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Research article

On inequalities of Bellman and Aczél type

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Abstract: The purpose of this paper is to prove some eigenvalue inequalities involving convex functions. These extend many remarkable inequalities, most of them related to the Bellman and Aczél inequalities.

Keywords: positive semidefinite matrix; weak majorization; convex functions; Bellman inequality; Aczél inequality
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1. Introduction

Let $\mathbb{B}(\mathcal{H})$ be the C^*- algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. When $\mathcal{H}$ is finite dimensional of dimension n, we identify $\mathbb{B}(\mathcal{H})$ with $\mathcal{M}_n$; the algebra of all complex $n\times n$ matrices. Let $\mathscr{H}_n$ be the set of all Hermitian matrices in $\mathcal{M}_n$. We denote by $\mathscr{H}_n(J)$ the set of all Hermitian matrices in $\mathcal{M}_n$ whose spectra are contained in an interval $J \subseteq \mathbb{R}$. The set of all positive definite matrices in $\mathcal{M}_n$ is denoted by $\mathscr{P}_n$. By I_n we denote the identity matrix of $\mathcal{M}_n$. We write $A \geq 0$ if A is a positive semidefinite matrix, and $A > 0$ if $A \geq 0$ is invertible (or strictly positive definite). A linear map $\Phi : \mathcal{M}_n \to \mathcal{M}_m$ is said to be positive if $0 \leq \Phi(A)$ when $0 \leq A$. If, in addition, $\Phi(I_n) = I_m$, it is said to be unital.

When $0 \leq t \leq 1$, the arithmetic mean and geometric mean of $A, B > 0$ are defined and denoted by

$$A\nabla_t B = (1-t)A + tB, \quad A\sharp_t B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}.$$

Let $x = (x_1, \ldots, x_n)$ be an element of $\mathbb{R}^n$. Let $x^{\downarrow}$ and $x^{\uparrow}$ be the vectors obtained by rearranging the coordinates of x in decreasing and increasing order respectively. Thus $x_1^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$ and $x_1^{\uparrow} \leq \cdots \leq x_n^{\uparrow}$. For $A \in \mathcal{M}_n$ with real eigenvalues, $\lambda(A)$ is a vector of the eigenvalues of A. Then, $\lambda^{\downarrow}(A)$ and $\lambda^{\uparrow}(A)$ can be defined as above. Let $x, y \in \mathbb{R}^n$. The weak majorization relation $x \prec_w y$ means $\sum_{j=1}^k x_j^{\downarrow} \leq \sum_{j=1}^k y_j^{\downarrow}, (1 \leq k \leq n)$. If further equality holds for $k = n$ then we have the majorization $x \prec y$. Similarly, the weak supermajorization relation $x \prec^w y$ means $\sum_{j=1}^k x_j^{\uparrow} \geq \sum_{j=1}^k y_j^{\uparrow}, (1 \leq k \leq n)$.

Let $A \in \mathscr{H}_n(J)$ have spectral decomposition $A = U^*\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)\, U$, where U is a unitary and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. Let f be a real valued function defined on J. Then $f(A)$ is defined by $f(A) = U^*\text{diag}(f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n))\, U$.

The scalar Bellman inequality [3] says that if p is a positive integer and $a, b, a_i, b_i\, (1 \le i \le n)$ are positive real numbers such that $\sum_{i=1}^n a_i^p \le a^p$ and $\sum_{i=1}^n b_i^p \le b^p$, then

$$\left(a^p - \sum_{i=1}^n t_i^p\right)^{\frac{1}{p}} + \left(b^p - \sum_{i=1}^n s_i^p\right)^{\frac{1}{p}} \le \left((a+b)^p - \sum_{k=1}^n (a_i + b_i)^p\right)^{\frac{1}{p}}.$$

A multiplicative analogue of this inequality is due to Aczél [1]. In 1956, he proved that

$$\left(a_1^2 - \sum_{i=2}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right) \le \left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2$$

where $a_i, b_i\, (1 \le i \le n)$ are positive real numbers such that $a_1^2 - \sum_{i=2}^n a_i^2 > 0$ or $b_1^2 - \sum_{i=2}^n b_i^2 > 0$.

The operator theory related to inequalities in Hilbert space is studied in many papers. In [11, Corollary 2.2], Morassaei et al. showed the following non-commutative version of classical Bellman inequality:

$$\Phi\left((I-A)^{\frac{1}{p}} \nabla_t (I-B)^{\frac{1}{p}}\right) \le (\Phi(I - A\nabla_t B))^{\frac{1}{p}}, \quad (0 \le t \le 1,\ p > 1)$$

where $A, B \in \mathbb{B}(\mathcal{H})$ are two contractions (i.e., $0 \le A, B \le I$) and $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ is a unital positive linear map. The reverse inequality holds when $\frac{1}{2} \le p \le 1$ or $p \le -1$ [13, Theorem 3]. Actually, this result follows from the following inequality

$$f(\Phi(A)\nabla_t \Phi(B)) \le \Phi(f(A))\nabla_t \Phi(f(B))$$

where f is an operator convex.

In [12, Theorem 2.2], Moslehian noted the following inequality for non-negative operator decreasing and operator concave f and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$:

$$f(A^p)\,\sharp_{\frac{1}{q}}\, f(B^q) \le f\left(A^p \sharp_{\frac{1}{q}} B^q\right). \tag{1.1}$$

This inequality may be considered as operator versions of Aczél inequality.

As it is mentioned in [6, Corollary 1.12], the function f on $[0, \infty)$ is operator concave if and only if f is operator monotone (increasing). So, the above inequality is valid just for the functions of type $f(t) = \alpha + \beta t$. In this paper, we give a matrix version of the inequality (1.1) for decreasing concave function f on $[0, \infty)$. Let f be a convex function (in the usual sense) on J, $A, B \in \mathscr{H}_n(J)$, and $0 \le t \le 1$. In this paper we prove that the eigenvalues of $f(\Phi(A)\nabla_t \Phi(B))$ are weakly majorized by the eigenvalues of $\Phi(f(A))\nabla_t \Phi(f(B))$. The results presented in this paper are motivated by the results in [2, 10, 12, 13].

2. Main results

We start from the well-known Jensen inequality [4, p. 281]: If $A \in \mathscr{H}_n(J)$ and f is a convex (resp. concave) function on J, then for any $x \in \mathbb{C}^m$ with $\|x\| = 1$,

$$f(\langle Ax, x\rangle) \le (resp. \ge) \langle f(A)x, x\rangle. \tag{2.1}$$

The following result provides an extension of (2.1).

Lemma 2.1. *Let $\Phi : \mathcal{M}_n \to \mathcal{M}_m$ be a unital positive linear map, f be a convex function on J, and $x \in \mathbb{C}^m$ with $\|x\| = 1$. Then*

$$f(\langle \Phi(A)x, x\rangle) \le \langle \Phi(f(A))x, x\rangle \tag{2.2}$$

for all $A \in \mathscr{H}_n(J)$.

Proof. We know that if f is a convex function on an interval J, then for each point $(s, f(s))$, there exists a real number C_s such that

$$f(s) + C_s(t-s) \le f(t), \quad \text{for all } t \in J. \tag{2.3}$$

(if f is differentiable at s, then $C_s = f'(s)$.)

Fix $s \in J$. Since J contains the spectra of the A, we may replace t in the above inequality by A, via a functional calculus to get

$$f(s)I_n + C_sA - C_ssI_n \le f(A).$$

Applying the positive linear mappings Φ, this implies

$$f(s)I_m + C_s\Phi(A) - C_ssI_m \le \Phi(f(A)). \tag{2.4}$$

The inequality (2.4) easily implies, for any $x \in \mathbb{C}^m$ with $\|x\| = 1$,

$$f(s) + C_s\langle \Phi(A)x, x\rangle - C_ss \le \langle \Phi(f(A))x, x\rangle. \tag{2.5}$$

On the other hand, since Φ is unital, we have $\langle \Phi(A)x, x\rangle \in J$ where $x \in \mathbb{C}^m$ with $\|x\| = 1$. Therefore, we may replace s by $\langle \Phi(A)x, x\rangle$ in (2.5). This yields (2.2). □

Remark 2.2. From inequality (2.3) one can infer that

$$\langle \Phi(f(A))x, x\rangle + \langle \Phi(A)x, x\rangle\langle \Phi(C_A)x, x\rangle - \langle \Phi(C_AA)x, x\rangle \le f(\langle \Phi(A)x, x\rangle). \tag{2.6}$$

This inequality can be regarded as a reverse of (2.2).
Actually, inequality (2.2) implies

$$f(A) + tC_A - C_AA \le f(t)I_n.$$

From the assumptions on Φ, we can write

$$\Phi(f(A)) + t\Phi(C_A) - \Phi(C_AA) \le f(t)I_m.$$

Consequently, for any unit vector $x \in \mathbb{C}^m$,

$$\langle \Phi(f(A))x, x\rangle + t\langle \Phi(C_A)x, x\rangle - \langle \Phi(C_AA)x, x\rangle \le f(t). \tag{2.7}$$

Now, (2.6) follows from (2.7) by putting $t = \langle \Phi(A)x, x\rangle$.

We repeat the following result from [7] for completeness.

Remark 2.3. Regarding the convexity of f on the interval J, we have

$$\begin{aligned} f((1-v)s+vt) &= f\left((1-2v)s+2v\frac{s+t}{2}\right) \\ &\leq (1-2v)f(s)+2vf\left(\frac{s+t}{2}\right) \\ &= (1-v)f(s)+vf(t)-2r\left(\frac{f(s)+f(t)}{2}-f\left(\frac{s+t}{2}\right)\right) \end{aligned}$$

for any $s,t\in J$ and $r=\min\{v,1-v\}$ with $0\leq v\leq\frac{1}{2}$. For the case $\frac{1}{2}\leq v\leq 1$, the same result is true.
Thus.

$$f((1-v)s+vt)\leq(1-v)f(s)+vf(t)-2r\left(\frac{f(s)+f(t)}{2}-f\left(\frac{s+t}{2}\right)\right)$$

holds for any $s,t\in J$ and $r=\min\{v,1-v\}$ with $0\leq v\leq 1$.
From the above inequality one can write

$$f(s+v(t-s))-f(s)\leq vf(t)-vf(s)-2r\left(\frac{f(s)+f(t)}{2}-f\left(\frac{s+t}{2}\right)\right).$$

Dividing by $v>0$, we get

$$\frac{f(s+v(t-s))-f(s)}{v}\leq f(t)-f(s)-2\frac{r}{v}\left(\frac{f(s)+f(t)}{2}-f\left(\frac{s+t}{2}\right)\right).$$

Now, if $v\to 0$, and by taking into account that for $0\leq v\leq\frac{1}{2}$, $r=v$ we infer

$$f(s)+f'(s)(t-s)\leq f(t)-2\left(\frac{f(s)+f(t)}{2}-f\left(\frac{s+t}{2}\right)\right).$$

This result can be considered as a refinement of inequality (2.3). Thus, if we apply the same arguments as in Lemma 2.1, we can obtain a sharper estimate than (2.2). Namely,

$$\begin{aligned} & f(\langle\Phi(A)x,x\rangle) \\ & \leq\langle\Phi(f(A))x,x\rangle \\ & \quad -2\left(\frac{f(\langle\Phi(A)x,x\rangle)+\langle\Phi(f(A))x,x\rangle}{2}-\left\langle\Phi\left(f\left(\frac{\langle\Phi(A)x,x\rangle I_n+A}{2}\right)\right)x,x\right\rangle\right). \end{aligned}$$

Lemma 2.4. *[4, p. 35] If $\lambda_j^{\downarrow}(A)$ denote the eigenvalues of $n\times n$ Hermitian matrix A arranged in decreasing order, then*

$$\sum_{j=1}^{k}\lambda_j^{\downarrow}(A)=\max\sum_{j=1}^{k}\left\langle Au_j,u_j\right\rangle,\ j=1,\ldots,n$$

where the maximum is taken over all choices of orthonormal vectors $u_1,\ldots,u_k$.

Theorem 2.5. *Let $\Phi:\mathcal{M}_n\to\mathcal{M}_m$ be a unital positive linear map, f be a convex (resp. concave) function on J, and $0\leq t\leq 1$. Then*

$$\lambda^{\downarrow}(f(\Phi(A)\nabla_t\Phi(B)))\prec_w\ (resp.\ \prec^w)\,\lambda^{\downarrow}(\Phi(f(A))\nabla_t\Phi(f(B)))$$

for all $A,B\in\mathscr{H}_n(J)$.

Proof. Since $x \prec_w y$ if and only if $(-x) \prec^w (-y)$, it suffices to consider the convex case. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $(1-t)\,\Phi(A) + t\Phi(B)$ and let $u_1, \ldots, u_n$ be the corresponding orthonormal eigenvectors arranged such that $f(\lambda_1) \geq \cdots \geq f(\lambda_n)$. Let $k = 1, \ldots, n$. Then

$$\begin{aligned}
&\sum_{j=1}^{k} \lambda_j^{\downarrow}\left(f\left((1-t)\,\Phi(A) + t\Phi(B)\right)\right) \\
&= \sum_{j=1}^{k} f\left(\left\langle (1-t)\,\Phi(A) + t\Phi(B)\, u_j, u_j \right\rangle\right) \\
&= \sum_{j=1}^{k} f\left((1-t)\left\langle \Phi(A)\, u_j, u_j \right\rangle + t\left\langle \Phi(B)\, u_j, u_j \right\rangle\right) \\
&\leq \sum_{j=1}^{k} \left[(1-t)\, f\left(\left\langle \Phi(A)\, u_j, u_j \right\rangle\right) + tf\left(\left\langle \Phi(B)\, u_j, u_j \right\rangle\right)\right] \quad \text{(since } f \text{ is convex)} \\
&\leq \sum_{j=1}^{k} \left[(1-t)\left\langle \Phi(f(A))\, u_j, u_j \right\rangle + t\left\langle \Phi(f(B))\, u_j, u_j \right\rangle\right] \quad \text{(by Lemma 2.1)} \\
&= \sum_{j=1}^{k} \left\langle (1-t)\,\Phi(f(A)) + t\Phi(f(B))\, u_j, u_j \right\rangle \\
&\leq \sum_{j=1}^{k} \lambda_j^{\downarrow}\left((1-t)\,\Phi(f(A)) + t\Phi(f(B))\right) \quad \text{(by Lemma 2.4).}
\end{aligned}$$

Therefore, we conclude

$$\sum_{j=1}^{k} \lambda_j^{\downarrow}\left(f\left(\Phi(A)\,\nabla_t \Phi(B)\right)\right) \leq \sum_{j=1}^{k} \lambda_j^{\downarrow}\left(\Phi(f(A))\,\nabla_t \Phi(f(B))\right)$$

so that we get the desired conclusion. □

If we choose $\Phi(X) = X$ in Theorem 2.5, we recover [2, Theorem 2.3], i.e.,

$$\lambda^{\downarrow}\left(f\left(A \nabla_t B\right)\right) \prec_w \lambda^{\downarrow}\left(f(A)\,\nabla_t f(B)\right).$$

Corollary 2.6. *Let $\Phi : \mathcal{M}_n \to \mathcal{M}_m$ be a unital positive linear map, and $0 \leq t \leq 1$, $p \geq 1$. Then*

$$\lambda^{\downarrow}\left(\left(\Phi\left(I_n - A \nabla_t B\right)\right)^p\right) \prec_w \lambda^{\downarrow}\left(\Phi\left((I_n - A)^p \nabla_t (I_n - B)^p\right)\right) \tag{2.8}$$

for all $A, B \in \mathscr{P}_n$ such that $0 \leq A, B \leq I_n$. In particular, for all unitarily invariant norm $\|\cdot\|_u$,

$$\left\|\left(\Phi\left(I_n - A \nabla_t B\right)\right)^p\right\|_u \leq \left\|\Phi\left((I_n - A)^p \nabla_t (I_n - B)^p\right)\right\|_u.$$

Inequality (2.8) can be regarded as a weak majorization version of [13, Theorem 3].

Theorem 2.7. *Let f be a decreasing concave function on $[0, \infty)$ and $0 \leq t \leq 1$. Then*

$$\lambda_j^{\downarrow}\left(f(A)\,\sharp_t f(B)\right) \leq \lambda_j^{\downarrow}\left(f\left(A \sharp_t B\right)\right), \quad j = 1, \ldots, n \tag{2.9}$$

for all $A, B \in \mathscr{P}_n$.

Proof. By the minimax principle, for any integer j less than or equal to the dimension of the space, we have a subspace $\mathscr{F}$ of dimension j such that

$$\begin{aligned}
&\lambda_j^{\downarrow}\left(f\left(A\right)\sharp_t f\left(B\right)\right)\\
&= \min_{x\in\mathscr{F};\|x\|=1}\left\langle f\left(A\right)\sharp_t f\left(B\right)x,x\right\rangle\\
&\le \min_{x\in\mathscr{F};\|x\|=1}\left\langle f\left(A\right)\nabla_t f\left(B\right)x,x\right\rangle \quad \text{(by the arithmetic-geometric mean inequality)}\\
&= \min_{x\in\mathscr{F};\|x\|=1}\left[\left(1-t\right)\left\langle f\left(A\right)x,x\right\rangle+t\left\langle f\left(B\right)x,x\right\rangle\right]\\
&\le \min_{x\in\mathscr{F};\|x\|=1}\left[\left(1-t\right)f\left(\left\langle Ax,x\right\rangle\right)+tf\left(\left\langle Bx,x\right\rangle\right)\right] \quad \text{(by (2.1))}\\
&\le \min_{x\in\mathscr{F};\|x\|=1} f\left(\left(1-t\right)\left\langle Ax,x\right\rangle+t\left\langle Bx,x\right\rangle\right) \quad \text{(since } f \text{ is concave)}\\
&\le \min_{x\in\mathscr{F};\|x\|=1} f\left(\left\langle Ax,x\right\rangle\sharp_t\left\langle Bx,x\right\rangle\right) \quad \text{(since } f \text{ is decreasing)}\\
&\le \min_{x\in\mathscr{F};\|x\|=1} f\left(\left\langle A\sharp_t Bx,x\right\rangle\right) \quad \text{(by [5, Lemma 8])}\\
&= \min_{x\in\mathscr{F};\|x\|=1}\left\langle f\left(A\sharp_t B\right)x,x\right\rangle\\
&\le \lambda_j^{\downarrow}\left(f\left(A\sharp_t B\right)\right)
\end{aligned}$$

and hence we have (2.9). □

Note that the above statement is equivalent to the existence of a unitary operator U satisfying in the following inequality:

$$f\left(A\right)\sharp_t f\left(B\right)\le Uf\left(A\sharp_t B\right)U^*. \tag{2.10}$$

Inequality (2.10) yields inequality

$$f\left(A^p\right)\sharp_{\frac{1}{q}} f\left(B^q\right)\le Uf\left(A^p\sharp_{\frac{1}{q}}B^q\right)U^*,\ \ \frac{1}{p}+\frac{1}{q}=1,\ p,q>1.$$

Furthermore, if $AB=BA$ we can write

$$f(A^p)^{\frac{1}{p}}f(A^q)^{\frac{1}{q}}\le Uf\left(AB\right)U^*.$$

Corollary 2.8. *Let $A,B\in\mathscr{P}_n$ be contractive (in the sense that $\|A\|,\|B\|\le 1$, where $\|\cdot\|$ is the usual operator norm). Then for $j=1,\ldots,n$*

$$\lambda_j^{\downarrow}\left(\left(I_n-A\right)\sharp\left(I_n-B\right)\right)\le\lambda_j^{\downarrow}\left(\left(I_n-A\sharp B\right)\right). \tag{2.11}$$

Proof. This inequality follows immediately from Theorem 2.7 by choosing $f\left(x\right)=1-x$ on $(0,1)$ and $t=1/2$. □

Remark 2.9. It has been shown in [8] that if A,B are contractive, then for $j=1,\ldots,n$

$$\lambda_j^{\downarrow}\left(\left(I_n-A^*A\right)\sharp\left(I_n-B^*B\right)\right)\le\lambda_j^{\downarrow}\left(\left|I_n-A^*B\right|\right).$$

We refer the reader to [9] for further results of this type of inequalities. If $A, B \in \mathscr{P}_n$ are contractive, this inequality implies that

$$\lambda_j^{\downarrow}\left((I_n - A)\,\sharp\,(I_n - B)\right) \leq \lambda_j^{\downarrow}\left(\left|I_n - A^{\frac{1}{2}}B^{\frac{1}{2}}\right|\right). \tag{2.12}$$

We remark that there is no ordering between (2.11) and (2.12). To see this, letting $A = \begin{bmatrix} 0.4 & -0.1 \\ -0.1 & 0.9 \end{bmatrix}$ and $B = \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 0.7 \end{bmatrix}$. Direct computation shows that

$$\lambda_1\left(I_2 - A\sharp B\right) \approx 0.7555, \quad \lambda_2\left(I_2 - A\sharp B\right) \approx 0.2355,$$

and

$$\lambda_1\left(I_2 - A^{\frac{1}{2}}B^{\frac{1}{2}}\right) \approx 0.7639, \quad \lambda_2\left(I_2 - A^{\frac{1}{2}}B^{\frac{1}{2}}\right) \approx 0.2199.$$

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Conflict of interest

The authors declare that there is no interest regarding the publication of this paper.

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