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Research article

A new fixed point algorithm for finding the solution of a delay differential equation

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Abstract: In this paper, we construct a new iterative algorithm and show that the newly introduced iterative algorithm converges faster than a number of existing iterative algorithms. We present a numerical example followed by graphs to validate our claim. We prove strong and weak convergence results for approximating fixed points of Suzuki generalized nonexpansive mappings. Again we reconfirm our results by example and table. Further, we utilize our proposed algorithm to solve delay differential equation.

Keywords: Suzuki generalized nonexpansive mappings; fixed point; contractive-like mappings; iteration process; strong and weak convergence; delay differential equation
Mathematics Subject Classification: 47H09, 47H10, 54H25

1. Introduction

Numerical reckoning fixed points for nonlinear operators is nowadays an active research problem of nonlinear analysis owing to its applications to: variational inequalities, equilibrium problems, computer simulation, image encoding and much more. Mann [19], Ishikawa [17] and Halpern [12] are the three basic iterative algorithms to approximate fixed points of nonexpansive mappings. Following these study, several authors constructed numerous algorithms to approximate the fixed points of different classes of nonlinear mappings mainly Noor iteration [20], Agarwal et al. iteration [4], SP iteration [21], Normal-S iteration [23], Abbas and Nazir iteration [1], Thakur et al. iterations [28, 29], Karakaya et al. iteration [18] and many others.

In 2008, Suzuki [26] introduced a new generalization of nonexpansive mappings and called the defining condition as Condition (C) which is also referred as Suzuki generalized nonexpansive mappings. A mapping $T : K \to K$ defined on a nonempty subset K of a Banach space E is said to satisfy the Condition (C) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$$

for all x and $y \in K$.

Suzuki proved that the mappings satisfying the Condition (C) is weaker than nonexpansive and also obtained few results regarding the existence of fixed points for such mappings. In 2011, Phuengrattana [22] used Ishikawa iteration to obtain some convergence results for mappings satisfying Condition (C) in uniformly convex Banach spaces. In the last few years, many authors have studied this particular class of mappings in various domain and have obtained many convergence results (e.g. [2, 3, 9, 10, 28, 30, 31, 35]).

Recently, Ullah and Arshad [31] introduced a new algorithm namely M-iteration algorithm as follows:

$$\begin{cases} d_1 \in K \\ b_n = (1-\alpha_n)d_n + \alpha_n T d_n \\ c_n = T b_n \\ d_{n+1} = T c_n \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. They proved some convergence results for Suzuki generalized nonexpansive mappings and showed that M-iteration converges faster than Picard-S [11] and S-iteration [4].

To achieve a better rate of convergence, we introduce a new iterative algorithm for approximating fixed points of Suzuki generalized nonexpansive mappings as follows:

$$\begin{cases} x_1 \in K \\ z_n = T x_n \\ y_n = T((1-\alpha_n)z_n + \alpha_n T z_n) \\ x_{n+1} = T y_n \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

The aim of this paper is to prove that newly defined iterative algorithm (1.2) converges faster than algorithm (1.1) for contractive-like mappings. Also, we prove some convergence results involving algorithm (1.2) for Suzuki generalized nonexpansive mappings. Further, we provide a numerical example to show that our iteration (1.2) converges faster than a number of existing iterative algorithms in respect of Suzuki generalized nonexpansive mappings. In the last section, we use our algorithm to find solution of a delay differential equation.

2. Preliminaries

For making our paper self contained, we collect some basic definitions and needed results.

Definition 2.1. A Banach space E is said to be uniformly convex if for each $\epsilon \in (0, 2]$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| > \epsilon$, we have

$$\left\|\frac{x+y}{2}\right\| < 1 - \delta.$$

Definition 2.2. A Banach space E is said to satisfy the Opial's condition if for any sequence $\{x_n\}$ in E which converges weakly to $x \in E$ i.e. $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.
Examples of Banach spaces satisfying this condition are Hilbert spaces and all l^p spaces $(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition.
A mapping $T : K \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in K and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in K$ and $Tx = y$.
Let K be a nonempty closed convex subset of a Banach E, and let $\{x_n\}$ be a bounded sequence in E. For $x \in E$ write:

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x - x_n\|.$$

The asymptotic radius of $\{x_n\}$ relative to K is given by

$$r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \ \in K\},$$

and the asymptotic center $A(K, \{x_n\})$ of $\{x_n\}$ is defined as:

$$A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r(K, \{x_n\})\}.$$

It is known that, in a uniformly convex Banach space, $A(K, \{x_n\})$ consists of exactly one point.
The following definitions about the rate of convergence were given by Berinde [7].
Definition 2.3. Let $\{a_n\}$ and $\{b_n\}$ be two real sequences converging to a and b respectively. Then, $\{a_n\}$ converges faster then $\{b_n\}$ if $\lim_{n\to\infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0$.
Definition 2.4. Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration processes converging to the same fixed point p. If $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero such that $\|u_n - p\| \leq a_n$ and $\|v_n - p\| \leq b_n$ for all $n \geq 1$, then we say that $\{u_n\}$ converges faster than $\{v_n\}$ to p if $\{a_n\}$ converges faster then $\{b_n\}$.
The following lemma due to Schu [24] is very useful in our subsequent discussion.
Lemma 2.1. Let E be a uniformly convex Banach space and $\{t_n\}$ be any sequence such that $0 < p \leq t_n \leq q < 1$ for some $p, r \in \mathbb{R}$ and for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be any two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\limsup_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r$ for some $r \geq 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Now, we list few lemmas involving Suzuki generalized nonexpansive mappings.
Lemma 2.2. ([26]) Let K be a nonempty subset of a Banach space E and $T : K \to K$ be any mapping. Then,
(i) If T is nonexpansive then T is Suzuki generalized nonexpansive mapping.
(ii) If T is Suzuki generalized nonexpanisve mapping such that $F(T) \neq \emptyset$, then T is a quasi-nonexpansive mapping.
(iii) If T is a Suzuki generalized nonexpansive mapping, then $\|x - Ty\| \leq 3\|x - Tx\| + \|x - y\|$ for all x and $y \in K$.

Lemma 2.3. ([27]) Let T be a Suzuki generalized nonexpansive mapping defined on a subset K of a Banach space E with the Opial property. If a sequence $\{x_n\}$ converges weakly to z and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $I - T$ is demiclosed at zero.

Lemma 2.4. ([26]) If T is a Suzuki generalized nonexpansive mapping defined on a compact convex subset K of a uniformly convex Banach space E then, T has a fixed point.

In 1972, Zamfirescu [34] introduced Zamfirescu mappings which serves as an important generalization for Banach contraction principle [5]. In 2004, Berinde [6] gave a more general class of mappings known as quasi-contractive mappings. Following this, Imoru and Olantiwo [16] gave the following definition:

Definition 2.5. A mapping $T : K \to K$ is known as contractive-like mapping if there exists a strictly increasing and continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and a constant $\delta \in [0, 1)$ such that for all $x, y \in K$, we have

$$\|Tx - Ty\| \leq \delta\|x - y\| + \varphi(\|x - Tx\|).$$

Clearly, the class of contractive-like mappings is wider than the class of quasi-contractive mappings.

3. Rate of convergence

In this section, first we show that our algorithm (1.2) converges faster than the M-iteration (1.1) for contractive-like mappings.

Theorem 3.1. Let T be a contractive-like mapping defined on a nonempty closed convex subset K of a Banach space E with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence defined by (1.2), then $\{x_n\}$ converges faster than the iterative algorithm (1.1).

Proof. From (1.1), for any $p \in F(T)$, we have

$$\begin{aligned} \|b_n - p\| &= \|(1-\alpha_n)d_n + \alpha_n T d_n - p\| \\ &\leq (1-\alpha_n)\|d_n - p\| + \alpha_n\delta\|d_n - p\| \\ &= (1-(1-\delta)\alpha_n)\|d_n - p\| \end{aligned}$$

and

$$\begin{aligned} \|c_n - p\| &= \|Tb_n - p\| \\ &\leq \delta\|b_n - p\| \\ &\leq \delta(1-(1-\delta)\alpha_n)\|d_n - p\|. \end{aligned}$$

As, $\{\alpha_n\}$ is a sequence in $(0, 1)$, we can find a constant $\alpha \in \mathbb{R}$ such that $\alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. So,

$$\begin{aligned} \|d_{n+1} - p\| &= \|Tc_n - p\| \\ &\leq \delta\|c_n - p\| \\ &\leq \delta^2(1-(1-\delta)\alpha_n)\|d_n - p\| \\ &\leq \delta^2(1-(1-\delta)\alpha)\|d_n - p\| \\ &\;\; . \\ &\;\; . \\ &\;\; . \\ &\leq \delta^{2n}(1-(1-\delta)\alpha)^n\|d_1 - p\|. \end{aligned}$$

Now, from (1.2) we get

$$\begin{aligned} \|z_n - p\| &= \|Tx_n - p\| \\ &\le \delta\|x_n - p\| \end{aligned}$$

and

$$\begin{aligned} \|y_n - p\| &= \|T((1-\alpha_n)z_n + \alpha_n Tz_n) - p\| \\ &\le \delta\|(1-\alpha_n)z_n + \alpha_n Tz_n - p\| \\ &\le \delta((1-\alpha_n)\|z_n - p\| + \alpha_n\|Tz_n - p\|) \\ &\le \delta((1-\alpha_n)\|z_n - p\| + \alpha_n\|z_n - p\|) \\ &= \delta(1-(1-\delta)\alpha_n)\|z_n - p\| \\ &\le \delta^2(1-(1-\delta)\alpha_n)\|x_n - p\|. \end{aligned}$$

So,

$$\begin{aligned} \|x_{n+1} - p\| &= \|Ty_n - p\| \\ &\le \delta\|y_n - p\| \\ &\le \delta^3(1-(1-\delta)\alpha_n)\|x_n - p\| \\ &\le \delta^3(1-(1-\delta)\alpha)\|x_n - p\| \\ &\quad . \\ &\quad . \\ &\quad . \\ &\le \delta^{3n}(1-(1-\delta)\alpha)^n\|x_1 - p\|. \end{aligned}$$

Let $b_n = \delta^{3n}(1-(1-\delta)\alpha)^n\|x_1 - p\|$ and $a_n = \delta^{2n}(1-(1-\delta)\alpha)^n\|d_1 - p\|$, then

$$\begin{aligned} \frac{b_n}{a_n} &= \frac{\delta^{3n}(1-(1-\delta)\alpha)^n\|x_1-p\|}{\delta^{2n}(1-(1-\delta)\alpha)^n\|d_1-p\|} \\ &\to 0 \quad as \quad n \to \infty. \end{aligned}$$

Hence, $\{x_n\}$ converges faster than $\{d_n\}$.

Now, we present a example of a contractive-like mapping which is not a contraction.

Example 1: Let $E = \mathbb{R}$ and $K = [0, 6]$. Let $T : K \to K$ be a mapping defined as

$$Tx = \begin{cases} \frac{x}{5} & x \in [0, 3) \\ \frac{x}{10} & x \in [3, 6]. \end{cases}$$

Proof: Clearly $x = 0$ is the fixed point of T. First, we prove that T is a contractive-like mapping but not a contraction. Since T is not continuous at $x = 3 \in [0, 6]$, so T is not a contraction. We show that T is a contractive-like mapping. For this, define $\varphi : [0, \infty) \to [0, \infty)$ as $\varphi(x) = \frac{x}{8}$. Then, φ is a strictly increasing as well as continuous function. Also, $\varphi(0) = 0$.

We need to show that

$$\|Tx - Ty\| \le \delta\|x - y\| + \varphi(\|x - Tx\|) \tag{A}$$

for all $x, y \in [0, 6]$ and δ is a constant in $[0, 1)$.

Before going ahead, let us note the following. When $x \in [0, 3)$, then

$$\|x - Tx\| = \left\|x - \frac{x}{5}\right\| = \frac{4x}{5}$$

and

$$\varphi(\frac{4x}{5}) = \frac{x}{10}. \tag{3.1}$$

Similarly, when $x \in [3, 6]$, then

$$\|x - Tx\| = \left\|x - \frac{x}{10}\right\| = \frac{9x}{10}$$

and

$$\varphi(\frac{9x}{10}) = \frac{9x}{80}. \tag{3.2}$$

Consider the following cases:

Case A: Let $x, y \in [0, 3)$, then using (3.1) we get

$$\begin{aligned} \|Tx - Ty\| &= \|\tfrac{x}{5} - \tfrac{y}{5}\| \\ &\leq \tfrac{1}{5}\|x - y\| \\ &\leq \tfrac{1}{5}\|x - y\| + \tfrac{x}{10} \\ &= \tfrac{1}{5}\|x - y\| + \varphi(\tfrac{4x}{5}) \\ &= \tfrac{1}{5}\|x - y\| + \varphi(\|x - Tx\|). \end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{5}$.

Case B: Let $x \in [0, 3)$ and $y \in [3, 6]$ then using (3.1) we get

$$\begin{aligned} \|Tx - Ty\| &= \|\tfrac{x}{5} - \tfrac{y}{10}\| \\ &= \|\tfrac{x}{10} + \tfrac{x}{10} - \tfrac{y}{10}\| \\ &\leq \tfrac{1}{10}\|x - y\| + \left\|\tfrac{x}{10}\right\| \\ &\leq \tfrac{1}{5}\|x - y\| + \varphi(\tfrac{4x}{5}) \\ &= \tfrac{1}{5}\|x - y\| + \varphi(\|x - Tx\|). \end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{5}$.

Case C: Let $x \in [3, 6]$ and $y \in [0, 3)$ then using (3.2) we get

$$\begin{aligned} \|Tx - Ty\| &= \|\tfrac{x}{10} - \tfrac{y}{5}\| \\ &= \|\tfrac{x}{5} - \tfrac{x}{10} - \tfrac{y}{5}\| \\ &\leq \tfrac{1}{5}\|x - y\| + \left\|\tfrac{x}{10}\right\| \\ &\leq \tfrac{1}{5}\|x - y\| + \left\|\tfrac{9x}{80}\right\| \\ &= \tfrac{1}{5}\|x - y\| + \varphi(\|x - Tx\|). \end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{5}$.

Case D: Let $x, y \in [3, 6]$ then using (3.2) we get

$$\begin{aligned} \|Tx - Ty\| &= \|\tfrac{x}{10} - \tfrac{y}{10}\| \\ &\leq \tfrac{1}{10}\|x - y\| + \left\|\tfrac{9x}{80}\right\| \\ &\leq \tfrac{1}{5}\|x - y\| + \left\|\tfrac{9x}{80}\right\| \\ &= \tfrac{1}{5}\|x - y\| + \varphi(\|x - Tx\|). \end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{5}$.

Consequently, (A) is satisfied for $\delta = \frac{1}{5}$ and $\varphi(x) = \frac{x}{8}$ in all the possible cases. Thus, T is a contractive-like mapping.

Now, using T, we show that our iterative algorithm (1.2) has a better rate of convergence. Set $\alpha_n = \beta_n = \gamma_n = \frac{n}{n+1}$ for each $n \in \mathbb{N}$. Then, we get the following Table 1, Table 2, Figure 1 and Figure 2 with the initial value 4.5.

Table 1. Sequences generated by Agarwal, Abbas, Thakur New, M and New Iteration.

Step	Agarwal Iteration	Abbas Iteration	Thakur New	M Iteration	New Iteration
1	4.5	4.5	4.5	4.5	4.5
2	0.4725	0.29475	0.06975	0.099	0.0108
3	0.0609	0.02087266667	0.001798	0.001848	0.00004032
4	0.006699	0.001367159667	0.000039556	0.000029568	1.29024×10^{-7}
5	0.0006538224	0.00008408578814	7.7213312×10^{-7}	4.257792×10^{-7}	$3.7158912 \times 10^{-10}$
6	0.00005811754667	4.92057575×10^{6}	$1.372681102 \times 10^{-8}$	5.677056×10^{-9}	$9.9090432 \times 10^{-13}$
7	$4.791732419 \times 10^{-6}$	$2.766998982 \times 10^{-7}$	$2.263523124 \times 10^{-10}$	$7.1368704 \times 10^{-11}$	$2.491416576 \times 10^{-15}$
8	$3.713592625 \times 10^{-7}$	$1.506285071 \times 10^{-8}$	$3.508460842 \times 10^{-12}$	$8.56424448 \times 10^{-13}$	$5.979399782 \times 10^{-18}$

Table 2. Sequences generated by Noor, Picard S, Thakur, M^* and New Iteration.

Step	Noor Iteration	Picard S Iteration	Thakur Iteration	M^* Iteration	New Iteration
1	4.5	4.5	4.5	4.5	4.5
2	2.49975	0.0945	0.3735	0.0999	0.0108
3	0.9650886667	0.002436	0.03574533333	0.00158064	0.00004032
4	0.2861487897	0.000053592	0.002645154667	0.000019599936	1.29024×10^{-7}
5	0.06902366645	$1.04611584 \times 10^{-6}$	0.0001606561138	$2.019577405 \times 10^{-7}$	$3.7158912 \times 10^{-10}$
6	0.0140603765	$1.859761493 \times 10^{-8}$	$8.330317014 \times 10^{-6}$	$1.795179916 \times 10^{-9}$	$9.9090432 \times 10{-}13$
7	0.002482824734	$3.066708748 \times 10^{-10}$	$3.790658541 \times 10^{-7}$	$1.412696685 \times 10^{-11}$	$2.491416576 \times 10^{-15}$
8	0.0003874758351	$4.75339856 \times 10^{-12}$	$1.544693355 \times 10^{-8}$	$1.003014646 \times 10^{-13}$	$5.979399782 \times 10^{-18}$
9	0.00005424449084	$6.995124794 \times 10^{-14}$	$5.724476775 \times 10^{-10}$	$6.518356911 \times 10^{-16}$	$1.381905727 \times 10^{-20}$
10	$6.89295594 \times 10^{-6}$	$9.84913571 \times 10^{-16}$	$1.952733517 \times 10^{-11}$	$3.921443518 \times 10^{-18}$	$3.09546883 \times 10^{-23}$

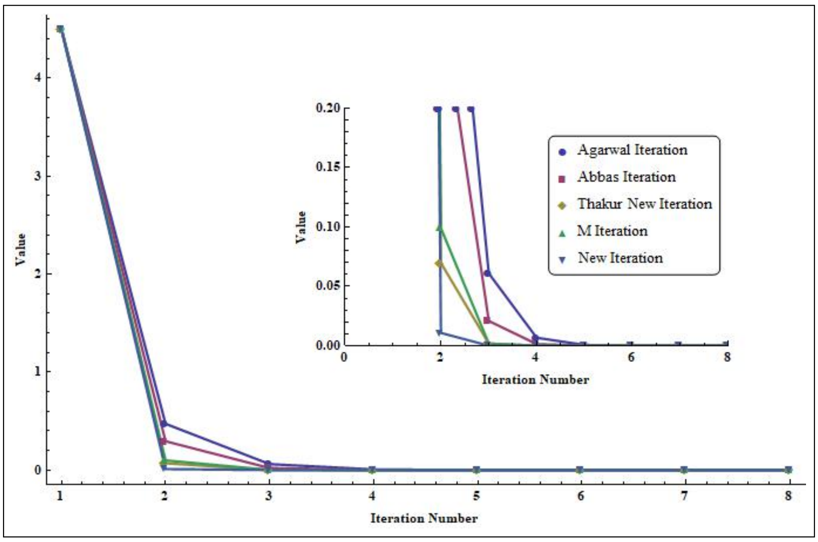

Figure 1. Graph corresponding to Table 1.

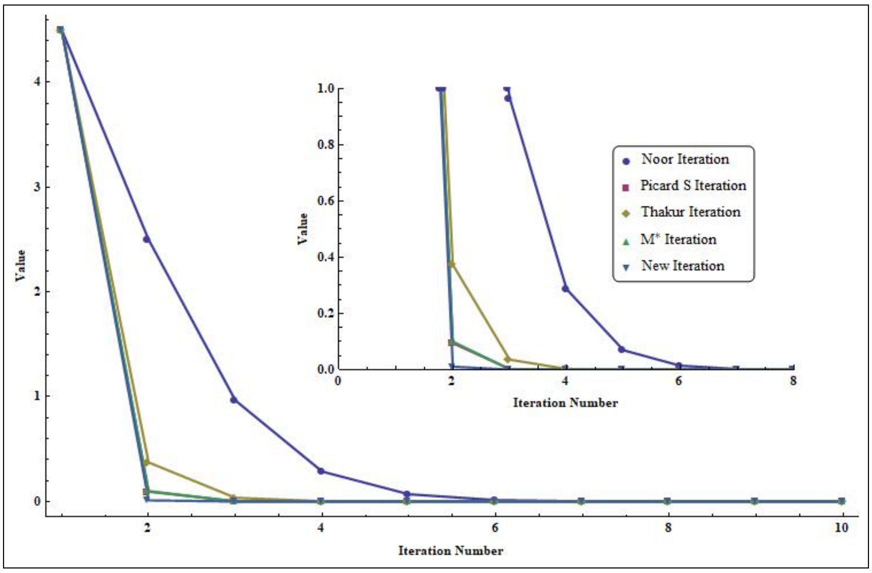

Figure 2. Graph corresponding to Table 2.

Clearly, our algorithm (1.2) converges at a faster rate for contractive-like mappings.

4. Convergence results

First, we prove few lemmas which will be useful in obtaining convergence results.

Lemma 4.1. Let T be a Suzuki generalized nonexpansive mapping defined on a nonempty closed convex subset K of a Banach space E with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the iterative sequence defined by the algorithm (1.2). Then, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$.

Proof. Let $p \in F(T)$ and $z \in K$. Since T is a Suzuki generalized nonexpansive mapping,
$\frac{1}{2}\|p - Tp\| = 0 \leq \|p - z\|$ implies that $\|Tp - Tz\| \leq \|p - z\|$.
Now we have,

$$\begin{aligned} \|z_n - p\| &= \|Tx_n - p\| \\ &\leq \|x_n - p\| \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \|y_n - p\| &= \|T((1-\alpha_n)z_n + \alpha_n Tz_n) - p\| \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|Tz_n - p\| \\ &\leq \|z_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{4.2}$$

Using (4.1) and (4.2), we get

$$\begin{aligned} \|x_{n+1} - p\| &= \|Ty_n - p\| \\ &\leq \|y_n - p\| \\ &\leq \|x_n - p\| \end{aligned}$$

Thus, $\{\|x_n - p\|\}$ is bounded and decreasing sequence of reals and hence $\lim_{n\to\infty} \|x_n - p\|$ exists.

Lemma 4.2. Let T be a Suzuki generalized nonexpansive mapping defined on a nonempty closed convex subset K of a uniformly convex Banach space E. Let $\{x_n\}$ be the iterative sequence defined by the algorithm (1.2). Then, $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Lemma 4.1, $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = c$.

From Eqs (4.1) and (4.2), we have

$$\limsup_{n\to\infty} \|y_n - p\| \leq c \tag{4.3}$$

and

$$\limsup_{n\to\infty} \|z_n - p\| \leq c. \tag{4.4}$$

Now,

$$c = \lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|Ty_n - p\|,$$

and

$$\|Ty_n - p\| \leq \|y_n - p\|.$$

So,

$$c \leq \liminf_{n\to\infty} \|y_n - p\|$$

which along with Eq (4.3) implies

$$\lim_{n\to\infty} \|y_n - p\| = c. \tag{4.5}$$

Since T is a Suzuki generalized nonexpansive mapping, we get

$$\|Tz_n - p\| \leq \|z_n - p\|.$$

From Eq (4.4), we obtain

$$\limsup_{n\to\infty} \|Tz_n - p\| \leq c. \tag{4.6}$$

Consider,

$$\begin{aligned} \lim_{n\to\infty} \|y_n - p\| &= \lim_{n\to\infty} \|T((1-\alpha_n)z_n + \alpha_n Tz_n) - p\| \\ &\leq \lim_{n\to\infty} \|(1-\alpha_n)(z_n - p) + \alpha_n(Tz_n - p)\|. \end{aligned}$$

Using Lemma 2.3, from Eqs (4.4), (4.5) and (4.6), we get

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{4.7}$$

Now, consider

$$\begin{aligned} \|y_n - Tz_n\| &= \|T((1-\alpha_n)z_n + \alpha_n Tz_n) - Tz_n\| \\ &\leq \|(1-\alpha_n)z_n + \alpha_n Tz_n - z_n\| \\ &= \alpha_n \|Tz_n - z_n\| \end{aligned}$$

which on using Eq (4.7) gives

$$\lim_{n\to\infty} \|y_n - Tz_n\| = 0. \tag{4.8}$$

Since,

$$\|z_n - y_n\| \leq \|z_n - Tz_n\| + \|Tz_n - y_n\|,$$

this together with Eqs (4.7) and (4.8) yields that

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{4.9}$$

Now, using Eqs (4.8) and (4.9), we have

$$\begin{aligned} \|Tx_{n+1} - x_{n+1}\| &= \|Tx_{n+1} - Ty_n\| \\ &\leq \|x_{n+1} - y_n\| \\ &= \|Ty_n - y_n\| \\ &= \|Ty_n - Tz_n + Tz_n - y_n\| \\ &\leq \|Ty_n - Tz_n\| + \|Tz_n - y_n\| \\ &\leq \|y_n - z_n\| + \|Tz_n - y_n\| \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Let $p \in A(K, \{x_n\})$, we have

$$\begin{aligned} r(Tp, \{x_n\}) &= \limsup_{n\to\infty} \|x_n - Tp\| \\ &\leq \limsup_{n\to\infty} (3\|Tx_n - x_n\| + \|x_n - p\|) \\ &= \limsup_{n\to\infty} \|x_n - p\| \\ &= r(p, \{x_n\}). \end{aligned}$$

This implies that $Tp \in A(K, \{x_n\})$. Since E is uniformly convex, $A(K, \{x_n\})$ is singleton, therefore we get $Tp = p$.

Theorem 4.1. Let T be a Suzuki generalized nonexpansive mapping defined on a nonempty closed convex subset K of a Banach space E which satisfies the Opial's condition with $F(T) \neq \emptyset$. If $\{x_n\}$ is the iterative sequence defined by the iterative algorithm (1.2), then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. Let $p \in F(T)$. Then, from Lemma 4.1 $\lim_{n\to\infty} \|x_n - p\|$ exists. In order to show the weak convergence of the algorithm (1.2) to a fixed point of T, we will prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. For this, let $\{x_{n_j}\}$ and $\{x_{n_k}\}$ be two subsequences of $\{x_n\}$ which converges weakly to u and v respectively. By Lemma 4.1, we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ and using the Lemma 2.3, we have $I - T$ is demiclosed at zero. So $u, v \in F(T)$.

Next, we show the uniqueness. Since $u, v \in F(T)$, so $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exists. Let $u \neq v$. Then, by Opial's condition, we obtain

$$\begin{aligned} \lim_{n\to\infty} \|x_n - u\| &= \lim_{j\to\infty} \|x_{n_j} - u\| \\ &< \lim_{j\to\infty} \|x_{n_j} - v\| \\ &= \lim_{n\to\infty} \|x_n - v\| \\ &= \lim_{k\to\infty} \|x_{n_k} - v\| \\ &< \lim_{k\to\infty} \|x_{n_k} - u\| \\ &= \lim_{n\to\infty} \|x_n - u\| \end{aligned}$$

which is a contradiction, so $u = v$. Thus, $\{x_n\}$ converges weakly to a fixed point of T.

Next, we establish some strong convergence results for iterative algorithm (1.2).

Theorem 4.2. Let T be a Suzuki generalized nonexpansive mapping defined on a nonempty compact convex subset K of a uniformly convex Banach space E. If $\{x_n\}$ is the iterative sequence defined by the iterative algorithm (1.2), then $\{x_n\}$ converges strongly to a fixed point of T.
Proof. Using Lemma 2.4 , we get $F(T) \neq \emptyset$. So, by Lemma 4.2, we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p for some $p \in K$. From Lemma 2.2(iii), we have

$$\|x_{n_k} - Tp\| \leq 3\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - p\|$$

for all $n \geq 1$. Letting $k \to \infty$, we get that $\{x_{n_k}\}$ converges to Tp. This implies that $Tp = p$, i.e., $p \in F(T)$. Further, $\lim_{n\to\infty} \|x_n - p\|$ exists by Lemma 4.1. So, p is the strong limit of the sequence $\{x_n\}$.
A mapping $T : K \to K$ is said to satisfy the Condition (A) ([25]) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = inf\{\|x - p\| : p \in F(T)\}$.

Theorem 4.3. Let T be a Suzuki generalized nonexpansive mapping defined on a nonempty closed convex subset K of a uniformly convex Banach space E such that $F(T) \neq \emptyset$ and $\{x_n\}$ be the sequence defined by (1.2). If T satisfies Condition (A), then $\{x_n\}$ converges strongly to a fixed point of T.
Proof. By Lemma 4.1, $\lim_{n\to\infty} \|x_n - p\|$ exists and $\|x_{n+1} - p\| \leq \|x_n - p\|$ for all $p \in F(T)$.
We get

$$\inf_{p\in F(T)} \|x_{n+1} - p\| \leq \inf_{p\in F(T)} \|x_n - p\|,$$

which yields

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)).$$

This shows that the sequence $\{d(x_n, F(T))\}$ is decreasing and bounded below, so $\lim_{n\to\infty} d(x_n, F(T))$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = r$ for some $r \geq 0$. If $r = 0$ then the result follows. Assume $r > 0$. Also, by Lemma 4.2 we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.
It follows from Condition (A) that

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

so that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$.
Since f is a non decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, therefore $\lim_{n\to\infty} d(x_n, F(T)) = 0$. So, we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\} \subset F(T)$ such that

$$\|x_{n_k} - y_k\| < \frac{1}{2^k}$$

for all $k \in \mathbb{N}$. Using (4.4), we obtain

$$\|x_{n_{k+1}} - y_k\| < \|x_{n_k} - y_k\| < \frac{1}{2^k}.$$

Therefore,

$$\begin{aligned} \|y_{k+1} - y_k\| &\leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - y_k\| \\ &\leq \tfrac{1}{2^{k+1}} + \tfrac{1}{2^k} \\ &< \tfrac{1}{2^{k-1}} \to 0 \qquad as \quad n \to \infty. \end{aligned}$$

This implies that $\{y_k\}$ is a cauchy sequence in $F(T)$. Since $F(T)$ is closed, so $\{y_k\}$ converges to a point $p \in F(T)$. Then, $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we get $x_n \to p \in F(T)$. This completes the proof.

5. Numerical example

In this section, first we will construct an example of a Suzuki generalized nonexpansive mapping which is not a nonexpansive mapping. Then, using that example, we will show that our iteration scheme (1.2) has a better speed of convergence than number of existing iteration schemes.

Example 2: Define a mapping $T : [0, 1] \to [0, 1]$ by

$$Tx = \begin{cases} 1 - x & x \in [0, \frac{1}{12}) \\ \frac{x+11}{12} & x \in [\frac{1}{12}, 1]. \end{cases}$$

First we show that T is not a nonexpansive map. For this, take $x = \frac{8}{100}$ and $y = \frac{1}{12}$. Then,

$$\|Tx - Ty\| = \left\|(1 - x) - (\frac{y + 11}{12})\right\| = \frac{52}{14400}$$

and

$$\|x - y\| = |x - y| = \frac{4}{1200}.$$

Clearly, $\|Tx - Ty\| > \|x - y\|$ which proves that T is not a nonexpansive mapping.

Now, we show that T satisfies the condition K. For this, consider the following cases:

Case-I: Let $x \in [0, \frac{1}{12})$, then $\frac{1}{2}\|x - Tx\| = \frac{1}{2}|2x - 1| = \frac{1}{2}(1 - 2x)$. For $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, we must have $\frac{1}{2}(1 - 2x) \leq \|x - y\|$, i.e., $\frac{1}{2}(1 - 2x) \leq |x - y|$. Here note that the case $y < x$ is not possible. So, we are left with only one case when $y > x$, which gives $\frac{1}{2}(1 - 2x) \leq y - x$, which yields $y \geq \frac{1}{2}$. So, $y \in [\frac{1}{2}, 1]$. Now, we have $x \in [0, \frac{1}{12})$ and $y \in [\frac{1}{2}, 1]$. So,

$$\|Tx - Ty\| = \left\|(1 - x) - \frac{y + 11}{12}\right\| = \left|\frac{12x + y - 1}{12}\right| < \frac{1}{12}$$

and

$$\|x - y\| = |x - y| > \frac{5}{12}.$$

Hence,

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|.$$

Case-II: Let $x \in [\frac{1}{12}, 1]$, then $\frac{1}{2}\|x - Tx\| = \frac{1}{2}|x - \frac{x+11}{12}| = \frac{11-11x}{24}$. For $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, we must have $\frac{11-11x}{24} \leq \|x - y\|$, i.e., $\frac{11-11x}{24} \leq |x - y|$. Here we have two possibilities.

A: When $x < y$, we get $\frac{11-11x}{24} \leq y - x$, i.e., $y \geq \frac{11+13x}{24}$. So, $y \in [\frac{145}{288}, 1] \subset [\frac{1}{12}, 1]$, which gives $\|Tx - Ty\| = \frac{1}{12}\|x - y\| \leq \|x - y\|$. Hence,

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|.$$

B: When $x > y$, then $\frac{11-11x}{24} \leq x - y$, i.e. $y \leq \frac{35x-11}{24}$ which gives $y \in [0, 1]$. Also, $\frac{24y+11}{35} \leq x$ which yields $x \in [\frac{11}{35}, 1]$. Here, for $x \in [\frac{11}{35}, 1]$ and $y \in [\frac{1}{12}, 1]$ Case IIA can be used. So, we only need to verify when $x \in [\frac{11}{35}, 1]$ and $y \in [0, \frac{1}{12})$. For this,

$$\|Tx - Ty\| = \left|\frac{x+11}{12} - (1-y)\right| = \frac{1}{12}|12y + x - 1| \leq \frac{1}{12}$$

and

$$\|x - y\| = |x - y| > \frac{97}{420}.$$

So, $\|Tx - Ty\| \leq \|x - y\|$. Thus, mapping T satisfies the Condition (C) for all the possible cases.
Now, using above example, we will show that iteration algorithm (1.2) converges faster than Thakur New, M and M* iteration. Let $\alpha_n = \beta_n = \frac{n}{n+10}$ for all $n \in \mathbb{N}$ and $x_1 = 0.02$, then we get the following Table 3 of iteration values and Figure 3.

Table 3. Comparison of the new method to other methods for Suzuki generalized nonexpansive mapping.

Step	Thakur New Iteration	M Iteration	M* Iteration	New Iteration
1	0.02	0.02	0.02	0.02
2	0.9976721763085	0.9938005050505	0.9937661654423	0.9998726851852
3	0.9999842461778	0.999963525348	0.9999634151701	0.9999999375787
4	0.9999998959391	0.9999998002857	0.999999800716	0.9999999999715
5	0.9999999993314	0.9999999989763	0.9999999989872	1.0000000000000
6	0.9999999999958	0.9999999999951	0.9999999999952	1.0000000000000
7	1.0000000000000	1.0000000000000	1.0000000000000	1.0000000000000

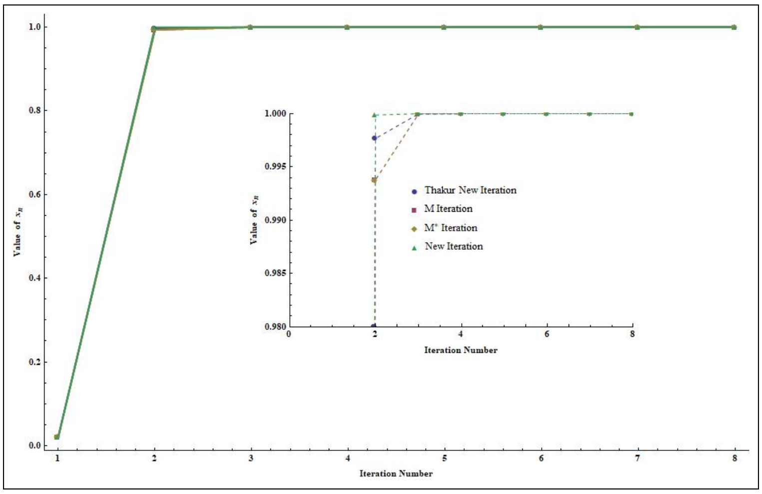

Figure 3. Graph corresponding to Table 3.

It is evident from above table and graph that our algorithm (1.2) converges at a better speed than the above mentioned schemes.

6. Application

In this section, we show that our iterative algorithm can be used to find a solution of a delay differential equation.

Many physical problems arising in various fields can be easily modeled with the help of ordinary differential equations. Later, it was recognized that a phenomena may have a delayed effect in a differential equation, leading to the development of concept of delay differential equations. Following this, numerous methods have been obtained to solve various kinds of delay differential equations (e.g. [13–15, 32, 33]).

In this paper, we consider the following delay differential equation

$$x'(t) = f(t, x(t), x(t-\tau)), \quad t \in [t_0, b] \tag{6.1}$$

with initial condition

$$x(t) = \psi(t), \quad t \in [t_0 - \tau, t_0]. \tag{6.2}$$

Now, we will show that the sequence generated by our iteration scheme (1.2) converges strongly to the solution of (6.1).

It is well known that $(C([a,b]), \|.\|_\infty)$ is a Banach space where $C([a,b])$ denotes the space of all continuous real valued functions on a closed interval $[a,b]$ and $\|.\|_\infty$ is a Chebyshev norm $\|x-y\|_\infty = \max_{t\in[a,b]} |x(t)-y(t)|$.

Assume that the following conditions are satisfied

(A_1) $t_0, b \in \mathbb{R}, \tau > 0$;

(A_2) $f \in C([t_0,b] \times \mathbb{R}^2, \mathbb{R})$;

(A_3) $\psi \in C([t_0-\tau, b], \mathbb{R})$;

(A_4) there exists $L_f > 0$ such that

$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le L_f \sum_{i=1}^{2} |u_i - v_i|, \quad \forall u_i, v_i \in \mathbb{R}, i = 1,2, t \in [t_0,b]$;

(A_5) $2L_f(b-t_0) < 1$.

We notice that the solution of (6.1)-(6.2) if it exists is of the following form

$$x(t) = \begin{cases} \psi(t), & t \in [t_0-\tau, t_0] \\ \psi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds, & t \in [t_0, b]. \end{cases}$$

Here, $x \in C([t_0-\tau, b], \mathbb{R}) \cap C^1([t_0,b], \mathbb{R})$.

Coman et al. [8] established the following results.

Theorem 6.1. Assume that conditions $(A_1)-(A_5)$ are satisfied. Then Problem (6.1) – (6.2) has a unique solution, say $p \in C([t_0-\tau, b], \mathbb{R}) \cap C^1([t_0,b], \mathbb{R})$ and

$$p = \lim_{n\to\infty} T^n(x) \qquad for \quad any \quad x \in C([t_0-\tau, b], \mathbb{R}).$$

Now, we prove the following result using our iterative process (1.2).

Theorem 6.2. Suppose that conditions $(A_1)-(A_5)$ are satisfied. Then the problem (6.1) – (6.2) has a unique solution say $p \in C([t_0-\tau, b], \mathbb{R}) \cap C^1([t_0,b], \mathbb{R})$ and sequence generated by the algorithm (1.2) converges to p.

Proof. Let $\{x_n\}$ be a iterative sequence generated by (1.2) for the following operator:

$$Tx(t) = \begin{cases} \psi(t), & t \in [t_0 - \tau, t_0] \\ \psi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds, & t \in [t_0, b], \end{cases}$$

where $\alpha_n \in (0,1)$ for all $n \in \mathbb{N}$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Denote by p the fixed point of T. We will show that $x_n \to p$ as $n \to \infty$.

For $t \in [t_0 - \tau, t_0]$, it is easy to see that $x_n \to p$ as $n \to \infty$.

For $t \in [t_0, b]$, we have

$$\begin{aligned}
\|z_n - p\|_\infty &= \|Tx_n - Tp\|_\infty \\
&= \max_{t\in[t_0-\tau,b]} |Tx_n(t) - Tp(t)| \\
&= \max_{t\in[t_0-\tau,b]} \left|\psi(t_0) + \int_{t_0}^{t} f(s, x_n(s), x_n(s-\tau)ds - \psi(t_0) - \int_{t_0}^{t} f(s, p(s), p(s-\tau))ds\right| \\
&\le \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} |f(s, x_n(s), x_n(s-\tau)) - f(s, p(s), p(s-\tau))|ds \\
&\le \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f(|x_n(s) - p(s)| + |x_n(s-\tau) - p(s-\tau)|)ds \\
&\le \int_{t_0}^{t} L_f\Big(\max_{t\in[t_0-\tau,b]} |x_n(s) - p(s)| + \max_{t\in[t_0-\tau,b]} |x_n(s-\tau) - p(s-\tau)|\Big)ds \\
&\le \int_{t_0}^{t} L_f(\|x_n - p\|_\infty + \|x_n - p\|_\infty)ds \\
&\le 2L_f(b - t_0)\|x_n - p\|_\infty, \qquad (6.3)
\end{aligned}$$

$$\begin{aligned}
\|y_n - p\|_\infty &= \|T((1-\alpha_n)z_n + \alpha_n Tz_n) - Tp\|_\infty \\
&= \max_{t\in[t_0-\tau,b]} |T((1-\alpha_n)z_n + \alpha_n Tz_n)(t) - Tp(t)| \\
&= \max_{t\in[t_0-\tau,b]} \Big|\psi(t_0) + \int_{t_0}^{t} f(s, ((1-\alpha_n)z_n + \alpha_n Tz_n)(s), ((1-\alpha_n)z_n + \alpha_n Tz_n)(s-\tau))ds \\
&\quad - \psi(t_0) - \int_{t_0}^{t} f(s, p(s), p(s-\tau))ds\Big| \\
&\le \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} |f(s, ((1-\alpha_n)z_n + \alpha_n Tz_n)(s), ((1-\alpha_n)z_n + \alpha_n Tz_n)(s-\tau)) \\
&\quad - f(s, p(s), p(s-\tau))|ds \\
&\le \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f(|((1-\alpha_n)z_n + \alpha_n Tz_n)(s) - p(s)| \\
&\quad + |((1-\alpha_n)z_n + \alpha_n Tz_n)(s-\tau) - p(s-\tau)|)ds \\
&\le \int_{t_0}^{t} L_f\Big(\max_{t\in[t_0-\tau,b]} |((1-\alpha_n)z_n + \alpha_n Tz_n)(s) - p(s)| \\
&\quad + \max_{t\in[t_0-\tau,b]} |((1-\alpha_n)z_n + \alpha_n Tz_n)(s-\tau) - p(s-\tau)|\Big)ds \\
&\le \int_{t_0}^{t} L_f(\|((1-\alpha_n)z_n + \alpha_n Tz_n) - p\|_\infty + \|((1-\alpha_n)z_n + \alpha_n Tz_n) - p\|_\infty)ds \\
&\le 2L_f(b - t_0)\|((1-\alpha_n)z_n + \alpha_n Tz_n) - p\|_\infty, \qquad (6.4)
\end{aligned}$$

$$
\begin{aligned}
\|(1-\alpha_n)z_n+\alpha_nTz_n-p\|_\infty &= \|(1-\alpha_n)z_n+\alpha_nTz_n-Tp\|_\infty\\
&\le (1-\alpha_n)\|z_n-p\|_\infty+\alpha_n\|Tz_n-Tp\|_\infty\\
&= (1-\alpha_n)\|z_n-p\|_\infty+\alpha_n\max_{t\in[t_0-\tau,b]}\Big|\psi(t_0)+\int_{t_0}^{t}f(s,z_n(s),z_n(s-\tau))ds\\
&\quad -\psi(t_0)-\int_{t_0}^{t}f(s,p(s),p(s-\tau))ds\Big|\\
&\le (1-\alpha_n)\|z_n-p\|_\infty\\
&\quad +\alpha_n\max_{t\in[t_0-\tau,b]}\int_{t_0}^{t}|f(s,z_n(s),z_n(s-\tau))-f(s,p(s),p(s-\tau))|ds\\
&\le (1-\alpha_n)\|z_n-p\|_\infty+\alpha_n\max_{t\in[t_0-\tau,b]}\int_{t_0}^{t}L_f(|z_n(s)-p(s)|\\
&\quad +|(z_n(s-\tau)-p(s-\tau)|)ds\\
&\le (1-\alpha_n)\|z_n-p\|_\infty+\alpha_n\int_{t_0}^{t}L_f(\|z_n-p\|_\infty+\|z_n-p\|_\infty)ds\\
&\le (1-\alpha_n)\|z_n-p\|_\infty+2\alpha_nL_f(b-t_0)\|z_n-p\|_\infty\\
&= \Big[1-\alpha_n(1-2L_f(b-t_0))\Big]\|z_n-p\|_\infty, \qquad (6.5)
\end{aligned}
$$

$$
\begin{aligned}
\|x_{n+1}-p\|_\infty &= \|Ty_n-Tp\|_\infty\\
&= \max_{t\in[t_0-\tau,b]}\Big|\int_{t_0}^{t}[f(s,y_n(s),y_n(s-\tau))-f(s,p(s),p(s-\tau))]\Big|\\
&\le \max_{t\in[t_0-\tau,b]}\int_{t_0}^{t}L_f(|y_n(s)-p(s)|+|y_n(s-\tau)-p(s-\tau)|)ds\\
&\le 2L_f(b-t_0)\|y_n-p\|_\infty \qquad (6.6)
\end{aligned}
$$

Using (6.3), (6.4), (6.5) and (6.6) we get

$$\|x_{n+1}-p\|_\infty \le 8L_f^3(b-t_0)^3\Big[1-\alpha_n(1-2L_f(b-t_0))\Big]\|x_n-p\|_\infty.$$

On using assumption (A_5), we have

$$\|x_{n+1}-p\|_\infty \le \Big[1-\alpha_n(1-2L_f(b-t_0))\Big]\|x_n-p\|_\infty.$$

Therefore, inductively we get

$$\|x_{n+1}-p\|_\infty \le \prod_{k=0}^{n}\Big[1-\alpha_k(1-2L_f(b-t_0))\Big]\|x_0-p\|_\infty.$$

Since $\alpha_n\in[0,1]$, for all $n\in\mathbb{N}$, assumption (A_5) yields

$$1-\alpha_n(1-2L_f(b-t_0))<1.$$

Using the fact that $e^{-x} \geq 1 - x$ for all $x \in [0, 1]$, we have

$$\|x_{n+1} - p\|_\infty \leq \|x_0 - p\|_\infty e^{-(1-2L_f(b-t_0))\sum_{k=0}^{n}\alpha_k},$$

which gives $\lim_{n\to\infty} \|x_n - p\|_\infty = 0$.
From the above theorem, we can say that our method will definitely converge to the unique solution of (6.1) which is a main advantage over the other methods available for the same.

7. Conclusion

In this study a new fixed iteration process (1.2) has been obtained which is utilized to approximate fixed point of Suzuki generalized nonexpansive mappings. Further, We show that our iteration process (1.2) converges faster than the recent M-iteration process (1.1) for contractive-like mappings. It must be noted here that Ullah and Arshad [31] did not give the rate of convergence of their process analytically. They claimed just by an example. However, we not only give the proof analytically but also validate with an example. Further, we performed convergence analysis and a non trivial example has been given to illustrate the convergence behaviour. In the last section, we applied our iteration process to find the solution of delay differential equation.

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Conflict of interest

The authors declare that they have no competing interests.

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