\right| \le \frac{|z||A-B|\cos\alpha}{1-B^2|z|^4}.$$

Therefore

$$\left(1-|B||z|^2\right)\left|\frac{f''(z)}{f'(z)}\right| \le |A-B||z|\cos\alpha.$$

Since $|B| \le 1$, so

$$\left(1-|z|^2\right)\left|\frac{f''(z)}{f'(z)}\right| \le |A-B||z|\cos\alpha.$$

The pre-Schwarzian norm for locally univalent functions is defined as

$$\|f\| = \sup_{z\in E}\left(1-|z|^2\right)\left|\frac{f''(z)}{f'(z)}\right|.$$

It is well-known that $\|f\| \le 6$, if f is univalent. Becker and Pommerenke [2] proved that if $\|f\| \le 1$, then f is univalent in E and this bound is sharp. Yamashita [20] proved that if f is convex, then $\|f\| \le 1$. The norm estimates for some subclasses of univalent functions are studied by many authors. For some details [3,9]. From Corollary 3.2, it is evident that for $f''(0)$ and $A = 1, B = -1$, we have $\|f\| \le 2\cos\alpha$ for Robertson functions. This result was proved by Ponnusamy et al. [15] also for $\alpha = 0$, we have $\|f\| \le 2$.

In the following result, we prove that the set $V_\lambda(z_0,A,B)$ is contained in a closed disc with centre $Q(\lambda,r)$ and radius $R(\lambda,r)$.

Corollary 3.3. *Consider the curve $\gamma : z(t),\ 0 \le t \le 1$ in E with $z(0) = 0$ and $z(1) = z_0$, then*

$$V_\lambda(z_0,A,B) \subset \bar{E}(Q(\lambda,r), R(\lambda,r)) = \{\omega \in \mathbb{C} : |\omega - Q(\lambda,r)| \le R(\lambda,r)\},$$

with

$$Q(\lambda, r) = \int_0^1 q(z(t), \lambda) z'(t)\, dt,$$

$$R(\lambda, r) = \int_0^1 r(z(t), \lambda) z'(t)\, dt,$$

where $q(z, \lambda)$ and $r(z, \lambda)$ are given in Proposition 3.1.

Proof. Suppose that $f \in C_\alpha[\lambda, A, B]$, then from proposition 3.1, we get

$$\begin{aligned} \left|\log f'(z_0) - Q(\lambda, r)\right| &= \left| \int_0^1 \left\{ \frac{f''(z)}{f'(z)} - q(z(t), \lambda) \right\} z'(t)\, dt \right| \\ &\leq \int_0^1 \left| \frac{f''(z)}{f'(z)} - q(z(t), \lambda) \right| |z'(t)|\, dt. \end{aligned}$$

Now using proposition 3.1, we get

$$\left|\log f'(z_0) - Q(\lambda, r)\right| \leq \int_0^1 r(z(t), \lambda) \left|z'(t)\right| dt = R(\lambda, r).$$

This shows $\log f'(z_0) \in \overline{D}(Q(\lambda, r), R(\lambda, r))$. Hence the required result. □

We need the following lemma which ensures the existence of normalized starlike function which is useful in the proof of next result.

Lemma 3.4. *For $\theta \in \mathbb{R}$ and $|\lambda| < 1$, the function*

$$G(z) = \int_0^z \frac{e^{i\theta}\zeta^2}{\left(1 + (\overline{\lambda} e^{i\theta} + B\lambda)\zeta + Be^{i\theta}\zeta^2\right)^2} d\zeta, \quad z \in E,$$

has zeros of order 2 at the origin and no zero elsewhere in E. Moreover, there exists a starlike normalized univalent function G_0 in E such that $G = \frac{1}{2} e^{i\theta} G_0^2$.

The above lemma is due to Ponnusamy et al. [7]. In the below proposition we show that $\log F'_{e^{i\theta},\lambda}(z_0)$ lies on the boundary of the set $V_\lambda(z_0, A, B)$.

Proposition 3.5. *Let $z_0 \in E \backslash \{0\}$. Then for $\theta \in (-\pi, \pi]$, we have $\log F'_{e^{i\theta},\lambda}(z_0) \in \partial V_\lambda(z_0, A, B)$. Further if $\log f'(z_0) = \log F'_{e^{i\theta},\lambda}(z_0)$ for $f \in C_\alpha[\lambda, A, B]$, then $f = F_{e^{i\theta},\lambda}$.*

Proof. Using 2.1, we have

$$F_{a,\lambda}(z)=\int_0^z\left(\exp\int_0^{\xi_2}\frac{e^{-i\alpha}\cos(A-B)\,\delta(a\xi_1,\lambda)}{1+B\xi_1\delta(a\xi_1,\lambda)}d\xi_1\right)d\xi_2.$$

Therefore

$$\begin{aligned}\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} &= \frac{e^{-i\alpha}\cos(A-B)\,\delta(az,\lambda)}{1+Bz\delta(az,\lambda)}\\ &= \frac{e^{-i\alpha}(A-B)\cos\alpha\ (az+\lambda)}{1+\left(\bar{\lambda}a+B\lambda\right)z+Baz^2}.\end{aligned}$$

From (3.5), it follows that

$$\begin{aligned}\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)}-D(z,\lambda) &= \frac{e^{-i\alpha}(A-B)\cos\alpha\left(1-|\lambda|^2\right)az}{\left\{1+\left(\bar{\lambda}a+B\lambda\right)z+Baz^2\right\}(1+Bz\lambda)}\\ \frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)}+E(z,\lambda) &= \frac{e^{-i\alpha}(A-B)\cos\alpha\left(|\lambda|^2-1\right)}{\left\{1+\left(\bar{\lambda}a+B\lambda\right)z+Baz^2\right\}\left(\bar{\lambda}+Bz\right)}.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)}-q(z,\lambda)\\ =\ &\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)}-\frac{D(z,\lambda)+|z|^2|\tau(z,\lambda)|^2E(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2}\\ =\ &\frac{1}{1-|z|^2|\tau(z,\lambda)|^2}\left\{\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)}-D(z,\lambda)-|z|^2|\tau(z,\lambda)|^2\left(\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)}+E(z,\lambda)\right)\right\}\\ =\ &\frac{e^{-i\alpha}(A-B)\cos\alpha\left(1-|\lambda|^2\right)\overline{\left(1+(\bar{\lambda}e^{i\theta}+B\lambda)z+Be^{i\theta}z^2\right)}}{\left\{1-B^2|z|^4+2B\left(1-|z|^2\right)Re\lambda z+|\lambda|^2|z|^2(B^2-1)\right\}\left\{1+\left(\bar{\lambda}a+B\lambda\right)z+Baz^2\right\}}.\end{aligned}$$

Putting $a=e^{i\theta}$, we get

$$\frac{F''_{e^{i\theta},\lambda}(z)}{F'_{e^{i\theta},\lambda}(z)}-q(z,\lambda)=\frac{r(z,\lambda)\,e^{i\theta}e^{-i\alpha}z}{|z|}\frac{\left|1+\left(\bar{\lambda}e^{i\theta}+B\lambda\right)z+Be^{i\theta}z^2\right|^2}{\left(1+\left(\bar{\lambda}e^{i\theta}+B\lambda\right)z+Be^{i\theta}z^2\right)^2}.$$

By using Lemma 3.4, we obtain

$$\frac{F''_{e^{i\theta},\lambda}(z)}{F'_{e^{i\theta},\lambda}(z)}-q(z,\lambda)=r(z,\lambda)\frac{e^{-i\alpha}G'(z)}{|G'(z)|}. \tag{3.7}$$

Using the argument of Lemma 3.4 that $G=2^{-1}e^{i\theta}G_0^2$, where G_0 is starlike in E with $G_0(0)=G_0'(0)-1=0$, for any $z_0\in E\backslash\{0\}$ the linear segment joining 0 and $G_0(z_0)$ lies entirely in $G_0(E)$. Let γ_0 be the curve defined by

$$\gamma_0 : z(t)=G_0^{-1}\left(tG_0(z_0)\right),\ t\in[0,1].$$

Since $G(z(t)) = 2^{-1}e^{i\theta}(G_0(z(t)))^2 = 2^{-1}e^{i\theta}(tG_0(z_0))^2 = t^2G(z_0)$. Differentiation w.r.t t gives us

$$G'(z(t))z'(t) = 2tG(z_0), \quad t \in [0,1]. \tag{3.8}$$

Therefore

$$\left\{\frac{F''_{e^{i\theta},\lambda}(z)}{F'_{e^{i\theta},\lambda}(z)} - q(z(t),\lambda)\right\} z'(t) = r(z(t),\lambda)\frac{e^{-i\alpha}G(z_0)}{|G(z_0)|}|z'(t)|.$$

This relation together with (3.7), we get

$$\begin{aligned} \log F'_{e^{i\theta},\lambda}(z) - Q(\gamma_0,\lambda) &= \int_0^1 \left(\frac{F''_{e^{i\theta},\lambda}(z)}{F'_{e^{i\theta},\lambda}(z)} - q(z,\lambda)\right) z'(t)\,dt \\ &= \int_0^1 r(z(t),\lambda)\frac{e^{-i\alpha}G'(z(t))z'(t)}{|G'(z(t))z'(t)|}|z'(t)|dt \\ &= \frac{e^{-i\alpha}G(z_0)}{|G(z_0)|}\int_0^1 r(z(t),\lambda)|z'(t)|dt \\ &= \frac{e^{-i\alpha}G(z_0)}{|G(z_0)|}R(\lambda,\gamma_0). \end{aligned} \tag{3.9}$$

This shows that $\log F'_{e^{i\theta},\lambda}(z) \in \partial E(Q(\gamma_0,\lambda), R(\gamma_0,r))$, where $Q(\lambda,\gamma_0)$ and $R(\lambda,\gamma_0)$ are defined as in Corollary 3.3. Also we have $\log F'_{e^{i\theta},\lambda}(z_0) \in V_\lambda(z_0,A,B)$, therefore $\log F'_{e^{i\theta},\lambda}(z_0) \in \partial V_\lambda(z_0,A,B)$.

Now we have to prove $\log f'(z_0) = \log F'(z_0)$ for some $f \in C_\alpha[\lambda,A,B]$, we have

$$h(t) = e^{-i\alpha}\frac{|G(z_o)|}{G(z_o)}\left\{\frac{f''(z(t))}{f'(z(t))} - q(z(t),\lambda)\right\} z'(t) \tag{3.10}$$

$$k(t) = e^{i\alpha}\frac{|G(z_o)|}{G(z_o)}\left\{\frac{F''(z(t))}{F'(z(t))} - q(z(t),\lambda)\right\} z'(t),$$

where $\gamma_0 : z(t),\ 0 \le t \le 1$. Then the function h is continuous and

$$|h(t)| = \left|\frac{f''(z(t))}{f'(z(t))} - q(z(t),\lambda)\right| \left|z'(t)\right|.$$

Using Proposition 3.1, we have

$$|h(t)| \le r(z(t),\lambda)\left|z'(t)\right|.$$

Now using Proposition 3.1, we get $|h(t)| \le r(z(t),\lambda)|z'(t)|$. Further from (3.9), we have From (3.7) and (3.8), this implies that $\frac{f''(z)}{f'(z)} = \frac{F''_{e^{i\theta},\lambda}(z)}{F'_{e^{i\theta},\lambda}(z)}$ on γ_0. The identity theorem for analytic functions yields us $f = F_{e^{i\theta},\lambda}$, $z \in E$. □

Main theorem

In our main result, we give precise description of regions of variability for the class $C_\alpha[\lambda, A, B]$ and show that the boundary $\partial V_\lambda(z_0, A, B)$ is a Jordan curve.

Theorem 3.6. *Let $\lambda \in E$ and $z_0 \in E\backslash\{0\}$. Then boundary $\partial V_\lambda(z_0, A, B)$ is the Jordan curve given by*

$$(-\pi, \pi] \ni \theta \mapsto \log F'_{e^{i\theta},\lambda}(z_0) = \int_0^{z_0} \frac{e^{-i\alpha}\cos\alpha\,(A-B)\,\delta(a\varsigma, \lambda)}{1 + B\varsigma\delta(a\varsigma, \lambda))} d\varsigma.$$

If $\log f'(z_0) = \log F'_{e^{i\theta},\lambda}(z_0)$ *for some* $f \in C_\alpha[\lambda, A, B]$ *and* $\theta \in (-\pi, \pi]$, *then* $f(z) = F_{e^{i\theta},\lambda}(z)$.

Proof. First we have to show that the curve

$$(-\pi, \pi] \ni \theta \mapsto \log F'_{e^{i\theta},\lambda}(z_0)$$

is simple. Let us assume that

$$\log F'_{e^{i\theta_1},\lambda}(z_0) = \log F'_{e^{i\theta_2},\lambda}(z_0)$$

for some $\theta_1, \theta_2 \in (-\pi, \pi]$ with $\theta_1 \neq \theta_2$. Then the use of Proposition 3.5 yield us that $F'_{e^{i\theta_1},\lambda}(z_0) = F'_{e^{i\theta_2},\lambda}(z_0)$, which further gives the following relation

$$\tau\left(\frac{w_{F_{e^{i\theta_1},\lambda}}(z)}{z}, \lambda\right) = \tau\left(\frac{w_{F_{e^{i\theta_2},\lambda}}(z)}{z}, \lambda\right).$$

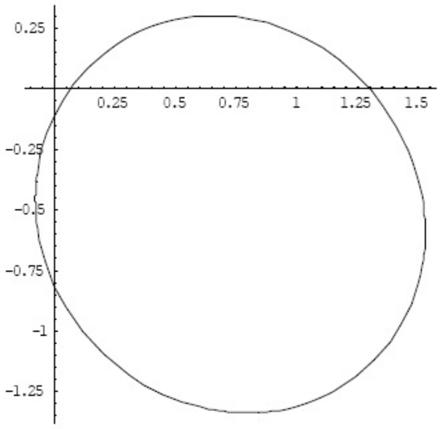
This implies that

$$\frac{B(ze^{i\theta_1} + \lambda) + \overline{\lambda}(1 + \overline{\lambda}e^{i\theta_1}z)}{1 + \overline{\lambda}e^{i\theta_1}z + \lambda B(ze^{i\theta_1} + \lambda)} = \frac{B(ze^{i\theta_2} + \lambda) + \overline{\lambda}(1 + \overline{\lambda}e^{i\theta_2}z)}{1 + \overline{\lambda}e^{i\theta_2}z + \lambda B(ze^{i\theta_2} + \lambda)}.$$

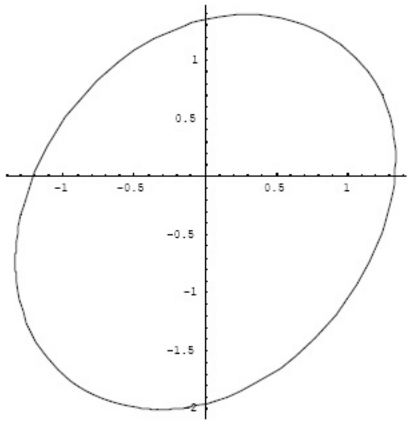
After some simplification, we obtain $ze^{i\theta_1} = ze^{i\theta_2}$, which leads us to a contradiction. Hence the curve is simple. Since $V_\lambda(z_0, A, B)$ is compact convex subset of $\mathbb{C}$ and has non-empty interior, therefore the boundary $\partial V_\lambda(z_0, A, B)$ is a simple closed curve. From Proposition 3.5 the curve $\partial V_\lambda(z_0, A, B)$ contains the curve $(-\pi, \pi] \ni \theta \mapsto \log F_{e^{i\theta},\lambda}(z_0)$. Since a simple closed curve cannot contain any simple closed curve other than itself. Thus $\partial V_\lambda(z_0, A, B)$ is given by $(-\pi, \pi] \ni \theta \mapsto \log F'_{e^{i\theta},\lambda}(z_0)$.

Geometric view of theorem

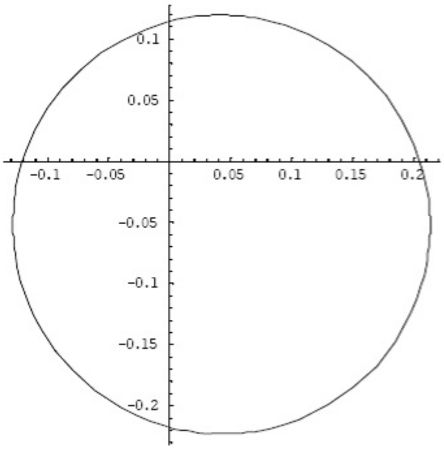
The following figures show us the geometric view of our main theorem with various choices of involved parameters.



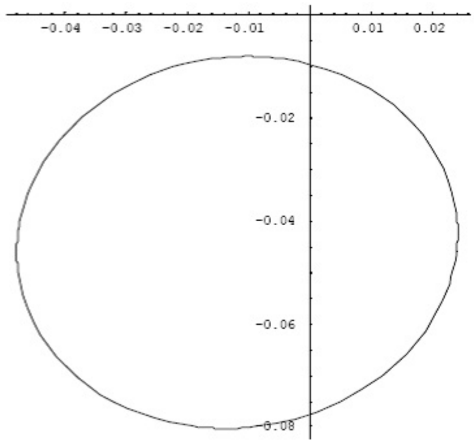
$$z_0 = 0.335192 - 0.787333i$$
$$\lambda = 0.0737292 + 0.466706i$$
$$A = 1, \; B = -1, \; \alpha = \frac{\pi}{4}$$



$$z_0 = -0.261209 + 0.926935i$$
$$\lambda = 0.0737292 + 0.466706i$$
$$\beta = -1.991244, \quad \gamma = 0.383292$$
$$A = 2(1-\beta)\cos\gamma e^{-i\gamma} - 1, \quad B = -1, \; \alpha = -\frac{\pi}{4}$$



$$z_0 = -0.41227 - 0.521734i$$
$$\lambda = -0.0875648 + 0.0714166i$$
$$A = 0.9868233 + 0.00835453i$$
$$B = -0.50, \; \alpha = \frac{\pi}{3}$$



$$z_0 = 0.771264 + 0.1512040i$$
$$\lambda = -0.391149 - 0.294747i$$
$$A = -0.2346400 - 0.180560i$$
$$B = -0.50, \quad \alpha = -\frac{\pi}{3}$$

□

Acknowledgments

Authors are thankful to the editor and anonymous referees for their valuable comments and suggestions.

Conflict of interest

The authors declare no conflicts of interest.

References

1. O. P. Ahuja, H. Silverman, *A survey on spiral-like and related function classes*, Math. Chronicle, **20** (1991), 39–66.
2. J. Becker, Ch. Pommerenke, *Schlichtheitskriterien und Jordangebiete*, J. Reine Angew. Math., **354** (1984), 74–94.
3. J. H. Choi, Y. C. Kim, S. Ponnusamy, et al. *Norm estimates for the Alexander transforms of convex functions of order alpha*, J. Math. Anal. Appl., **303** (2005), 661–668.
4. S. Dineen, *The Schwarz lemma*, Oxford Math. Monogr., Clarendon Press Oxford, 1989.
5. P. L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften, New York, Berlin, Heidelberg, Tokyo, Springer-Verlag, 1983.
6. A. W. Goodman, *Univalent Functions, Vols. I and II*, Mariner Publishing Co. Tampa, Florida, 1983.
7. W. U. Haq, *Variability regions for Janowski convex functions*, Complex Var. Elliptic Equ., **59** (2014), 355–361.
8. J. Janowski, *Some extremal problems for certain families of analytic functions I*, Ann. Polon. Math., **28** (1973), 297–326.
9. Y. C. Kim, S. Ponnusamy, T. Sugawa, *Mapping properties of nonlinear integral operators and pre-Schwarzian derivatives*, J. Math. Anal. Appl., **299** (2004), 433–447.
10. Y. C. Kim, T. Sugawa, *Correspondence between spirallike functions and starlike functions*, Math. Nach., **285** (2012), 322–331.
11. S. Ponnusamy, A. Vasudevarao, *Regions of variability for functions with positive real part*, Ann. Polon. Math., **99** (2010), 225–245.
12. S. Ponnusamy, A. Vasudevarao, *Region of variability of two subclasses of univalent functions*, J. Math. Anal. Appl., **332** (2007) 1323–1334.
13. S. Ponnusamy, A. Vasudevarao, M. Vuorinen, *Region of variability for certain classes of univalent functions satisfying differential inequalities*, Complex var. Elliptic Equ., **54** (2009), 899–922.
14. S. Ponnusamy, A. Vasudevarao, H. Yanagihara, *Region of variability for close-to-convex functions*, Complex Var. Elliptic Equ., **53** (2008), 709–716.
15. S. Ponnusamy, A. Vasudevarao, H. Yanagihara, *Region of variability of univalent functions $f(z)$ for which $zf'(z)$ is spirallike*, Houston J. Math., **34** (2008), 1037–1048.

16. M. Raza, W. U. Haq, S. Noreen, *Regions of Variability for Janowski Functions*, Miskloc Math. Notes, **16** (2015), 1117–1127.

17. M. S. Robertson, U*nivalent functions* $f(z)$ *for which* $zf'(z)$ *is spirallike*, Mich. Math. J., **16** (1969), 97–101.

18. A. Y. Sen, Y. Polatoglu, M. Aydogan, *Distortion theorem and the radius of convexity for Janowski-Robertson functions*, Stud. Univ. Babeş-Bolyai Math., **57** (2012), 291–294.

19. L. Spacek, *Contribution a la theorie des fonctions univalentes (in Czech)*, Časopis Pest. Mat., **62** (1933), 12–19.

20. S. Yamashita, *Norm estimates for function starlike or convex of order alpha*, Hokkaido Math. J., **28** (1999), 217–230.

21. H. Yanagihara, *Regions of variability for convex function*, Math. Nach., **279** (2006), 1723–1730.

22. H. Yanagihara, *Regions of variability for functions of bounded derivatives*, Kodai Math. J., **28** (2005), 452–462.