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Research article

On the uniqueness of meromorphic functions that share small functions on annuli

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Abstract: In this paper, we aim to investigate the uniqueness of meromorphic functions that share small functions on annuli. As a matter of fact, we give several uniqueness theorems about meromorphic functions sharing four or three distinct small functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. To some extent, our theorems extend the previous work by T. B. Cao, H. X. Yi and H. Y. Xu, and also generalize the work by N. Wu and Q. Ge.

Keywords: uniqueness; meromorphic functions; transcendental; annuli; sharing small functions
Mathematics Subject Classification: 30D30, 30D35

1. Introduction and main result

In this article, we assume that the readers are familiar with the basic results and the standard notations of Nevanlinna's value distribution theory [16, 18]. Let f and g be two non-constant moromorphic functions, and let a be a complex number or a small function with respect to f and g. Then, we say that f and g share a IM (or CM) provided that $f - a$ and $g - a$ have the same zeros ignoring (or counting) multiplicities.

It is well known that R. Nevanlinna [11] proved the five-value theorem in 1926: For two non-constant meromorphic functions f and g in the complex plane $\mathbb{C}$, we have $f \equiv g$ providing that f and g share a_j IM for $j = 1, 2, 3, 4, 5$, where $a_j (j = 1, 2, 3, 4, 5)$ are five distinct values. In 2000 and 2001, Y. H. Li, J. Y. Qiao [9] and H. X. Yi [17] extended this very work to the case of sharing five small functions, proving the five small functions theorem: Let f and g be two non-constant meromorphic functions in the complex plane $\mathbb{C}$, and $a_j (j = 1, 2, 3, 4, 5)$ be five distinct small functions with respect to f and g. If f and g share $a_j (j = 1, 2, 3, 4, 5)$ IM in $\mathbb{C}$, then $f \equiv g$. In 2003 and 2004, J. H. Zheng [19, 20] proved the five value theorem in one angular domain. In 2011, H. F. Liu and Z. Q. Mao [10] further gave the five small functions theorem in one angular domain.

Next we will mainly discuss the uniqueness theory of meromorphic functions on annuli. For the basic results and necessary notations as $T_0(r, f), m_0(r, f), N_0(r, f)$, the readers can refer to [4–7, 13–15]. Here, let f, g, α be meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Then, α is named as a small function with respect to f on $\mathbb{A}$ providing that $T_0(r, \alpha) = o(T_0(r, f))$ as $r \to \infty$ except for the set $\triangle_r$ such that $\int_{\triangle_r} r^{\lambda-1}dr < +\infty$ for $R_0 = +\infty$, or $T_0(r, \alpha) = o(T_0(r, f))$ as $r \to R_0$ except for the set $\triangle_r'$ such that $\int_{\triangle_r'} \frac{dr}{(R_0-r)^{\lambda+1}} < +\infty$ for $R_0 < +\infty$. For a nonconstant meromorphic function f on the annulus $\mathbb{A}$, it is called as a transcendental meromorphic function on the annulus $\mathbb{A}$ if

$$\limsup_{r\to\infty} \frac{T_0(r, f)}{\log r} = \infty, \quad 1 < r < R_0 = +\infty$$

or

$$\limsup_{r\to R_0} \frac{T_0(r, f)}{-\log(R_0 - r)} = \infty, \quad 1 < r < R_0 < +\infty,$$

respectively. Therefore, for a transcendental meromorphic function on the annulus $\mathbb{A}$, $S(r, f) = o(T_0(r, f))$ holds for all $1 < r < R_0$ except for the set $\triangle_r$ such that $\int_{\triangle_r} r^{\lambda-1}dr < +\infty$ or the set $\triangle_r'$ such that $\int_{\triangle_r'} \frac{dr}{(R_0-r)^{\lambda+1}} < +\infty$, respectively. Additionally we denote by $\overline{N}_C(r, \alpha)$ ($\overline{N}_D(r, \alpha)$) the reduced counting function of common zeros (different zeros) of $f - \alpha$ and $g - \alpha$ on $\mathbb{A}$. Then, it is obvious that f and g share α IM if $\overline{N}_D(r, \alpha) = 0$. Furthermore, we say f and g share α "IM" provided that $\overline{N}_D(r, \alpha) = o(T_0(r, f)) + o(T_0(r, g))$.

Recently, T. B. Cao, H. X. Yi and H. X. Xu [1, 2] obtained the following five-value theorem on the annulus $\mathbb{A}$:

Theorem A [1, 2] Let f and g be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Let a_j $(j = 1, 2, 3, 4, 5)$ be five distinct complex numbers in $\mathbb{C}\bigcup\{\infty\}$. If f and g share a_j IM for $j = 1, 2, 3, 4, 5$, then $f \equiv g$. (In fact, this result for the case $R_0 = +\infty$ was proved by A. A. Kondratyuk and I. Laine [6]).

In 2015, N. Wu and Q. Ge [12] further proved the five small functions theorem on the annulus $\mathbb{A}$ as follows:

Theorem B [12] Let f and g be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Let $a_j (j = 1, 2, 3, 4, 5)$ be five distinct small functions with respect to f and g on the annulus $\mathbb{A}$. If f and g share a_j IM for $j = 1, 2, 3, 4, 5$, then $f \equiv g$.

Naturally, it is an interesting question to investigate whether Theorem B holds if f and g share less than five small functions. In this paper, we mainly deal with this question, and propose the following theorems, which partly generalize the five value theorem and the five small functions theorem on annuli.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Let $a_i \equiv a_i(z) (i = 1, 2, 3, 4, 5)$ be five distinct small functions with respect to f and g on $\mathbb{A}$. If f and g share $a_i (i = 1, 2, 3, 4)$ "IM" and*

$$\overline{N}_C(r, a_5) \ne S(r, f),$$

then $f \equiv g$, where $\overline{N}_C(r, a_5)$ is the reduced counting function of the common zeros of $f - a_5$ and $g - a_5$ (ignoring multiplicities) on $\mathbb{A}$.

Theorem 1.2. *Let f and g be two transcendental meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Let $a_i \equiv a_i(z) (i = 1, 2, 3, 4, 5)$ be five distinct small functions with respect to f and g on $\mathbb{A}$. If f and g share $a_i (i = 1, 2, 3)$ "IM" and*

$$\overline{N}_C(r, a_5) - \overline{N}_D(r, a_4) \neq S(r, f),$$

then $f \equiv g$, where $\overline{N}_D(r, a_j)$ are the reduced counting functions of the different zeros of $f - a_j$ and $g - a_j$ on $\mathbb{A}$.

2. Preliminaries

In 2005, A. Y. Khrystiyanyn and A. A. Kondratyuk [4, 5] proposed the following properties of meromorphic functions on annuli:

Lemma 2.1. [4] Let f be a non-constant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < r < R_0 \le +\infty$, then the following properties always hold:

(i) $T_0(r, f) = T_0\left(r, \frac{1}{f}\right)$,
(ii) $\max\{T_0(r, f_1 \cdot f_2), T_0(r, f_1/f_2), T_0(r, f_1 + f_2)\} \le T_0(r, f_1) + T_0(r, f_2) + O(1)$,
(iii) $T_0\left(r, \frac{1}{f-a}\right) = T_0(r, f) + O(1),\ for\ every\ fixed\ a \in \mathbb{C}$.

Lemma 2.2. [5] Let f be a non-constant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 \le +\infty$, and let $\lambda \ge 0$. Then

(i) if $R_0 = +\infty$, then $m_0\left(r, \frac{f'}{f}\right) = O(\log(rT_0(r, f)))$ for $R \in (1; +\infty)$ except for the set $\triangle_r$ such that $\int_{\triangle_r} r^{\lambda-1} dr < +\infty$;

(ii) if $R_0 < +\infty$, then $m_0\left(r, \frac{f'}{f}\right) = O\left(\log\left(\frac{T_0(r,f)}{R_0 - r}\right)\right)$ for $r \in (1; R_0)$ except for the set $\triangle_r'$ such that $\int_{\triangle_r'} \frac{dr}{(R_0-r)^{\lambda+1}} < +\infty$.

In addition, A. Y. Khrystiyanyn and A. A. Kondratyuk [5] proved the second fundamental theorem on annuli. Furthermore, T. B. Cao, H. X. Yi and H. Y. Xu [2] provided another form of the second fundamental theorem on the the annulus $\mathbb{A}$:

Lemma 2.3. [2] Let f be a non-constant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$ in which $1 < R_0 \le +\infty$. Let $a_1, a_2, \ldots, a_q$ be q distinct complex numbers in the extended complex plane $\overline{\mathbb{C}}$. Then

$$(q-2)T_0(r, f) < \sum_{j=1}^{q} \overline{N}_0\left(r, \frac{1}{f - a_j}\right) + S(r, f).$$

Motivated and inspired by the ideas of [3, 8, 17], we propose the following lemmas.

Lemma 2.4. *Let f be a transcendental meromorphic function on $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$ in which $1 < R_0 \le +\infty$, and let $a_1 \equiv a_1(z)$ and $a_2 \equiv a_2(z)$ be two distinct small functions with respect to f on $\mathbb{A}$.*

Set

$$L(f,a_1,a_2)=\begin{vmatrix} f & f' & 1\\ a_1 & a_1' & 1\\ a_2 & a_2' & 1\end{vmatrix},$$

then, we have

$$m_0\left(r,\frac{L(f,a_1,a_2)f^k}{(f-a_1)(f-a_2)}\right)=S(r,f),$$

where k=0,1.

Proof. It follows from the determinant nature that

$$\frac{L(f,a_1,a_2)}{(f-a_1)(f-a_2)}=\frac{f'-a_2'}{f-a_2}-\frac{f'-a_1'}{f-a_1}.$$

This implies

$$m_0\left(r,\frac{L(f,a_1,a_2)}{(f-a_1)(f-a_2)}\right)=S(r,f)$$

by applying Lemma 2.2.

Next we can deduce

$$\frac{L(f,a_1,a_2)f}{(f-a_1)(f-a_2)}=(a_1'-a_2')+a_2\frac{f'-a_2'}{f-a_2}-a_1\frac{f'-a_1'}{f-a_1}$$

by some simple computing. It follows that

$$m_0\left(r,\frac{L(f,a_1,a_2)f}{(f-a_1)(f-a_2)}\right)=S(r,f).$$

Lemma 2.4 is proved. □

Lemma 2.5. *Let f and g be two transcendental meromorphic functions on $\mathbb{A}$, and let $a_1=0$, $a_2=1$, $a_3=\infty$, $a_4=a(z)$ be four distinct small functions respect to f and g on $\mathbb{A}$, in which $a(z)\not\equiv 0,1,\infty$. Set*

$$H\equiv\frac{L(f,0,1)(f-g)L(g,1,a)}{f(f-1)(g-1)(g-a)}-\frac{L(g,0,1)(f-g)L(f,1,a)}{g(g-1)(f-1)(f-a)}.$$

Then we get

$$T_0(r,H)\le\sum_{i=1}^{4}\overline{N}_D(r,a_i)+S(r,f)+S(r,g),$$

where $\overline{N}_D(r,a_i)$ is the reduced counting function of the different zeros of $f-a_i$ and $g-a_i$ on $\mathbb{A}$.

Proof. Here, we consider the counting function $N_0(r,H)$. It is obvious that the poles of H only come from the zeros, 1-points, poles of f or g, and the zeros of $f-a$ or $g-a$ on $\mathbb{A}$. Firstly, let z_4 be a common zero of $f-a$ and $g-a$ on $\mathbb{A}$ with multiplicity p and q respectively, satisfying that $a(z_4)\neq 0,1,\infty$. Applying the determinate nature, we have

$$H\equiv(f-g)\left[\left(\frac{f'}{f-1}-\frac{f'}{f}\right)\left(\frac{g'-a'}{g-a}-\frac{g'}{g-1}\right)-\left(\frac{g'}{g-1}-\frac{g'}{g}\right)\left(\frac{f'-a'}{f-a}-\frac{f'}{f-1}\right)\right]$$

$$\equiv \quad (f-g)\left[\frac{f'}{f-1}\frac{g'-a'}{g-a}+\frac{g'}{g}\frac{f'-a'}{f-a}+\frac{f'}{f}\frac{g'}{g-1}-\frac{g'}{g-1}\frac{f'-a'}{f-a}-\frac{f'}{f}\frac{g'-a'}{g-a}-\frac{g'}{g}\frac{f'}{f-1}\right].$$

It follows that $H(z_4)\neq\infty$ since z_4 is a zero of $(f-g)$, and a simple pole or an analytic point of

$$\left[\frac{f'}{f-1}\frac{g'-a'}{g-a}+\frac{g'}{g}\frac{f'-a'}{f-a}+\frac{f'}{f}\frac{g'}{g-1}-\frac{g'}{g-1}\frac{f'-a'}{f-a}-\frac{f'}{f}\frac{g'-a'}{g-a}-\frac{g'}{g}\frac{f'}{f-1}\right]. \tag{2.1}$$

Secondly, let z_3 (resp. z_1, z_2) be a common pole (resp. zero, 1-points) of f and g on $\mathbb{A}$ with multiplicity p and q respectively, satisfying that $a(z_4)\neq 0,1,\infty$ (resp. $a(z_1)\neq 0,1,\infty$, $a(z_2)\neq 0,1,\infty$). Without loss of generality, assume that $p\geq q$. By simple computation, we can write H as

$$\frac{a(a-1)f'g'(f-g)^2}{f(f-1)(f-a)g(g-1)(g-a)}+\frac{a'(f-g)[f(f-1)(g-a)g'-g(g-1)(f-a)f']}{f(f-1)(f-a)g(g-1)(g-a)}. \tag{2.2}$$

Noting that z_3 is a pole of $a(a-1)f'g'(f-g)^2$ with multiplicity $3p+q+2$ at most, a pole of $a'(f-g)[f(f-1)(g-a)g'-g(g-1)(f-a)f']$ with multiplicity $3p+2q+1$ at most, and a pole of $f(f-1)(f-a)g(g-1)(g-a)$ with multiplicity $3p+3q$, we obtain $H(z_3)\neq\infty$. In the same manner, we can get $H(z_1)\neq\infty, H(z_2)\neq\infty$. Therefore, the poles of H only come from the different zeros of $f,g,f-1,g-1,f-a,g-a$ and the different poles of f,g on $\mathbb{A}$. In order to analyze these different zeros and different poles, we distinguish the following distinct cases.

Case 1. Let ζ_1 be a zero of f, which is neither a zero of g, a, and $a-1$ nor a pole of a. Then, from (2.1) and (2.2) we find that ζ_1 is a pole of H with multiplicity at most 1 if $g(\zeta_1)\neq 1,\infty,a(\zeta_1)$; and otherwise ζ_1 is a pole of H with multiplicity at most 2.

Case 2. Let ζ_2 be a zero of $f-1$, which is neither a zero of $g-1$, a, and $a-1$ nor a pole of a. By (2.1) and (2.2) we know that ζ_2 is a pole of H with multiplicity at most 1 if $g(\zeta_2)\neq 0,\infty,a(\zeta_2)$; and otherwise ζ_2 is a pole of H with multiplicity at most 2.

Case 3. Let ζ_3 be a pole of f, which is neither a pole of g and a nor a zero of a and $a-1$. Then, it is clear that ζ_3 is a pole of H with multiplicity at most 1 if $g(\zeta_3)\neq 0,1,a(\zeta_3)$; and otherwise ζ_3 is a pole of H with multiplicity at most 2.

Case 4. Let ζ_4 be a zero of $f-a$, which is neither a zero of $g-a$, a, and $a-1$ nor a pole of a. It is obvious that that ζ_4 is a pole of H with multiplicity at most 1 if $g(\zeta_4)\neq 0,1,\infty$; and otherwise ζ_4 is a pole of H with multiplicity at most 2.

In view of the discussion above, we deduce that

$$N_0(r,H)\leq\sum_{i=1}^{4}\overline{N}_D(r,a_i)+N_0(r,\frac{1}{a})+N_0(r,\frac{1}{a-1})+N_0(r,a)=\sum_{i=1}^{4}\overline{N}_D(r,a_i)+S(r,f).$$

Moreover, it is a direct consequence of Lemma 2.4 that $m_0(r,H)=S(r,f)+S(r,g)$, which implies that

$$T_0(r,H)=m_0(r,H)+N_0(r,H)\leq\sum_{i=1}^{4}\overline{N}_D(r,a_i)+S(r,f)+S(r,g).$$

Lemma 2.5 is proved. □

3. The proof of theorem 1.1

By Lemma 2.1 and Lemma 2.3, we derive that $T_0(r, f) \le 3T_0(r, g)+S(r, f)$ and $T_0(r, g) \le 3T_0(r, f)+ S(r, g)$ noting that f and g share $a_i(i = 1, 2, 3, 4)$ "IM". Then, it is obvious that $S(r, f) = S(r, g)$.

By applying the quasi-M$\ddot{o}$bius transformation

$$\frac{f-a_1}{f-a_3}\frac{a_2-a_3}{a_2-a_1},$$

we can assume that $a_1(z) = 0$, $a_2(z) = 1$, $a_3(z) = \infty$, $a_4(z) = a(z)$, $a_5(z) = b(z)$, where a, b are two distinct small functions of f and g on $\mathbb{A}$ satisfying $a, b \not\equiv 0, 1, \infty$. As in Lemma 2.5, we set

$$H \equiv \frac{L(f,0,1)(f-g)L(g,1,a)}{f(f-1)(g-1)(g-a)} - \frac{L(g,0,1)(f-g)L(f,1,a)}{g(g-1)(f-1)(f-a)}.$$

Since f and g share $0, 1, \infty, a$ "IM", we can deduce $T_0(r, H) = S(r, f)$ by the virtue of Lemma 2.5.

It is obvious that a common zero of $f - b$ and $g - b$ must be a zero of H when it is not a zero of $b, b - 1, b - a$. We assume that $H \not\equiv 0$, then, we get

$$\overline{N}_C(r,b) \le N_0\left(r,\frac{1}{H}\right) + S(r,f) \le T_0(r,H) + S(r,f) = S(r,f).$$

This contradict $\overline{N}_C(r, b) \neq S(r, f)$. It follows that $H \equiv 0$.

In the following we assume that $f \not\equiv g$. From $H \equiv 0$ we have

$$\frac{L(f,0,1)L(g,1,a)}{f(g-a)} \equiv \frac{L(g,0,1)L(f,1,a)}{g(f-a)}.$$

It follows that

$$\frac{f'}{f}\left[a' - (a-1)\frac{g'-a'}{g-a}\right] \equiv \frac{g'}{g}\left[a' - (a-1)\frac{f'-a'}{f-a}\right]. \tag{3.1}$$

If a is a constant, then from (3.1) we get

$$\frac{f'}{f}\left[-(a-1)\frac{g'}{g-a}\right] \equiv \frac{g'}{g}\left[-(a-1)\frac{f'}{f-a}\right],$$

which further yields $f \equiv g$. This is a contradiction, we consequently have $a' \not\equiv 0$. Note that the equation (3.1) can be written as

$$\frac{f'[a'(g-a)-(a-1)(g'-a')]}{g'[a'(f-a)-(a-1)(f'-a')]} - 1 = \frac{f(g-a)}{g(f-a)} - 1.$$

This implies

$$\frac{f'-g'}{f-g} = \frac{g'}{g-1} - \frac{-a[a'(f-a)-(a-1)(f'-a')]}{a'g(g-1)(f-a)}. \tag{3.2}$$

Since $\overline{N}_C(r, b) \neq S(r, f)$, there exist a point z_0 satisfying that z_0 is a common zero of $f - b$ and $g - b$, but not a zero or pole of $a, a', b, b - 1, b - a$. It follows that z_0 is a simple pole of the left side of (3.2), but not a pole of the right side of (3.2), which is impossible. Hence Theorem 1.1 is proved.

4. The proof of theorem 1.2

To the contrary, we suppose that $f \not\equiv g$. Similarly to the proof of Theorem 1.1, we can get

$$T_0(r, H) \leq \overline{N}_D(r, a) + S(r, f)$$

by utilizing Lemma 2.5. If $H \not\equiv 0$, then we have

$$\overline{N}_C(r, b) \leq N_0\left(r, \frac{1}{H}\right) + S(r, f) \leq T_0(r, H) + S(r, f) \leq \overline{N}_D(r, a) + S(r, f),$$

which contradict $\overline{N}_C(r, b) - \overline{N}_D(r, a) \neq S(r, f)$. We consequently obtain $H \equiv 0$, and then the equations (3.1) and (3.2) still hold.

Note that $\overline{N}_C(r, b) - \overline{N}_D(r, a) \neq S(r, f)$ implies $\overline{N}_C(r, b) \neq S(r, f)$. So there exists a point z_0 satisfying that z_0 is a common zero of $f - b$ and $g - b$, but not a zero or pole of $a, a', b, b-1, b-a$. Clearly, z_0 is a simple pole of the left side of (3.2), but not a pole of the right side of (3.2). This is impossible. Therefore we have proved Theorem 1.2.

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Conflict of interest

The authors declare no conflicts of interest.

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