Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(4): 3138–3155.
DOI: 10.3934/math.2020202
Received: 16 December 2019
Accepted: 16 March 2020
Published: 24 March 2020

Research article

On the construction, properties and Hausdorff dimension of random Cantor one p^{th} set

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Abstract: In 1883, German Mathematician George Cantor introduced Cantor ternary set which is a self-similar fractal. K. J. Falconer (1990) defined random Cantor set with statistical self-similarity. The purpose of this paper is to introduce generalized random Cantor sets (one 5^{th}, one 7^{th} and in general one p^{th}). Some properties and results of random Cantor one p^{th} set have also been obtained. We compute Hausdorff dimension of random Cantor one p^{th} sets and show that Hausdorff dimension of these random Cantor sets is less than that of Hausdorff dimension of Cantor one p^{th} sets, calculated by Ashish et al. (2013).

Keywords: nonlinear dynamics; random construction; random Cantor one p^{th} set; Hausdorff dimension; martingale; probability
Mathematics Subject Classification: 54H20, 37H99, 60B05, 49K45

1. Introduction

The complex characterization of dynamic modelling has been the hot topic in diverse applications of physics [1–3], mathematical biology [4–11], networks systems [12–18], etc. Especially the fractals have received great attention in the literature. The notion of fractals occupies an important place in understanding the structures of objects found in nature [19–21]. Benoit B. Mandelbrot defined fractals as self-similar objects either deterministic or statistical. Fractals which have different scales of self-similarity (statistical self-similarity) are examples of random fractals.

Cantor ternary set which was defined by George Cantor [22] in 1883 is an example of a classical self-similar fractal. During the period 1879–1884, George Cantor published a series of papers [22–27] in which he discussed many problems in the area of set theory. For detailed study of Cantor ternary set, one may refer to Peitegen et al. [28], Devaney [29], Beardon [30], Falconar [31, 32]and the references therein. Kumar et al. [33] introduced 5-adic Cantor one-fifth set and studied its application in string theory. Furher, Ashish et al. [34] calculated the Hausdorff dimension of a self-similar Cantor middle one half set and Cantor one-fifth set.

Recently, focus of the researchers is on random Cantor set which is an example of statistical self-similar fractal. The construction and Hausdorff dimension of a random Cantor set have been discussed in the books of Falconer [31, 32]. He proved some results on random fractals. In 2009, Pestana et al. [35] computed Hausdorff dimension of a random Cantor set. In 2015, Islam et al. [36] showed that generalized Cantor set is both measurable set and Borel set. Recently in 2017, Changhao Chen [37] determined the almost sure Hausdorff, Packing, Box and Assouad dimensions of a class of random Cantor sets.

In this paper, we give some basic definitions and lemmas in Section 2 that have been taken into account during our study. Section 3 is dedicated to the construction of random Cantor one p^{th} sets. Some properties of random Cantor one p^{th} set are driven in Section 4. We prove our main results in Section 5. In Section 6, we find the general formula to calculate the Hausdorff dimension of random Cantor one p^{th} sets and show that Hausdorff dimension of these random Cantor sets is less than that of Hausdorff dimension of the Cantor one p^{th} sets. Finally, we summarize our findings in Section 7.

2. Preliminaries

This section deals with some definitions and lemmas which are prerequisite for further work.

Definition 2.1. (Random Cantor Set) [31] $F = \bigcap_{i=1}^{\infty} I_i$ is a random Cantor set, where $[0, 1] = I_0 \supset I_1 \supset ...$ is a decreasing sequence of closed sets. The set I_i is the union of 2^i disjoint closed i^{th} level sub-intervals with random length. We suppose that each i^{th} level interval I consists two $(i + 1)^{th}$ level intervals I_L and I_R, expressing the left and right hand ends of I, respectively. Now, we impose statistical self-similarity by the requirement that the ratios $\frac{|I_L|}{|I|}$ have independent and identical probability distribution for every basic interval I of the construction, and similarly for the ratios $\frac{|I_R|}{|I|}$. Thus obtained random Cantor set F is statistically self-similar, in that the distribution of the set $F \cap I$ is same as that of F, but scaled by a factor $|I|$, for each interval I in the construction.

Definition 2.2. [38] The outer measure of a set K is denoted by $m^*(K)$ and given by

$$m^*(K) = \inf \left\{ \sum_{i=1}^{\infty} l(I_i) | K \subseteq \bigcup_{i=1}^{\infty} I_i \right\}.$$

Definition 2.3. [38] A set A is said to be measurable if

$$m^*(K) = m^*(K \cap A) + m^*(K \cap A^c)$$

holds for any set K.

Definition 2.4. [38] A Borel set is the set that can be formed from open or closed sets by repeatedly taking countable unions, countable intersections and relative complements.

Definition 2.5. [32] μ is said to be a measure on $\mathbb{R}$ if μ assigns a non - negative number including ∞ to each subset of $\mathbb{R}$ and satisfy

(i) $\mu(\phi) = 0$,

(ii) $C \subseteq D \Rightarrow \mu(C) \leq \mu(D)$,

(iii) if $E_i,\ i = 1, 2, ...$ is a countable sequence of pairwise disjoint sets, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Here, $\mu(E)$ is the measure or size of the set E.

Definition 2.6. [32] The support of a measure μ is the smallest closed set Y for which $\mu(\mathbb{R} \setminus Y) = 0$ and it is denoted by spt μ.

Definition 2.7. [32] A mass distribution is a measure μ defined on $\mathbb{R}^n$ which satisfy $0 < \mu(\mathbb{R}) < \infty$. Also, $\mu(E)$ is called the mass of the set E.

Let H_i be a collection of disjoint Borel subsets of a set I with $I = H_0$, and for each $i = 1, 2, ...,$ we construct H_i in such a way that each set E in H_i contains a finite number of sets of H_{i+1} and itself is contained in one of the sets of H_{i-1}. Let I_i be the union of sets in H_i for $i = 1, 2, ...$. Moreover, the collection of the sets that are contained in H_i together with subsets of $(\mathbb{R}^n \setminus I_i)$ for some i is denoted by H.

Lemma 2.8. *([32], Proposition 1.7) Consider μ, defined on a collection of sets H as described above, then the definition of μ can be extended to all subsets of* $\mathbb{R}^n$ *so that μ becomes a measure. If K is a Borel set, then the value of* $\mu(K)$ *is uniquely determined. Also, the support of μ, i.e. spt* $\mu \subset I_\infty = \bigcap_{i=1}^{\infty} \bar{I}_i$.

Definition 2.9. [39] An experiment is known as a random experiment if the outcomes cannot be predicted with certainty.

Definition 2.10. [39] The collection of all possible outcomes of a random experiment is said to be a sample space, denoted by Ω.

Definition 2.11. [39] An event A is a subset of the sample space Ω which belongs to a collection $\mathcal{D}$ of subsets of Ω and satisfy

(a) $\Omega \in \mathcal{D}$,

(b) $A \in \mathcal{D} \implies \mathcal{D} \setminus A \in \mathcal{D}$,

(c) $A_j \in \mathcal{D} \implies \bigcup_{j=1}^{\infty} A_j \in \mathcal{D}$, for $1 \leq j < \infty$.

The collection $\mathcal{D}$ is said to be an event space.

Definition 2.12. [32] Consider a δ-cover $\{U_i\}$ of a Borel set K which covers K, i.e., $K \subset \cup_i U_i$, where $0 < |U_i| \leq \delta$. Define

$$H^r_\delta(K) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^r : U_i \textit{ is open}, 0 < |U_i| \leq \delta \textit{ and } K \subset \cup_i U_i\right\},$$

for each $\delta > 0$ and $r \geq 0$. Then, the r-dimensional Hausdorff measure $H^r(K)$ is given by the relation

$$H^r(K) = \lim_{\delta \to 0} H^r_\delta(K).$$

Moreover, the Hausdorff dimension of set K is defined by

$$dim_H(K) = \sup\{r : H^r(K) > 0\}.$$

Definition 2.13. [32] For $t \geq 0$, the t-potential at a point x of $\mathbb{R}^n$ resulting from the mass distribution μ on $\mathbb{R}^n$ is defined as

$$\phi_s(x) = \int \frac{d\mu(y)}{|x-y|^t}.$$

The t-energy of mass distribution μ is given by

$$I_s(\mu) = \int \phi_s(x) d\mu(x) = \int \frac{d\mu(y)d\mu(x)}{|x-y|^t}.$$

Definition 2.14. [32] A transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a similarity of ratio $\lambda > 0$ if $|T(x) - T(y)| = \lambda\,|x-y|$ for all $x, y \in \mathbb{R}^n$, i.e. a similarity transforms sets into geometrically similar ones with all lengths multiplied by the factor λ.

Lemma 2.15. *([32], Theorem 4.13) Let K be a subset of* $\mathbb{R}^n$*. If there is a mass distribution* μ *on K with t-energy of* μ *less than* ∞*, i.e.* $I_t(\mu) < \infty$*, then* $H^t(K) = \infty$ *and* $dim_H K \geq t$*.*

Lemma 2.16. *([32], Theorem 9.3) Suppose that the similarities* S_k *on* $\mathbb{R}^n$ *satisfy the open set condition, i.e., there exists a non empty bounded open set V such that*

$$\bigcup_{k=1}^{m} S_k(V) \subset V,$$

and ratios $0 < r_k < 1$ *for* $1 \leq k \leq m$*. If F is given by the relation*

$$F = \bigcup_{k=1}^{m} S_k(F),$$

with iterated function system $\{S_1, S_2, ..., S_m\}$*, then* $dim_H F = s$*, where s satisfy the equation*

$$\sum_{k=1}^{m} r_k^s = 1.$$

3. Construction of random Cantor one p^{th} set

In this section, we construct random Cantor one 5^{th} set, random Cantor one 7^{th} set and in general random Cantor one p^{th} set. Throughout the paper, we consider p as an odd number greater than 1, i.e., $p = 3, 5, 7, \ldots$.

3.1. Random Cantor one 5^{th} Set

Let us consider constants a, b and c such that $0 < a \leq b \leq c < \frac{1}{3}$. Let Ω be the collection of all decreasing sequences of sets $[0, 1] = H_0 \supset H_1 \supset H_2 \supset ...$. Here, the set H_i contains 3^i disjoint closed intervals $E_{k_1,k_2,...k_i}$, where $k_j = 1$ *or* 2 *or* 3 ($1 \leq j \leq i$) as shown in Figure 1. We see that the interval $E_{k_1,k_2,...k_i}$ of H_i consists the three sub - intervals $E_{k_1,k_2,...k_i,1}$, $E_{k_1,k_2,...k_i,2}$ and $E_{k_1,k_2,...k_i,3}$ of H_{i+1} in such a way that left hand ends of $E_{k_1,k_2,...k_i}$ and $E_{k_1,k_2,...k_i,1}$ remain same. Similarly, the right hand ends of $E_{k_1,k_2,...k_i}$ and $E_{k_1,k_2,...k_i,3}$ coincide. Let us suppose that $M_{k_1,k_2,...,k_i} = \frac{|E_{k_1,k_2,...k_i}|}{|E_{k_1,k_2,...k_{i-1}}|}$ and $a \leq M_{k_1,k_2,...,k_i} \leq c, \forall\, k_1, k_2, ..., k_i$. Here, the ratios $M_{k_1,k_2,...,k_i}$ are taken as random independent variables. Now, we impose statistical self similarity on our construction by considering that the length ratios $\frac{|E_{k_1,k_2,...k_i,1}|}{|E_{k_1,k_2,...k_i}|}$ and $\frac{|E_{k_1,k_2,...k_i,2}|}{|E_{k_1,k_2,...k_i}|}$ have the same statistical distribution as do the ratios $\frac{|E_{k_1,k_2,...k_i,3}|}{|E_{k_1,k_2,...k_i}|}$ for each $k_1, k_2, ...k_i$. Thus, from above construction, we say that random Cantor one 5^{th} set $F_{\frac{1}{5}}$ has statistical self-similarity and is given by

$$F_{\frac{1}{5}} = \bigcap_{i=1}^{\infty} H_i. \tag{3.1}$$

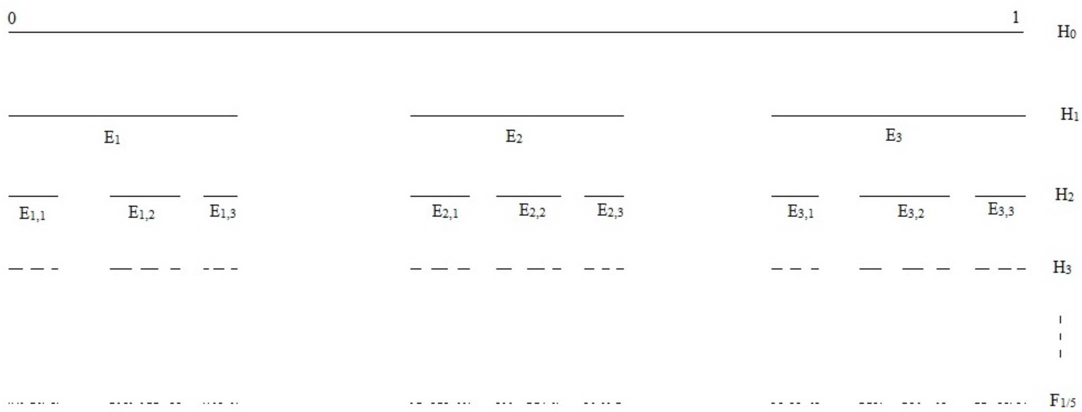

Figure 1. Random Cantor one 5^{th} set.

3.2. Random Cantor one 7^{th} Set

Now consider $[0, 1] = H_0 \supset H_1 \supset H_2 \supset ...$ as a decreasing sequence of closed intervals. H_i is the union of 4^i disjoint closed i^{th} level intervals. Then, random Cantor one 7^{th} set is defined as follows

$$F_{\frac{1}{7}} = \bigcap_{i=1}^{\infty} H_i, \tag{3.2}$$

where each i^{th}-level interval H_i contains 4^i disjoint closed intervals $E_{k_1,k_2,...k_i}$, where $k_j = 1$ *or* 2 *or* 3 *or* 4 ($1 \leq j \leq i$) as shown in Figure 2. We take the length of each interval random and for every interval $E_{k_{1,2,...i}}$ we impose the same statistical self-similarity as imposed in the construction of random Cantor one 5^{th} set.

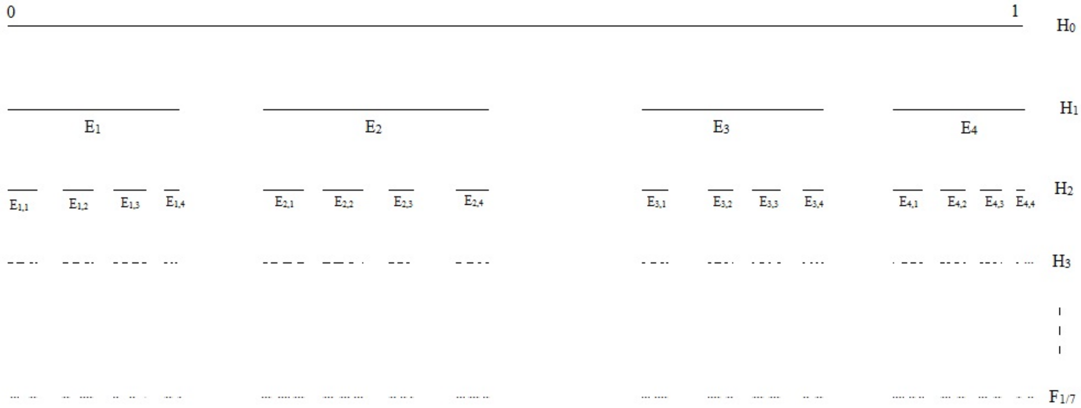

Figure 2. Random Cantor one 7^{th} set.

3.3. Random Cantor one p^{th} Set

By analogue we construct random Cantor one p^{th} set $F_{\frac{1}{p}}$ and impose the same statistical self-similarity as imposed in above constructions. Let $[0, 1] = H_0 \supset H_1 \supset H_2 \supset ...$ be a decreasing sequence of closed intervals. Here, H_i is the union of $(\frac{p+1}{2})^i$ disjoint closed intervals of i^{th} level intervals. The random Cantor one p^{th} set is given by

$$F_{\frac{1}{p}} = \bigcap_{i=1}^{\infty} H_i,$$

where each i^{th}-level interval H_i contains $(\frac{p+1}{2})^i$ disjoint closed intervals $E_{k_1,k_2,...k_i}$, $k_j = 1 \text{ or } 2 \text{ or } ... \text{ or } \frac{p+1}{2}$ $(1 \leq i \leq \frac{p+1}{2})$ and $p = 3, 5, 7, ...$ as shown in Figure 3. The length of each interval is taken random.

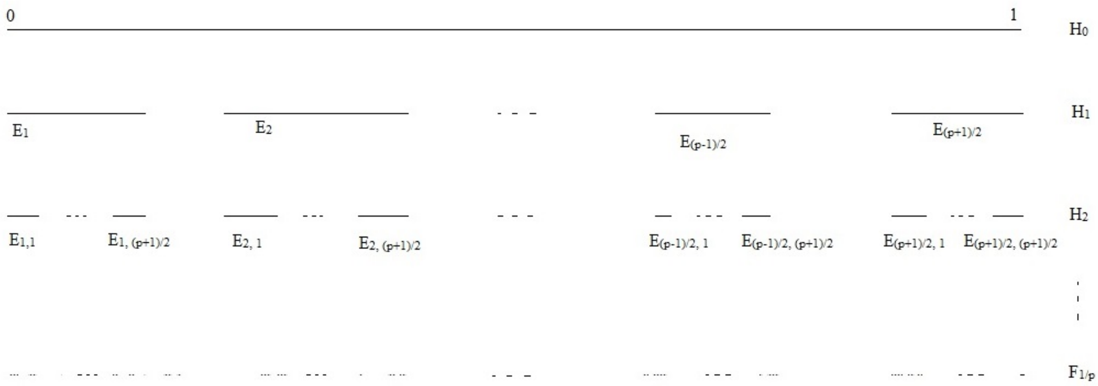

Figure 3. Random Cantor one p^{th} set.

Now, we describe this construction in terms of probability. Let us consider constants $a_1, a_2, ..., a_{\frac{p+1}{2}}$ such that $0 < a_1 \leq a_2 \leq ... \leq a_{\frac{p-1}{2}} < a_{\frac{p+1}{2}}$. Let Ω be the collection of all decreasing sequences of sets $[0, 1] = H_0 \supset H_1 \supset H_2 \supset ...$. Here, the set H_i contains $(\frac{p+1}{2})^i$ disjoint closed intervals $E_{k_1,k_2,...k_i}$, where $k_j = 1 \text{ or } 2 \text{ or } 3 ... \text{ or } \frac{p+1}{2}$ $(1 \leq j \leq i)$ as shown in Figure 3. We see that the interval $E_{k_1,k_2,...k_i}$ of H_i

comprises $(\frac{p+1}{2})$ sub - intervals $E_{k_1,k_2,...k_i,1}, E_{k_1,k_2,...k_i,2}, ..., E_{k_1,k_2,...k_i,\frac{p+1}{2}}$ of H_{i+1} in such a way that left hand ends of $E_{k_1,k_2,...k_i}$ and $E_{k_1,k_2,...k_i,1}$ remain same. Similarly, the right hand ends of $E_{k_1,k_2,...k_i}$ and $E_{k_1,k_2,...k_i,\frac{p+1}{2}}$ coincide. Let us suppose that $M_{k_1,k_2,...,k_i} = \frac{|E_{k_1,k_2,...k_i}|}{|E_{k_1,k_2,...k_{i-1}}|}$ and $a_1 \leq M_{k_1,k_2,...,k_i} \leq a_{\frac{p+1}{2}}, \forall\ k_1, k_2, ..., k_i$. Here, the ratios $M_{k_1,k_2,...,k_i}$ are considered as random independent variables. Now, we impose some statistical self similarity on our construction by considering that the length ratios $\frac{|E_{k_1,k_2,...,k_i,1}|}{|E_{k_1,k_2,...,k_i}|}, \frac{|E_{k_1,k_2,...,k_i,2}|}{|E_{k_1,k_2,...,k_i}|}, ..., \frac{|E_{k_1,k_2,...,k_i,\frac{p-1}{2}}|}{|E_{k_1,k_2,...,k_i}|}$ have the same statistical distribution as do the ratios $\frac{|E_{k_1,k_2,...,k_i,\frac{p+1}{2}}|}{|E_{k_1,k_2,...,k_i}|}$ for each $k_1, k_2, ...k_i$. In this way, we obtain a random Cantor one p^{th} set $F_{\frac{1}{p}}$ given by

$$F_{\frac{1}{p}} = \bigcap_{i=1}^{\infty} H_i. \tag{3.3}$$

4. Properties of random Cantor one p^{th} set

4.1. Random Cantor one p^{th} set is disconnected and contains no intervals

The random Cantor one p^{th} set $F_{\frac{1}{p}}$ is disconnected, since in its construction it contains only points and no intervals.

4.2. Random Cantor one p^{th} set is nowhere dense

A set K is said to be nowhere dense if closure of K has empty interior, i.e., there are no open sets in its closure. The closure of K is the union of itself and the set of its limit points. Since random Cantor one p^{th} set has every point as a limit point. So, the closure of random Cantor one p^{th} set is the set itself. The random Cantor one p^{th} set has empty interior. Thus, random Cantor one p^{th} set is nowhere dense.

4.3. Random Cantor one p^{th} set is both a Borel set and a measurable set

Since, arbitrary intersection of closed sets is closed set. Then, by our construction $F_{\frac{1}{p}} = \bigcap_{i=1}^{\infty} H_i$ is a closed set. Thus, by the definition of Borel set $F_{\frac{1}{p}}$ is a Borel set. Also, every Borel set is measurable set. Hence, random Cantor one p^{th} set is both a Borel set and a measurable set.

5. Main results

Before proving the Theorem 5.1, let Ω be the collection of all decreasing sequences of sets $[0, 1] = H_0 \supset H_1 \supset H_2 \supset ...$. Here, the set H_i contains $(\frac{p+1}{2})^i$ disjoint closed intervals $E_{k_1,k_2,...k_i}$, where $k_j = 1$ *or* 2 *or* 3 ... *or* $\frac{p+1}{2}$ $(1 \leq j \leq i)$ as shown in Figure 3. The interval $E_{k_1,k_2,...k_i}$ of H_i comprises $(\frac{p+1}{2})$ sub - intervals $E_{k_1,k_2,...k_i,1}, E_{k_1,k_2,...k_i,2}, ..., E_{k_1,k_2,...k_i,\frac{p+1}{2}}$ of H_{i+1} in such a way that left hand ends of $E_{k_1,k_2,...k_i}$ and $E_{k_1,k_2,...k_i,1}$ remain same. Similarly, the right hand ends of $E_{k_1,k_2,...k_i}$ and $E_{k_1,k_2,...k_i,\frac{p+1}{2}}$ coincide. Let us suppose that $M_{k_1,k_2,...,k_i} = \frac{|E_{k_1,k_2,...k_i}|}{|E_{k_1,k_2,...k_{i-1}}|}$ with $k_j = 1$ *or* 2 *or* 3 ... *or* $\frac{p+1}{2}$ $(1 \leq j \leq i)$. Here, the ratios $M_{k_1,k_2,...,k_i}$ are considered as independent random variables. Now, we impose some statistical self similarity on our construction by considering that for each $n = 1, 2, ..., \frac{p+1}{2}$, the variables $M_{k_1,k_2,...,k_i,n} = \frac{|E_{k_1,k_2,...,k_i,n}|}{|E_{k_1,k_2,...,k_i}|}$

have the same statistical distribution, where $p = 3, 5, 7, ..., \frac{p+1}{2}$, e.i. the length ratios $\frac{|E_{k_1,k_2,...,k_i,1}|}{|E_{k_1,k_2,...,k_i}|}$, $\frac{|E_{k_1,k_2,...,k_i,2}|}{|E_{k_1,k_2,...,k_i}|}, ..., \frac{|E_{k_1,k_2,...,k_i,\frac{p-1}{2}}|}{|E_{k_1,k_2,...,k_i}|}$ have the same statistical distribution as do the ratios $\frac{|E_{k_1,k_2,...,k_i,\frac{p+1}{2}}|}{|E_{k_1,k_2,...,k_i}|}$ for every sequence $k_1, k_2, ...k_i$, where $k_j = 1$ *or* 2 *or* 3 ... *or* $\frac{p+1}{2}$ $(1 \leq j \leq i)$ (see Subsection 3.3 and Figure 3).

Theorem 5.1. *The random Cantor one p^{th} set $F_{\frac{1}{p}}$, constructed in Subsection 3.3 has Hausdorff dimension r i.e., $dim_H F_{\frac{1}{p}} = r$, where r is the solution of the expectation equation*

$$\mathbf{E}(M_1^r + M_2^r + ... + M^r_{(\frac{p+1}{2})}) = 1. \tag{5.1}$$

Also, $F_{\frac{1}{p}}$ has probability 1.

Proof. For $E \in H_i$, we mean that the interval E is the i^{th} -level interval $E_{k_1,k_2,...k_i}$ of H_i. For such type of intervals, we take random variables $E_{L_1} = E_{k_1,k_2,...k_i,1}$, $E_{L_2} = E_{k_1,k_2,...k_i,2}$ and $E_{L_{\frac{p+1}{2}}} = E_{k_1,k_2,...k_i,\frac{p+1}{2}}$. Also, let $\mathbf{E}(Y|\mathcal{D}_i)$ be the conditional expectation of a random variable Y given $\mathcal{D}_i$ (independent random variables), where $\mathcal{D}_i = M_{k_1,k_2,...k_j}$ for all sequences $k_1, k_2, ...k_j$ with $j \leq i; i = 1, 2, ..., \frac{p+1}{2}$. Let $E_{k_1,k_2,...k_i}$ be an interval of H_i. Then for $r > 0$

$$\begin{aligned} &\mathbf{E}(|E_{k_1,k_2,...,k_i,1}|^r + |E_{k_1,k_2,...,k_i,2}|^r + ... + |E_{k_1,k_2,...,k_i,\frac{p+1}{2}}|^r \big| \mathcal{D}_i) \\ &= \mathbf{E}(M^r_{k_1,k_2,...k_i,1} + M^r_{k_1,k_2,...,k_i,2} + ... + M^r_{k_1,k_2,...,k_i,\frac{p+1}{2}})|E_{k_1,k_2,...,k_i}|^r \\ &= \mathbf{E}(M_1^r + M_2^r + ... + M^r_{\frac{p+1}{2}})|E_{k_1,k_2,...,k_i}|^r. \end{aligned}$$

Taking summation over all the intervals in H_i, since ratios are identically distributed, we have

$$\mathbf{E}\left(\sum_{E \in H_{i+1}} |E|^r \big| \mathcal{D}_i\right) = \sum_{E \in H_i} |E|^r \mathbf{E}(M_1^r + M_2^r + ... + M^r_{\frac{p+1}{2}}). \tag{5.2}$$

Thus, the unconditional expectation satisfy

$$\mathbf{E}\left(\sum_{E \in H_{i+1}} |E|^r\right) = \mathbf{E}\left(\sum_{E \in H_i} |E|^r\right) \mathbf{E}(M_1^r + M_2^r + ... + M^r_{\frac{p+1}{2}}). \tag{5.3}$$

As r is the solution of (5.1), (5.2) reduces to

$$\mathbf{E}\left(\sum_{E \in H_{i+1}} |E|^r \big| \mathcal{D}_i\right) = \sum_{E \in H_i} |E|^r. \tag{5.4}$$

(5.4) gives that the sequence given by

$$Y_i = \sum_{E \in H_i} |E|^r, \tag{5.5}$$

of random variables is a martingale with respect to H_i. Thus Y_i converges to a random variable Y with probability 1 as $i \to \infty$ satisfying $\mathbf{E}(Y) = \mathbf{E}(Y_0) = \mathbf{E}(1^r) = 1$. Particularly, $0 \leq Y < \infty$ with probability 1 and $Y = 0$ with probability q, where $q < 1$. But $Y = 0$ iff all the $(\frac{p+1}{2})$ sums $\sum_{E \in H_i \cap E_1} |E|^r$, $\sum_{E \in H_i \cap E_2} |E|^r$ and $\sum_{E \in H_i \cap E_{\frac{p+1}{2}}} |E|^r$ converge with probability 1 as $i \to \infty$ to 0, where $E_1, E_2, ..., E_{\frac{p+1}{2}}$

are closed intervals of H_1. Also, this happens with probability $q^{\frac{p+1}{2}}$ due to our statistical self-similar construction. Hence, $q = q^{\frac{p+1}{2}} \Rightarrow q = 0$. Thus, $0 < Y < \infty$ with probability 1. Thus, there exists random numbers $N_1, N_2, ..., N_{\frac{p+1}{2}}$ such that

$$0 < N_1 \leq N_2 \leq ... \leq N_{\frac{p-1}{2}} \leq Y_i = \sum_{E \in H_i} |E|^r \leq N_{\frac{p+1}{2}} < \infty \;\; \forall\, i. \tag{5.6}$$

We get

$$|E| \leq (\frac{p+1}{2})^{-i} \;\; for \;\; all \;\; E \in H_i.$$

So, $H^r_\delta(F_{\frac{p+1}{2}}) \leq \sum_{E \in H_i} |E|^r \leq N_{\frac{p+1}{2}}$ if $(\frac{p+1}{2})^{-i} < \delta \Rightarrow -i \log \frac{p+1}{2} < \log \delta$.
i.e. $i > \frac{-\log \delta}{\log \frac{p+1}{2}}$ which gives $H^r(F) \leq N_{\frac{p+1}{2}}$.
Thus, $dim_H F_{\frac{p+1}{2}} \leq r$, with probability 1.
To prove the reverse inequality, a random mass distribution μ on random set $F_{\frac{1}{p}}$ is introduced. Let us consider a random variable $\mu(E)$ for $E \in H_i$ as follows:

$$\mu(E) = \lim_{j \to \infty} \left\{ \sum |K|^r : \; K \in H_j \; and \; K \subset E \right\}$$

Also, from (5.5), this limit exists, where $0 < \mu(E) < \infty$ having probability 1. Further, if $E \in H_i$,

$$\mathbf{E}(\mu(E)|\mathcal{D}_i) = |E|^r. \tag{5.7}$$

Then, $\mu = \mu(E_{L_1}) + \mu(E_{L_2}) + ... + \mu(E_{L_{\frac{p+1}{2}}})$, i.e. μ is additive on i^{th} - level intervals for all i. By using Lemma 2.8, the mass distribution μ can be extended to a mass distribution with support contained in $\cap_{i=0}^{\infty} H_i = F_{\frac{1}{p}}$.
Now, we estimate the expectation of the t-energy of μ and fix $0 < t < r$. For $x_1, x_2, ..., x_{\frac{p+1}{2}} \in F_{\frac{p+1}{2}}$, let $x_1 \wedge x_2 \wedge ... \wedge x_{\frac{p+1}{2}}$ be an i^{th}-level common interval of $x_1, x_2, ..., x_{\frac{p+1}{2}}$ for some greatest integer i. The $(i+1)^{th}$-level sub-intervals $E_{L_1}, E_{L_2}, ..., E_{L_{\frac{p+1}{2}}}$ of an i^{th}-level interval E are set apart with a distance of at least $d|E|$ with $d = 1 - (\frac{p+1}{2}) a_{\frac{p+1}{2}}$, where $a_1, a_2, ..., a_{\frac{p+1}{2}}$ are constants such that $0 < a_1 \leq a_2 \leq ... \leq a_{\frac{p+1}{2}} < \frac{p+1}{2}$.
Thus,

$$\underbrace{\int \int ... \int_{x_1 \wedge x_2 \wedge ... \wedge x_{\frac{p+1}{2}} = E}}_{\frac{p+1}{2}} (|x_1 - x_2|^{-t} + |x_2 - x_3|^{-t} + ... + |x_{\frac{p-1}{2}} - x_{\frac{p+1}{2}}|^{-t}) d\mu(x_1) d\mu(x_2) ... d\mu(x_{\frac{p+1}{2}})$$

$$= \frac{p+1}{2} \int_{x_1 \in E_{L_1}} \int_{x_2 \in E_{L_2}} |x_1 - x_2|^{-t} d\mu(x_1) d\mu(x_2) + \cdots$$
$$+ \frac{p+1}{2} \int_{x_{\frac{p-1}{2}} \in E_{L_{\frac{p-1}{2}}}} \int_{x_{\frac{p+1}{2}} \in E_{L_{\frac{p+1}{2}}}} |(x_{\frac{p-1}{2}} - x_{\frac{p+1}{2}}|^{-t} d\mu(x_{\frac{p-1}{2}}) d\mu(x_{\frac{p+1}{2}})$$

$$\leq \frac{p+1}{2} d^{-t} |E|^{-t} \mu(E_{L_1}) \mu(E_{L_2}) + \frac{p+1}{2} d^{-t} |E|^{-t} \mu(E_{L_2}) \mu(E_{L_3}) + \cdots + \frac{p+1}{2} d^{-t} |E|^{-t} \mu(E_{L_{\frac{p-1}{2}}}) \mu(E_{L_{\frac{p+1}{2}}})$$

$$= \frac{p-1}{2} d^{-t}|E|^{-t}[\mu(E_{L_2})\{\mu(E_{L_1})+\mu(E_{L_3})\}+\mu(E_{L_4})\{\mu(E_{L_3})+\mu(E_{L_5})\}+\cdots+\mu(E_{L_{\frac{p-1}{2}}})\{\mu(E_{L_{\frac{p-3}{2}}})+\mu(E_{L_{\frac{p+1}{2}}})\}]$$

If $I \in H_i$,

$$\mathbf{E}[\underbrace{\int\int\cdots\int_{x_1\wedge x_2\wedge\ldots\wedge x_{\frac{p+1}{2}}=E}}_{\frac{p+1}{2}}(|x_1-x_2|^{-t}+|x_2-x_3|^{-t}+\ldots+|x_{\frac{p-1}{2}}-x_{\frac{p+1}{2}}|^{-t})d\mu(x_1)d\mu(x_2)\ldots d\mu(x_{\frac{p+1}{2}})|\mathcal{D}_{i+1}]$$

$$\leq \frac{p+1}{2} d^{-t}|E|^{-t}\{\mathbf{E}(\mu(E_{L_1})|\mathcal{D}_{i+1})\mathbf{E}(\mu(E_{L_2})|\mathcal{D}_{i+1}) + \mathbf{E}(\mu(E_{L_2})|\mathcal{D}_{i+1})\mathbf{E}(\mu(E_{L_3})|\mathcal{D}_{i+1}) + \cdots$$
$$+\mathbf{E}(\mu(E_{L_{\frac{p-1}{2}}})|\mathcal{D}_{i+1})\mathbf{E}(\mu(E_{L_{\frac{p+1}{2}}})|\mathcal{D}_{i+1})\}$$

$$\leq \frac{p+1}{2} d^{-t}|E|^{-t}\{|E_{L_1}|^r|E_{L_2}|^r + |E_{L_2}|^r|E_{L_3}|^r + \cdots + |E_{L_{\frac{p-1}{2}}}|^r|E_{L_{\frac{p+1}{2}}}|^r\}$$

$$= \frac{p+1}{2} d^{-t}|E|^{-t}\{|E_{L_2}|^r(|E_{L_1}|^r + |E_{L_3}|^r) + |E_{L_4}|^r(|E_{L_3}|^r + |E_{L_5}|^r) + \cdots + |E_{L_{\frac{p-1}{2}}}|^r(|E_{L_{\frac{p-3}{2}}}|^r + |E_{L_{\frac{p+1}{2}}}|^r)\}$$

$$= \frac{p+1}{2} d^{-t}|E|^{-t}\left(\frac{p-1}{2}|E|^{2r}\right)$$

$$= \frac{p^2-1}{4} d^{-t}|E|^{2r-t}.$$

Using (5.7), since expectation is independent from $\mathcal{D}_i$ and using unconditional property of expectation, we have the inequality

$$\mathbf{E}\{\underbrace{\int\int\cdots\int_{x_1\wedge x_2\wedge\ldots\wedge x_{\frac{p+1}{2}}=E}}_{\frac{p+1}{2}}(|x_1-x_2|^{-t}+|x_2-x_3|^{-t}+\ldots+|x_{\frac{p-1}{2}}-x_{\frac{p+1}{2}}|^{-t})d\mu(x_1)d\mu(x_2)\ldots d\mu(x_{\frac{p+1}{2}})\}$$

$$\leq \frac{p^2-1}{4} d^{-t}\mathbf{E}(|E|^{2r-t}).$$

Taking summation over $E \in H_i$,

$$\mathbf{E}\{\underbrace{\int\int\cdots\int_{x_1\wedge x_2\wedge\ldots\wedge x_{\frac{p+1}{2}}=E}}_{\frac{p+1}{2}}(|x_1-x_2|^{-t}+|x_2-x_3|^{-t}+\ldots+|x_{\frac{p-1}{2}}-x_{\frac{p+1}{2}}|^{-t})d\mu(x_1)d\mu(x_2)\ldots d\mu(x_{\frac{p+1}{2}})\}$$

$$\leq \frac{p^2-1}{4} d^{-t}\mathbf{E}\left(\sum_{E\in H_i}|E|^{2r-t}\right)$$

$$= \frac{p^2-1}{4} d^{-t}\delta^i,$$

where $\delta = \mathbf{E}(M_1^{2r-t} + M_2^{2r-t} + ... + M_{\frac{p+1}{2}}^{2r-t})$. Then, by using repeatedly (5.3), we have $\delta < 1$. Then,

$$\mathbf{E}\{\underbrace{\int\int ... \int_{x_1 \wedge x_2 \wedge ... \wedge x_{\frac{p+1}{2}} = E}}_{\frac{p+1}{2}} (|x_1 - x_2|^{-t} + |x_2 - x_3|^{-t} + ... + |x_{\frac{p-1}{2}} - x_{\frac{p+1}{2}}|^{-t}) d\mu(x_1) d\mu(x_2) ... d\mu(x_{\frac{p+1}{2}})\}$$

$$= \mathbf{E}\{\sum_{i=0}^{\infty} \sum_{E \in H_i} \underbrace{\int\int ... \int_{x_1 \wedge x_2 \wedge ... \wedge x_{\frac{p+1}{2}} = E}}_{\frac{p+1}{2}} (|x_1 - x_2|^{-t} + |x_2 - x_3|^{-t} + ... + |x_{\frac{p-1}{2}} - x_{\frac{p+1}{2}}|^{-t}) d\mu(x_1) d\mu(x_2) ... d\mu(x_{\frac{p+1}{2}})\}$$

$$\leq \frac{p^2 - 1}{4} d^{-t} \sum_{i=0}^{\infty} \sum_{E \in H_i} \delta^i < \infty.$$

Thus, with probability 1, μ has finite t-energy. Also, since $0 < \mu(F_{\frac{1}{p}}) = \mu([0, 1])$ and has probability 1, therefore, using Lemma 2.15, we have $dim_H F_{\frac{1}{p}} \geq t$. This gives $dim_H F_{\frac{1}{p}} \geq r$ with probability 1. Hence, Hausdorff dimension of random Cantor one p^{th} set is r, i.e., $dim_H F_{\frac{1}{p}} = r$ with probability 1. □

Corollary 5.2. *If the random ratios $M_1, M_2, ..., M_{\frac{p+1}{2}}$ are constants instead of variables, then* (5.1) *reduces to*

$$\mathbf{E}(M_1^r + M_2^r + ... + M_{\frac{p+1}{2}}^r) = M_1^r + M_2^r + ... + M_{\frac{p+1}{2}}^r = 1, \tag{5.8}$$

which is similarity dimension formula for a self-similar fractal.

Corollary 5.3. *For $p = 5$, the random cantor one 5^{th} set $F_{\frac{1}{5}}$ given by (3.1) satisfy $dim_H F_{\frac{1}{5}} = r$, where r is the solution of the expectation equation*

$$\mathbf{E}(M_1^r + M_2^r + M_3^r) = 1,$$

with probability 1.

Corollary 5.4. *Let* **P** *be the probability measure defined on a family of subsets of ω in such a way that the ratios $M_{k_1,k_2,...,k_i} = \frac{|E_{k_1,k_2,...k_i}|}{|E_{k_1,k_2,...k_{i-1}}|}$, with $k_j = 1$ or 2 or... or $\frac{p+1}{2}, 1 \leq i \leq \frac{p+1}{2}$ are random variables. We take V for random number of positive ratios $M_1, M_2, ..., M_{\frac{p+1}{2}}$. If q is the probability of being empty of random cantor set $F_{\frac{1}{p}}$ described above, then the polynomial equation*

$$h(t) = \sum_{i=0}^{\frac{p+1}{2}} \mathbf{P}(V = i) t^i = t, \tag{5.9}$$

has $t = q$ as its smallest non - negative solution.

Proof. We prove this corollary by combining the Theorem 5.1 and Lemma 2.16. We see that if there is positive probability that $V = 0$, then there is a positive probability that $H_1 = \phi$ and therefore, we have $F_{\frac{1}{p}} = \phi$. This emptiness happens if each of the component sets in H_1 becomes empty. By the statistical self similarity of the construction, if the probability of this happening is q, then $q = h(q)$. Moreover, if q is any non negative solution of (5.9), then by induction $q \geq \mathbf{P}(H_i = \phi)\ \forall\ i$. This happened only when $i = 0$ and if it holds for some i, then as h is increasing, $q = h(q) \geq h(\mathbf{P}(H_i = \phi)) = \mathbf{P}(H_{i+1} = \phi)$. If $F_{\frac{1}{p}} = \phi$, then $H_i = \phi$ for some i, so $q \geq \mathbf{P}(F_{\frac{1}{p}} = \phi)$, thus the probability of being empty of random Cantor set is the least non-negative solution of $q = h(q)$. □

Before moving on the next result, let us consider that the interval [0, 1] is divided into p sub intervals each of length $\frac{1}{p}$, $p = 3, 5, 7, ...$. Now, we construct the random Cantor one p^{th} set $F_{\frac{1}{p}}$ by tossing a unbiased coin and including the interval if head appears on the coin. Let u be the probability of getting head.

Theorem 5.5. *The probability of an empty random Cantor one p^{th} set which is constructed by tossing a unbiased coin and including the interval if head appears on the coin is 1. i.e.,*

$$\mathbf{P}(F_{\frac{1}{p}} = \phi) = 1.$$

Proof. The random Cantor one p^{th} set $F_{\frac{1}{p}}$ is empty i.e., $F_{\frac{1}{p}} = \phi$ if following $(\frac{p+1}{2} + 1)$ events happen :
A_0 : None of the intervals $[0, \frac{1}{p}], [\frac{2}{p}, \frac{3}{p}], \cdots, [\frac{p-1}{p}, 1]$ is included.
A_1 : Exactly one interval is included and $F_{\frac{1}{p}}$ is eventually empty below that interval.
A_2 : Exactly two of them are included and $F_{\frac{1}{p}}$ is eventually empty below both of them.

$$\cdots$$
$$\cdots$$
$$\cdots$$

$A_{\frac{p+1}{2}}$: All $(\frac{p+1}{2})$ intervals are included and $F_{\frac{1}{p}}$ is eventually empty below all of them.
As u is the probability of getting head and random Cantor one p^{th} set is constructed by including the intervals if coin shows head. Let v be the probability of being empty of random Cantor one p^{th} set. i.e., $\mathbf{P}(F_{\frac{1}{p}} = \phi) = v$.
The above events have following probabilities:

$$\mathbf{P}(A_0) = (1-u)^{\frac{p+1}{2}},$$

$$\mathbf{P}(A_1) = \binom{\frac{p+1}{2}}{1} u^1 (1-u)^{(\frac{p+1}{2}-1)} v,$$

$$\mathbf{P}(A_2) = \binom{\frac{p+1}{2}}{2} u^2 (1-u)^{(\frac{p+1}{2}-2)} v^2,$$

$$\cdots$$
$$\cdots$$
$$\cdots$$

$$\mathbf{P}(A_{\frac{p+1}{2}}) = \binom{\frac{p+1}{2}}{\frac{p+1}{2}} u^{\frac{p+1}{2}} (1-u)^{(\frac{p+1}{2}-\frac{p+1}{2})} v^{\frac{p+1}{2}},$$

i.e.,

$$\mathbf{P}(A_{\frac{p+1}{2}}) = \binom{\frac{p+1}{2}}{n} u^n (1-u)^{(\frac{p+1}{2}-n)} v^n;\ n = 0, 1, 2, ..., \frac{p+1}{2},$$

where p is an odd number greater than 1.
From Corollary 5.4, $v = \mathbf{P}(F_{\frac{1}{p}} = \phi)$ is the solution of the equation

$$t = (1-u)^{\frac{p+1}{2}} + \binom{\frac{p+1}{2}}{1} u^1 (1-u)^{(\frac{p+1}{2}-1)} t + \cdots + u^{\frac{p+1}{2}} t^{\frac{p+1}{2}}$$

or

$$t = \sum_{n=0}^{\frac{p+1}{2}} \binom{\frac{p+1}{2}}{n} u^n (1-u)^{(\frac{p+1}{2}-n)} t^n. \tag{5.10}$$

Now, we find the nature of solutions of (5.10).
For $p = 3$, solutions of (5.10) are 1 and $(\frac{1-u}{u})^2$. In this case, $(\frac{1-u}{u})^2 > 1$ for some $u \in [0, 1]$ which is not possible since v a probability. i.e., $0 \le v \le 1$. Only possible solution is 1.
Now, for $p = 5$, solutions of (5.10) are 1, $\frac{u(2u-3)+\sqrt{4u-3u^2}}{2u^2}$ and $\frac{u(2u-3)-\sqrt{4u-3u^2}}{2u^2}$. For $u < \frac{1}{2}$, $\frac{u(2u-3)+\sqrt{4u-3u^2}}{2u^2} > 1$ and $\frac{u(2u-3)-\sqrt{4u-3u^2}}{2u^2} < -1$ Thus, in this case also, the only possible solution is 1.
For $p = 7$ and $u = \frac{1}{5}$; we obtain the real roots of (5.10) as $1,\ 1.755$. Again, the only possible solution is 1. For $p = 9$ and $u = \frac{1}{6}$; we obtain the real roots of (5.10) as $1, -15.362$ and 1.548. Thus, the only possible solution is 1.
Hence, in general, we can say that the only possible solution of (5.10) is 1 for any p. This implies that

$$\mathbf{P}(F_{\frac{1}{p}} = \phi) = 1.$$

Also, we see $\mathbf{P}(F_{\frac{1}{p}} = \phi) \to 0$ as $u \to 1$. □

6. Hausdorff dimension of random Cantor one p^{th} set

Since any empty random set is dimensionless. So, we calculate the Hausdorff dimension of a non empty random Cantor one p^{th} set. We divide the unit interval $[0, 1]$ into p equal sub-intervals and construct random Cantor one p^{th} set by including intervals randomly. Here, we construct our random Cantor one p^{th} set by tossing a unbiased coin and including the interval if head appears on the coin.

Theorem 6.1. *The Hausdorff dimension r of a nonempty random Cantor one p^{th} set $F_{\frac{1}{p}}$ which is constructed by tossing a unbiased coin and including the interval if head appears on the coin, given by $r = \frac{\log(\frac{p+1}{2}u)}{\log p}$. i.e.,*

$$dim_H(F_{\frac{1}{p}}) = \frac{\log(\frac{p+1}{2}u)}{\log p}, \tag{6.1}$$

where u is the probability of getting head.

Proof. Let u be the probability of getting head. Each interval has length $\frac{1}{p}$ i.e. constant. To construct random Cantor one p^{th} set, following $\frac{p+1}{2}$ events happen :
A_1 : Exactly one interval from $[0, \frac{1}{p}], [\frac{2}{p}, \frac{3}{p}], \cdots, [\frac{p-1}{p}, 1]$ is included.
A_2 : Exactly two of them are included.

$\cdots$

$\cdots$

...

$A_{\frac{p+1}{2}}$: All $(\frac{p+1}{2})$ intervals are included.
As u is the probability of getting head and random Cantor one p^{th} set is constructed by including the intervals if coin shows head. The above events have following probabilities:

$$\mathbf{P}(A_1) = \binom{\frac{p+1}{2}}{1} u^1 (1-u)^{(\frac{p+1}{2}-1)},$$

$$\mathbf{P}(A_2) = \binom{\frac{p+1}{2}}{2} u^2 (1-u)^{(\frac{p+1}{2}-2)},$$

...

...

...

$$\mathbf{P}(A_{\frac{p+1}{2}}) = \binom{\frac{p+1}{2}}{\frac{p+1}{2}} u^{\frac{p+1}{2}} (1-u)^{(\frac{p+1}{2}-\frac{p+1}{2})},$$

i.e.,

$$\mathbf{P}(A_{\frac{p+1}{2}}) = \binom{\frac{p+1}{2}}{n} u^n (1-u)^{(\frac{p+1}{2}-n)};\ n = 1, 2, ..., \frac{p+1}{2}.$$

Let r be the Hausdorff dimension of random Cantor one p^{th} set. Then by Theorems 5.1 and 5.5, r satisfy the equation

$$\mathbf{E}(A_1^r + ... + A^r_{(\frac{p+1}{2})}) = 1, \tag{6.2}$$

where $\{A_n,\ n = 1, 2, ..., \frac{p+1}{2}\}$ are the events. Using expectation properties and Corollary 5.4, (6.2) reduces to

$$p^{-r}\mathbf{P}(A_1) + 2p^{-r}\mathbf{P}(A_2) + 3p^{-r}\mathbf{P}(A_3) + \cdots + (\frac{p+1}{2})p^{-r}\mathbf{P}(A_{\frac{p+1}{2}}) = 1.$$

This implies

$$p^{-r}\{\binom{\frac{p+1}{2}}{1} u^1 (1-u)^{\frac{p+1}{2}-1} + 2\binom{\frac{p+1}{2}}{2} u^2 (1-u)^{(\frac{p+1}{2}-2)} + 3.\binom{\frac{p+1}{2}}{3} u^3 (1-u)^{(\frac{p+1}{2}-3)}$$

$$+ \cdots + (\frac{p+1}{2}) u^{\frac{p+1}{2}} (1-u)^0\} = 1. \tag{6.3}$$

(6.3) reduces to

$$p^{-r}\frac{(p+1)}{2}u\{(1-u)^{\frac{p+1}{2}-1} + \frac{(p-1)}{2}u^1(1-u)^{(\frac{p+1}{2}-2)} + \frac{(p-1)(p-3)}{2^3}u^2(1-u)^{(\frac{p+1}{2}-3)} + \cdots + u^{\frac{p-1}{2}}\} = 1.$$

By solving this, we have

$$p^{-r}\frac{(p+1)}{2}u = 1$$

$$\Rightarrow \frac{(p+1)}{2}u = p^r$$

$$\Rightarrow r\log p = \log(\frac{(p+1)}{2}u)$$

$$\Rightarrow r = \frac{\log(\frac{(p+1)}{2}u)}{\log p}. \tag{6.4}$$

Hence, Hausdorff dimension r of a random Cantor one p^{th} set $F_{\frac{1}{p}}$ where $p = 3, 5, 7, ...$. is given by (6.4). □

6.1. Hausdorff dimension of random Cantor set

Put $p = 3$ in (6.3), we have

$$3^{-r}\left\{2u(1-u) + 2u^2\right\} = 1$$

$$\Rightarrow 3^{-r}2u = 1$$

$$\Rightarrow r = \frac{\log(2u)}{\log 3}.$$

For $u = \frac{2}{3}$, we have $r = 0.2619$ which is Hausdorff dimension of a random Cantor set.
To obtain the Hausdorff dimension of classical Cantor set, we take $u = 1$. Then $r = \frac{\log(2)}{\log 3} = 0.6309$.

6.2. Hausdorff dimension of random Cantor one 5^{th} set

Substituting $p = 5$ in (6.3), we have

$$5^{-r}\left\{3u(1-u)^2 + 6u^2(1-u) + 3u^3\right\} = 1$$

$$5^{-r}3u\left\{(1-u)^2 + 2u(1-u) + u^2\right\} = 1$$

$$\Rightarrow 5^{-r}3u = 1$$

$$\Rightarrow r = \frac{\log(3u)}{\log 5}.$$

Now, for $u = \frac{3}{5}$, $r = 0.3652$. For $u = 1$, we have $r = \frac{\log(3)}{\log 5} = 0.6826$ which is Hausdorff dimension of Cantor one 5^{th} set.

Remark 6.2. The Subsections 6.1 and 6.2 show that the Hausdorff dimension of a random Cantor one p^{th} set is less than that of the Hausdorff dimension of a Cantor one p^{th} set.

7. Conclusions

In this paper, we construct random Cantor one p^{th} sets. Some properties, results and Hausdorff dimension of random Cantor one p^{th} sets have been obtained. The following conclusions are drawn out from our paper:

1. We generalize the random Cantor set and construct random Cantor one p^{th} set.
2. Similar like Cantor one p^{th} set, the random Cantor one p^{th} set is connected, nowhere dense, Borel and measurable set.

3. Theorem 1 may be used to obtain the Hausdorff dimension for random fractals, i.e., random Seirpinski Gasket, random Koch Curve etc.

4. An empty random Cantor one p^{th} set has probability 1.

5. We have obtained a general formula $\frac{\log(\frac{p+1}{2}u)}{\log p}$ to compute the Hausdorff dimension of random Cantor one p^{th} set, where u is the probability of getting head (see, Section 6).

6. Hausdorff dimension of a random Cantor one p^{th} set is less than that of Hausdorff dimensions of the corresponding Cantor one p^{th} set.

Conflict of interest

The authors declare no conflict of interest in this paper.

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