Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(4): 3035–3055.
DOI: 10.3934/math.2020197
Received: 29 October 2019
Accepted: 16 March 2020
Published: 20 March 2020

Research article

A comparison study of two modified analytical approach for the solution of nonlinear fractional shallow water equations in fluid flow

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Abstract: In this study, a comparison between the modified homotopy analysis transform method (MHATM) and residual power series method (RPSM) have been given for solving time-fractional coupled shallow water equations (SWEs). The time-fractional coupled SWEs are a system of PDEs that describe the flow below a pressure surface in a fluid is considered. Rigorous convergence analysis and error estimated have been exhibited for both the featured methods. The results obtained by MHATM and RPSM are then compared with well-known exact solutions. To show the effectiveness and advantage of the featured techniques the numerical simulation of coupled SWEs has been represented graphically with tabulated data. However, the results indicate that MHATM provides more accurate value than RPSM for solving fractional coupled SWEs.

Keywords: fractional shallow water equation; fractional power series; RPSM; HATM; homotopy polynomials; optimal value
Mathematics Subject Classification: 26A33, 34A08, 46M22

1. Introduction

Recent research focused on the applications of the Fractional Calculus (FC) in various fields which includes the modeling and analysis of complex real-world problems. Very recently, numerous papers appeared with the various type of applications of fractional calculus [1–22]. Further, computational aspects of various problems can be found in [23–26].

The SWEs are a system of PDEs that illustrate the flow below a pressure surface in a fluid, the motion of water bodies and flow in vertically well-mixed water bodies. The general characteristics of shallow water flows are that the vertical dimension is much smaller than the typical horizontal scale, the fluid is homogeneous and incompressible, the flow is steady, and the pressure distribution is hydrostatic. The SWEs can be utilized to model the hydrodynamics of lakes, ocean currents, tidal flats, coastal zones and to study dredging feasibility. Differently, it can also be used to investigate several physical phenomena [27]. Many geophysical flows are modelled by variants of the SWEs.

The popular form of SWEs can be derived from the Benny system.

The Benny equations are characterized as [28]

$$\begin{aligned} &\frac{\partial u(x,y,t)}{\partial t} + u(x,y,t)\frac{\partial u(x,y,t)}{\partial t} - \frac{\partial u(x,y,t)}{\partial y}\int_0^y \frac{\partial u(x,\tau,t)}{\partial y}d\tau + \frac{\partial h(x,t)}{\partial x} = 0,\\ &\frac{\partial h(x,t)}{\partial t} + \frac{\partial}{\partial x}\int_0^y \frac{\partial u(x,\tau,t)}{\partial y}d\tau = 0, \end{aligned} \tag{1.1}$$

where $h(x,t)$, $u(x,y,t)$ denotes the free surface and the horizontal velocity component respectively and y is the rigid bottom. If the horizontal velocity component u is independent of the height h the (1.1) reduced to the equation system in the classical water theory corresponding to the case of irrational motion. The Equivalent wave motion is the coupled SWEs as

$$\begin{aligned} &D_t h(x,t) + u(x,t)D_x h(x,t) + h(x,t)D_x u(x,t) = 0,\\ &D_t u(x,t) + u(x,t)D_x u(x,t) + D_x h(x,t) = 0, \end{aligned} \tag{1.2}$$

subject to the ICs

$$h(x,0) = f(x), u(x,0) = g(x). \tag{1.3}$$

In this article, let us suppose the time-fractional order coupled SWEs of the form

$$\begin{aligned} &D_t^{\lambda} h(x,t) + u(x,t)D_x h(x,t) + h(x,t)D_x u(x,t) = 0,\\ &D_t^{\beta} u(x,t) + u(x,t)D_x u(x,t) + D_x h(x,t) = 0. \end{aligned} \tag{1.4}$$

subject to the initial conditions

$$h(x,0) = f(x), u(x,0) = g(x). \tag{1.5}$$

Here $0 < \lambda \leq 1$ and $0 < \beta \leq 1$ are the parameters representing the orders of the fractional time derivative. The fractional derivative is considered in the Caputo sense [29, 30].

Recently, Kumar [31] gave the solutions of time-fractional nonlinear SWEs by using the HPM. The essential target of this work is to present a comparative study between the HATM [32–34] with modification and RPSM [35–39] through the solution of fractional SWEs (1.4). The MHATM is a combination of LTM and the HAM [40–44] with homotopy polynomials [45]. Where as RPSM is an analytical method based on power series expansion without linearization, perturbation or discretization. The benefits of the RSPM as compared to the other classical power series techniques is that the RPSM does not require any conversion while switching from the low-order to the higher-order. Also, it can switch from simple linearity to complex nonlinearity. This means that the RSPM can apply directly to the problems by considering a suitable initial guess approximation.

2. RPSM for time-fractional SWEs

Let us assume the time-fractional SWEs (1.4) with the ICs

$$h(x,0)=\frac{1}{9}\left(x^2-2x+1\right) \quad \text{and} \quad u(x,0)=\frac{2}{3}(1-x). \tag{2.1}$$

In this case, the exact solution of Eq (1.4) for standard motion, i.e. $\lambda=1$ and $\beta=1$, is given by [28]

$$h(x,t)=\frac{(x-1)^2}{9(t-1)^2} \quad \text{and} \quad u(x,t)=\frac{2(x-1)}{3(t-1)}. \tag{2.2}$$

Further, starting with the ICs

$$\begin{aligned} f_0(x)&=h(x,0)=\tfrac{1}{9}\left(x^2-2x+1\right),\\ g_0(x)&=u(x,0)=u(x,t)=\tfrac{2(x-1)}{3(t-1)}, \end{aligned} \tag{2.3}$$

and with the k_1^{th} and k_2^{th} residual functions for SWEs as

$$\begin{aligned} \operatorname{Res}^h_{k_1}(x,t)&=\tfrac{\partial^\lambda h_{k_1}}{\partial t^\lambda}+u_{k_1}\tfrac{\partial h_{k_1}}{\partial x}+h_{k_1}\tfrac{\partial u_{k_1}}{\partial x}, \quad k_1=1,2,3,...,\\ \operatorname{Res}^u_{k_2}(x,t)&=\tfrac{\partial^\beta u_{k_2}}{\partial t^\beta}+u_{k_1}\tfrac{\partial u_{k_1}}{\partial x}+\tfrac{\partial h_{k_1}}{\partial x}, \quad k_2=1,2,3,.... \end{aligned} \tag{2.4}$$

In addition, taking into account those forms of $f_0(x)$ and $g_0(x)$ and using (2.4), the k^{th} truncated series of the multiple FPS expansion of $h(x,t)$ and $u(x,t)$ at $t=0$ should be

$$\begin{aligned} h_k(x,t)&=f(x)+f_1(x)\frac{t^\lambda}{\Gamma(1+\lambda)}+\sum_{n=2}^{k}f_n(x)\frac{t^{n\lambda}}{\Gamma(1+n\lambda)}, k=2,3,4,...,\\ u_k(x,t)&=g(x)+g_1(x)\frac{t^\beta}{\Gamma(1+\beta)}+\sum_{n=2}^{k}g_n(x)\frac{t^{n\beta}}{\Gamma(1+n\beta)}, k=2,3,4,..., \end{aligned} \tag{2.5}$$

respectively. To ascertain the first unknown coefficients, $f_1(x)$ and $g_1(x)$, in the expansion of (2.5), substitute the 1st truncated series $h_1(x)$ and $u_1(x)$ into the 1-st residual functions given in (2.4), to obtain

$$\begin{aligned} \operatorname{Res}^h_1(x,t)&=\tfrac{\partial^\lambda h_1}{\partial t^\lambda}+u_1\tfrac{\partial h_1}{\partial x}+h_1\tfrac{\partial u_1}{\partial x},\\ \operatorname{Res}^u_1(x,t)&=\tfrac{\partial^\beta u_1}{\partial t^\beta}+u_1\tfrac{\partial u_1}{\partial x}+\tfrac{\partial h_1}{\partial x}. \end{aligned} \tag{2.6}$$

But since $h_1(x,t)=f(x)+f_1(x)\frac{t^\lambda}{\Gamma(1+\lambda)}$ and $u_1(x,t)=g(x)+g_1(x)\frac{t^\beta}{\Gamma(1+\beta)}$. Then Eq (2.6) leads to the following result:

$$\begin{aligned} \operatorname{Res}^h_1(x,t)=f_1(x)&+\Big(g(x)+g_1(x)\frac{t^\beta}{\Gamma(1+\beta)}\Big)\frac{\partial\Big(f(x)+f_1(x)\frac{t^\lambda}{\Gamma(1+\lambda)}\Big)}{\partial x}\\ &+\Big(f(x)+f_1(x)\frac{t^\lambda}{\Gamma(1+\lambda)}\Big)\frac{\partial\Big(g(x)+g_1(x)\frac{t^\beta}{\Gamma(1+\beta)}\Big)}{\partial x},\\ \operatorname{Res}^u_1(x,t)=g_1(x)&+\Big(g(x)+g_1(x)\frac{t^\beta}{\Gamma(1+\beta)}\Big)\frac{\partial\Big(g(x)+g_1(x)\frac{t^\beta}{\Gamma(1+\beta)}\Big)}{\partial x}\\ &+\Big(f(x)+f_1(x)\frac{t^\lambda}{\Gamma(1+\lambda)}\Big). \end{aligned} \tag{2.7}$$

Now, based on the result of (2.5) for $n = 1$, the substitution of $t = 0$ through (2.7) yields

$$\begin{aligned} f_1(x) &= \tfrac{2}{9} - \tfrac{4x}{9} + \tfrac{2x^2}{9}, \\ g_1(x) &= \tfrac{2}{3} - \tfrac{2x}{3}. \end{aligned} \tag{2.8}$$

Therefore, the 1st RPS approximate solution of (1.4) can be represented as

$$\begin{aligned} h_1(x,t) &= \frac{1}{9}\left(x^2 - 2x + 1\right) + \left(\frac{2}{9} - \frac{4x}{9} + \frac{2x^2}{9}\right)\frac{t^\lambda}{\Gamma(1+\lambda)}, \\ u_1(x,t) &= \frac{2}{3}(1-x) - \left(\frac{2}{3} - \frac{2x}{3}\right)\frac{t^\lambda}{\Gamma(1+\lambda)}. \end{aligned} \tag{2.9}$$

In a similar way, the second unknown coefficients $f_2(x)$ and $g_2(x)$ can be obtained by substituting the 2nd truncated series $h_2(x,t) = f(x) + f_1(x)\frac{t^\lambda}{\Gamma(1+\lambda)} + f_2(x)\frac{t^{2\lambda}}{\Gamma(1+2\lambda)}$ and $u_2(x,t) = g(x) + g_1(x)\frac{t^\beta}{\Gamma(1+\beta)} + g_2(x)\frac{t^{2\beta}}{\Gamma(1+2\beta)}$ of (2.5) into 2-nd residual functions $\text{Res}_2^h(x,t) = \frac{\partial^\lambda h_2}{\partial t^\lambda} + u_2\frac{\partial h_2}{\partial x} + h_2\frac{\partial u_2}{\partial x}$ and $\text{Res}_2^u(x,t) = \frac{\partial^\beta u_2}{\partial t^\beta} + u_2\frac{\partial u_2}{\partial x} + \frac{\partial h_2}{\partial x}$ of (2.4) to get the following discretized form:

$$\begin{aligned} \text{Res}_2^h(x,t) = {} & f_1(x) + g(x)f'(x) + g'(x)f(x) + \left(f_2(x) + gf_1'(x) + f_1g'(x)\right)\frac{t^\lambda}{\Gamma(1+\lambda)} \\ & + \left(g_1f'(x) + f_1g_1'(x)\right)\frac{t^\beta}{\Gamma(1+\beta)} + \left(g'(x)f_2'(x) + f_2(x)g'(x)\right)\frac{t^{2\lambda}}{\Gamma(1+2\lambda)} \\ & + \left(g_2(x)f'(x) + f(x)g_2'\right)\frac{t^{2\beta}}{\Gamma(1+2\beta)} + \left(g_1f_1' + f_1g_1'(x)\right)\frac{t^{\lambda+\beta}}{\Gamma(1+\lambda)\Gamma(1+\beta)} \\ & + \left(g_1f_2' + f_2g_1'(x)\right)\frac{t^{2\lambda+\beta}}{\Gamma(1+2\lambda)\Gamma(1+\beta)} + \left(g_2f_1' + f_1g_2'(x)\right)\frac{t^{\lambda+2\beta}}{\Gamma(1+\lambda)\Gamma(1+2\beta)} \\ & + \left(g_2f_2' + f_2g_2'(x)\right)\frac{t^{2\lambda+2\beta}}{\Gamma(1+2\lambda)\Gamma(1+2\beta)} \\ \text{Res}_2^u(x,t) = {} & g_1(x) + g'(x)g(x) + f'(x) + \left(g_2(x) + gg_1'(x) + g_1g''(x)\right)\frac{t^\beta}{\Gamma(1+\beta)} \\ & + f_1'(x)\frac{t^\lambda}{1+\lambda} + f_2'(x)\frac{t^{2\lambda}}{1+2\lambda} + \left(gg_2' + g_1g_1'(x) + g_2g'(x)\right)\frac{t^{2\beta}}{\Gamma(1+2\beta)} \\ & + \left(g_1g_2' + g_2g_1'(x)\right)\frac{t^{3\beta}}{\Gamma(1+3\beta)} + \left(g_2g_2'(x)\right)\frac{t^{4\beta}}{\Gamma(1+4\beta)} \\ & + \left(2f_1(x)g_2(x) + 2f_2(x)g_1(x)\right)'\frac{t^{3\beta}}{\Gamma(1+3\beta)} + 2f_2(x)g_2(x)\frac{t^{4\beta}}{\Gamma(1+4\beta)}. \end{aligned} \tag{2.10}$$

Now, operating D_t^λ one time on both sides of (2.7) gives the $\lambda-$ th time fractional derivative of $\text{Res}_2^h(x,t)$ and $\text{Res}_2^u(x,t)$. Then, from (2.5) when $n = 2$, substituting $t = 0$ through (2.10) yields

$$\begin{aligned} f_2(x) &= \tfrac{2}{3} - \tfrac{4x}{3} + \tfrac{2x^2}{3}, \\ g_2(x) &= \tfrac{4}{3} - \tfrac{4x}{3}. \end{aligned} \tag{2.11}$$

Hence, the 2nd RPS approximate solution of (1.4) of the form

$$
\begin{aligned}
h_2(x,t) &= \frac{1}{9}\left(x^2-2x+1\right)+\Big(\frac{2}{9}-\frac{4x}{9}+\frac{2x^2}{9}\Big)\frac{t^{\lambda}}{\Gamma(1+\lambda)} \\
&+\left(\frac{2}{3}-\frac{4x}{3}+\frac{2x^2}{3}\right)\frac{t^{2\lambda}}{\Gamma(1+2\lambda)}, \\
u_2(x,t) &= \frac{2}{3}(1-x)-\left(\frac{2}{3}-\frac{2x}{3}\right)\frac{t^{\beta}}{\Gamma(1+\beta)}+\left(\frac{4}{3}-\frac{4x}{3}\right)\frac{t^{2\beta}}{\Gamma(1+2\beta)}.
\end{aligned}
\tag{2.12}
$$

By applying the same steps for $n = 3$, we get following form of $f_3(x)$ and $g_3(x)$

$$
\begin{aligned}
f_3(x) &= \tfrac{20}{9}-\tfrac{40x}{9}+\tfrac{20x^2}{9}, \\
g_3(x) &= \tfrac{32}{9}-\tfrac{32x}{9}.
\end{aligned}
\tag{2.13}
$$

In fact, from (2.13) and based on Preceding results for $f_0(x)$, $g_0(x)$, $f_1(x)$, $g_1(x)$, $f_2(x)$ and $g_2(x)$ the 3rd RPS approximation solution of Eq (1.4) will be ready to summarized as follows

$$
\begin{aligned}
h_3(x,t) &= \frac{1}{9}\left(x^2-2x+1\right)+\Big(\frac{2}{9}-\frac{4x}{9}+\frac{2x^2}{9}\Big)\frac{t^{\lambda}}{\Gamma(1+\lambda)} \\
&+\left(\frac{2}{3}-\frac{4x}{3}+\frac{2x^2}{3}\right)\frac{t^{2\lambda}}{\Gamma(1+2\lambda)}+\Big(\frac{20}{9}-\frac{40x}{9}+\frac{20x^2}{9}\Big)\frac{t^{3\lambda}}{\Gamma(1+3\lambda)}, \\
u_3(x,t) &= \frac{2}{3}(1-x)-\left(\frac{2}{3}-\frac{2x}{3}\right)\frac{t^{\beta}}{\Gamma(1+\beta)}+\left(\frac{4}{3}-\frac{4x}{3}\right)\frac{t^{2\beta}}{\Gamma(1+2\beta)} \\
&+\left(\frac{32}{9}-\frac{32x}{9}\right)\frac{t^{3\beta}}{\Gamma(1+3\beta)}.
\end{aligned}
\tag{2.14}
$$

Continuing in this approach, the rest of the components of $f_n(x)$ and $g_n(x)$ for $n \geq 4$, can be completely acheived and the series solution is thus completely determined. Finally, the solution of Eq (1.4) is given by

$$
h(x,t) = \sum_{n=0}^{\infty} f_n(x)\frac{t^{n\lambda}}{\Gamma(1+n\lambda)}, \quad u(x,t) = \sum_{n=0}^{\infty} g_n(x)\frac{t^{n\beta}}{\Gamma(1+n\beta)}
\tag{2.15}
$$

Convergence study and error estimate

Theorem 1. *Let us take the coupled fractional differential equation (1.4) with the initial conditions given by (2.1) and assume that $D_t^{\lambda}h(x,t)$ and $D_t^{\beta}u(x,t)$ be the Caputo derivative with $D_t^{\lambda}h(x,t), D_t^{\beta}u(x,t) \in C([0,M]\times[0,L])$, where $C([0,M]\times[0,L])$ be the set of all continuous functions over the interval $[0,M]\times[0,L]$, then the approximate solutions $\tilde{h}(x,t)$ and $\tilde{u}(x,t)$ of the coupled fractional differential equation (1.4) are*

$$
\tilde{h}(x,t) \cong \sum_{n=0}^{N} A_n t^{n\lambda} \quad \text{and} \quad \tilde{u}(x,t) \cong \sum_{n=0}^{N} B_n t^{n\beta},
$$

where

$$
A_n = \frac{D^{n\lambda}h(x,t_0)}{\Gamma(n\lambda+1)} \quad \text{and} \quad B_n = \frac{D^{n\beta}u(x,t_0)}{\Gamma(n\beta+1)}.
$$

Furthermore, ∃ value δ, where $0 \leq \delta \leq t$ so that the errors $E_N^1(x,t)$ and $E_N^2(x,t)$ on the Banach space $(C[0,M]\times[0,L], \|.\|)$ for the approximate solutions $\tilde{h}(x,t)$ and $\tilde{u}(x,t)$ have the form

$$\left\|E_N^1(x,t)\right\| = Sup_{\,0\leq x\leq M,\ 0\leq t\leq L}\left|\frac{D^{(N+1)\lambda}h(x,0+)}{\Gamma((N+1)\lambda+1)}t^{n\lambda}\right| \textit{and}$$

$$\left\|E_N^2(x,t)\right\| = Sup_{\,0\leq x\leq M,\ 0\leq t\leq L}\left|\frac{D^{(N+1)\beta}u(x,0+)}{\Gamma((N+1)\beta+1)}t^{n\beta}\right| \textit{respectively, if}\ \ \delta\to 0+$$

Proof. First part of the proof is follows for the approximate solution $\tilde{h}(x,t)$.
In this case, the error term:

$$E_N^1(x,t) = h(x,t) - \tilde{h}(x,t),$$

where

$$h(x,t) = \sum_{n=0}^{\infty}\frac{D^{n\lambda}h(x,0)}{\Gamma(n\lambda+1)}t^{n\lambda} \ \text{ and } \ \tilde{h}(x,t) = \sum_{n=0}^{N}\frac{D^{n\lambda}h(x,0)}{\Gamma(n\lambda+1)}t^{n\lambda}.$$

For $0<\alpha<1$,

$$\begin{aligned}
&J_t^{n\lambda}D_t^{n\lambda}h(x,t) - J_t^{(n+1)\lambda}D_t^{(n+1)\lambda}h(x,t)\\
&= J_t^{n\lambda}\Big(D_t^{n\lambda}h(x,t) - J_t^{\lambda}D_t^{\lambda}(D_t^{n\lambda}h(x,t))\Big)\\
&\ = J_t^{n\lambda}\Big(D_t^{n\lambda}h(x,0)),\\
&\ = \frac{D^{n\lambda}h(x,0)}{\Gamma(n\lambda+1)}t^{n\lambda}.
\end{aligned}$$

The N-th order approximation for $h(x,t)$ is

$$\begin{aligned}
\tilde{h}(x,t) &= \sum_{n=0}^{N}\frac{D^{n\lambda}h(x,0)}{\Gamma(n\lambda+1)}t^{n\lambda}\\
&= \sum_{n=0}^{N}J_t^{n\lambda}D_t^{n\lambda}h(x,t) - J_t^{(n+1)\lambda}D_t^{(n+1)\lambda}h(x,t), \ \ \text{using above}\\
&= h(x,t) - \sum_{n=0}^{N}J_t^{(n+1)\lambda}D_t^{(n+1)\lambda}h(x,t).
\end{aligned}$$

Therefore, we have the following error term

$$\begin{aligned}
E_N^1(x,t) &= h(x,t) - \tilde{h}(x,t),\\
&= \sum_{n=0}^{N}J_t^{(n+1)\lambda}D_t^{(n+1)\lambda}h(x,t)
\end{aligned}$$

$$
\begin{aligned}
&= J_t^{(N+1)\lambda} D_t^{(N+1)\lambda} h(x,t) \\
&= \frac{1}{\Gamma((N+1)\lambda)} \int_0^t \frac{D^{(N+1)\lambda} h(x,\zeta)}{(t-\zeta)^{1-(N+1)\lambda}} d\zeta \\
&= \frac{D^{(N+1)\lambda} h(x,\delta)}{\Gamma((N+1)\lambda)} \int_0^t \frac{d\zeta}{(t-\zeta)^{1-(N+1)\lambda}}, \text{ In view of the integral mean value theorem} \\
&= \frac{D^{(N+1)\lambda} h(x,\delta)}{\Gamma((N+1)\lambda)} t^{(N+1)\lambda}.
\end{aligned}
$$

Now, the error term on the Banach space $(C[0, M] \times [0, L], \|.\|)$ is

$$
\begin{aligned}
\left\| E_N^1(x,t) \right\| &= \text{Sup}_{\ 0 \le x \le M,\ 0 \le t \le L} \left| h(x,t) - \tilde{h}(x,t) \right| \\
&= \text{Sup}_{\ 0 \le x \le M,\ 0 \le t \le L} \left| \frac{D^{(N+1)\lambda} h(x,\delta)}{\Gamma((N+1)\lambda)} t^{(N+1)\lambda} \right| \\
&= \text{Sup}_{\ 0 \le x \le M,\ 0 \le t \le L} \left| \frac{D^{(N+1)\lambda} h(x,0+)}{\Gamma((N+1)\lambda)} t^{(N+1)\lambda} \right| \ \text{ as } \ \delta \to 0+ \ ,
\end{aligned}
$$

As $N \to \infty$, $\left\| E_N^1(x,t) \right\| \to 0$, hence $h(x,t)$ can be approximate as

$$
h(x,t) = \sum_{n=0}^{\infty} \frac{D^{n\lambda} h(x,0)}{\Gamma(n\lambda+1)} t^{n\lambda} \cong \sum_{n=0}^{N} \frac{D^{n\lambda} h(x,0)}{\Gamma(n\lambda+1)} t^{n\lambda} = \tilde{h}(x,t),
$$

with the error term $\left\| E_N^1(x,t) \right\|$.
Following the similar argument, for the approximate solution $\tilde{u}(x,t)$ we can also find the error $\left\| E_N^2(x,t) \right\| = \|u(x,t) - \tilde{u}(x,t)\|$. □

3. Modified homotopy analysis transform method

3.1. The analytical technique

Here, we consider a general fractional partial differential equation for the discussion of MHATM method as

$$
D_t^{\lambda} h(x,t) + R[x]h(x,t) + N[x]h(x,t) = g(x,t), \quad t > 0, \quad x \in \mathbb{R}, \quad 0 < \lambda \le 1, \tag{3.1}
$$

where $R[x]$, $N[x]$, $g(x,t)$ and $h(x,t)$ is defined as above.

Now methodology discussed in [32, 33], applied to Eq (3.1) we obtained the *mth*-order deformation equation

$$
\begin{aligned}
h_m(x,t) \quad &= (\chi_m + \hbar) h_{m-1} - \hbar(1-\chi_m) \textstyle\sum_{i=0}^{j-1} t^i h^{(i-1)}(0) + \hbar L^{-1}\Big(\frac{1}{s^{\lambda}} L\Big(R_{m-1}[t] h_{m-1}(t) \\
&\quad + \textstyle\sum_{k=0}^{m-1} P_k(h_0, h_1, \ldots, h_m) - g(x,t), \Big)\Big)
\end{aligned} \tag{3.2}
$$

where P_k are the homotopy polynomial.

The expression in nonlinear operator form has been modified in homotopy analysis transforms method for the convenience. That is, the nonlinear term $N[x,t]h(x,t)$ is expanded in terms of homotopy polynomials as

$$N[h(x,t)] = N\Big(\sum_{k=0}^{m-1} h_m(x,t)\Big) = \sum_{m=0}^{\infty} P_m h^m. \tag{3.3}$$

Next, from the Eq (3.2), we find the various $h_m(x,t)$ for $m \geq 1$ and the series solution of Eq (3.1) is thus entirely determined

$$h(x,t) = \sum_{m=0}^{\infty} h_m(x,t). \tag{3.4}$$

3.2. Implementation of the MHATM to time-fractional SWEs

Applying the Laplace transform (LT) on both the sides of (1.4), we get

$$\begin{aligned} &s^{\lambda} L[h(x,t)] - s^{\lambda-1} h(x,0) + L[2uh_x - hu_x] = 0, \\ &s^{\beta} L[u(x,t)] - s^{\beta-1} u(x,0) + L[uu_x + h_x] = 0. \end{aligned} \tag{3.5}$$

After some simplification and applying the inverse Laplace transform on (3.5), we have

$$\begin{aligned} h(x,t) &= \tfrac{1}{9}\left(x^2 - 2x + 1\right) + L^{-1}\left(s^{-\lambda} L[uh_x + hu_x]\right), \\ u(x,t) &= \tfrac{2(1-x)}{3} + L^{-1}\left(s^{-\beta} L[uu_x + h_x]\right). \end{aligned} \tag{3.6}$$

Next, for this case the system of nonlinear operator as follows:

$$\begin{aligned} N[\phi(x,t;q)] &= L[\phi(x,t;q)] - \tfrac{1}{9}\left(x^2 - 2x + 1\right) + s^{-\lambda} L[\Phi\phi_x + \phi\Phi_x], \\ N[\Phi(x,t;q)] &= L[\phi(x,t;q)] - \tfrac{2(1-x)}{3} + s^{-\beta} L[2\Phi(\Phi)_x + \phi_x]. \end{aligned} \tag{3.7}$$

Which leads to the $mth-$order deformation equations as

$$\begin{aligned} L[h_m(x,t) - \chi_m h_{m-1}(x,t)] &= \hbar R_m(\vec{h}_{m-1}, x, t), \\ L[u_m(x,t) - \chi_m u_{m-1}(x,t)] &= \hbar R_m(\vec{u}_{m-1}, x, t). \end{aligned} \tag{3.8}$$

Applying the inverse Laplace transform to both sides of (3.8) yields

$$\begin{aligned} h_m(x,t) &= \chi_m h_{m-1}(x,t) + \hbar q L^{-1}[R_m(\vec{h}_{m-1}, x, t)], \\ u_m(x,t) &= \chi_m u_{m-1}(x,t) + \hbar q L^{-1}[R_m(\vec{u}_{m-1}, x, t)], \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} R_m(\vec{h}_{m-1}, x, t) &= L[h_{m-1}(x,t)] - (1-\chi_m)\frac{1}{9}\left(x^2 - 2x + 1\right) \\ &\quad + s^{-\lambda}[2P_m + \ P_m^1], \\ R_m(\vec{u}_{m-1}, x, t) &= L[u_{m-1}(x,t)] - (1-\chi_m)\frac{2}{3}(1-x) \\ &\quad + s^{-\beta}[(P_m^2 + (u_{m-1})_x], \end{aligned} \tag{3.10}$$

P_m, P_m^1, and P_m^2 are the homotopy polynomials given as

$$\begin{aligned} P_m &= \frac{1}{\Gamma m}\left[\frac{\partial^m}{\partial q^m} N[(q\Phi(x,t;q))(q\phi(x,t;q))_x]\right]_{q=0}, \\ P_m^1 &= \frac{1}{\Gamma m}\left[\frac{\partial^m}{\partial q^m} N[(q\phi(x,t;q))(q\Phi(x,t;q))_x]\right]_{q=0}, \\ P_m^2 &= \frac{1}{\Gamma m}\left[\frac{\partial^m}{\partial q^m} N[(q\Phi(x,t;q))(q\Phi(x,t;q))_x]\right]_{q=0}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \phi(t;q) &= \phi_0 + q\phi_1 + q^2\phi_2 + q^3\phi_3 + ..., \\ \Phi(t;q) &= \Phi_0 + q\Phi_1 + q^2\Phi_2 + q^3\Phi_3 + ... \end{aligned} \tag{3.12}$$

The solutions of $mth-$order deformation Eq (3.8) becomes

$$\begin{aligned} h_m(x,t) &= (\chi_m + \hbar)h_{m-1} - (1-\chi_m)\frac{1}{9}\left(x^2 - 2x + 1\right) \\ &\quad + \hbar L^{-1}(s^{-\lambda}[P_m + P_m^1]), \\ u_m(x,t) &= (\chi_m + \hbar)u_{m-1} - (1-\chi_m)\frac{2}{3}(1-x) \\ &\quad + \hbar L^{-1}(s^{-\beta}[(P_m^2 + (u_{m-1})_x]). \end{aligned} \tag{3.13}$$

By putting the initial approximation (2.1) into the iterative scheme (3.13), we successively obtain

$$\begin{aligned} h_1 &= \frac{-\hbar\, t^\lambda}{\Gamma(1+\lambda)}\frac{2}{9}(x-1)^2, \\ u_1 &= \frac{-\hbar\, t^\beta}{\Gamma(1+\beta)}\frac{2}{3}(1-x), \\ h_2 &= \frac{-\hbar(1+\hbar)\, t^\lambda}{\Gamma(1+\lambda)}\frac{2}{9}(x-1)^2 + \frac{\hbar^2\, t^{2\lambda}}{\Gamma(1+2\lambda)}\frac{4}{9}(x-1)^2 + \frac{\hbar^2\, t^{\lambda+\beta}}{\Gamma(1+\lambda+\beta)}\frac{2}{9}(x-1)^2, \\ u_2 &= \frac{-\hbar(1+\hbar)\, t^\beta}{\Gamma(1+\beta)}\frac{2}{3}(1-x) + \frac{\hbar^2\, t^{2\beta}}{\Gamma(1+2\beta)}\frac{8}{9}(1-x) + \frac{\hbar^2\, t^{\lambda+\beta}}{\Gamma(1+\lambda+\beta)}\frac{4}{9}(1-x). \end{aligned}$$

Similarly way the remaining term of the series $h_m(x,t)$ and $u_m(x,t)$, for $m \geq 3$ can be completely achieved. Finally, the solution of Eq (1.4) can be given in the form

$$h(x,t) = \sum_{m=0}^{\infty} h_m(x,t), \qquad u(x,t) = \sum_{m=0}^{\infty} u_m(x,t). \tag{3.14}$$

3.3. Convergence study and error estimate

In this subsection, we examine the convergence analysis and error estimate of the MHATM for (1.4) with respect to the initial condition (2.1)

Theorem 2. *Suppose that $h_m(x,t), u_m(x,t), h(x,t)$ and $u(x,t)$ be defined in Banach space $(C[0,1], \|.\|)$. Then the series solution $\{h_m(x,t)\}_{m=0}^{\infty}$ and $\{u_m(x,t)\}_{m=0}^{\infty}$ given by (3.14) convergence to the solutions of (1.4), if there exist $0<\mu<1$, such that $\|h_{n+1}\| \le \mu \|h_n\|$ and $\|u_{n+1}\| \le \mu \|u_n\|$, for $n \in N$.*

Proof. We have $(C[0,1], \|.\|)$ is the Banach space of all continuous functions on $[0,1]$ with the norms,

$$\|h(x,t)\| = \max_{\forall x,t\in[0,1]} |h(x,t)| \text{ and } \|u(x,t)\| = \max_{\forall x,t\in[0,1]} |u(x,t)|.$$

Define that $\{S_n\}$ is the sequence of partial sum as,

$$\begin{aligned}
&S_0 = h_0(x,t),\\
&S_1 = h_0(x,t) + h_1(x,t),\\
&S_2 = h_0(x,t) + h_1(x,t) + h_2(x,t),\\
&.\\
&.\\
&.\\
&S_m = h_0(x,t) + h_1(x,t) + h_2(x,t) + \dots + h_m(x,t).
\end{aligned}$$

It is sufficient to show that $\{S_m\}_{m=0}^{\infty}$ is a Cauchy sequence in Banach space $(C[0,1], \|.\|)$. For $m,n \in N, m \ge n$, we have

$$\begin{aligned}
\|S_m - S_n\| &= \|(S_m - S_{m-1}) + (S_{m-1} - S_{m-2}) + \dots + (S_{n+1} - S_n)\|\\
&\le \|(S_m - S_{m-1})\| + \|(S_{m-1} - S_{m-2})\| + \dots + \|(S_{n+1} - S_n)\|\\
&= \|h_m(x,t)\| + \|h_{m-1}(x,t)\| + \dots + \|h_n(x,t)\|\\
&\le \mu^m \|u_0(x,t)\| + \mu^{m-1} \|u_0(x,t)\| + \dots + \mu^{n+1} \|u_0(x,t)\|\\
&= \frac{1-\mu^{m-n}}{1-\mu}\mu^{n+1} \|u_0(x,t)\|.
\end{aligned}$$

Since $0<\mu<1$, we have $1-\mu^{m-n} < 1$; then,

$$\|S_m - S_n\| \le \frac{\mu^{n+1}}{1-\mu} \max \|u_0(x,t)\|.$$

Since $\|u_0(x,t)\|$ is bounded,

$$\lim_{m,n\to\infty} \|S_m - S_n\| = 0.$$

Therefore $\{S_m\}_{m=0}^{\infty}$ is a Cauchy sequence in the Banach space $(C[0,1], \|.\|)$, so the series solution defined in (3.14), converges. Similarly, we can show for $u(x,t)$ case. This completes the proof. □

Theorem 3. *The maximum absolute truncation error of the series solution Eq (3.14) for Eq (1.4) w.r.to the initial conditions 2.1 is estimated to be*

$$\left| h(x,t) - \sum_{i=0}^{m} h_i(x,t) \right| \le \frac{\mu^{n+1}}{1-\mu} \|h_0(x,t)\| \text{ and } \left| u(x,t) - \sum_{i=0}^{m} u_i(x,t) \right| \le \frac{\mu^{n+1}}{1-\mu} \|u_0(x,t)\|.$$

Proof. From theorem 4.1 , for $m \geq n$ we have

$$\|S_m - S_n\| = \frac{1-\mu^{m-n}}{1-\mu}\mu^{n+1}\, \|u_0(x,t)\|\,. \tag{3.15}$$

Now, as $m \to \infty$ then $S_m \to u(x,t)$. So,

$$|u(x,t) - S_n| \leq \frac{\mu^{n+1}}{1-\mu}\mu^{n+1}\, \|u_0(x,t)\|\,. \tag{3.16}$$

Since $0 < \mu < 1$, we have $1 - \mu^{m-n} < 1$. Therefore the above inequality becomes,

$$\left|h(x,t) - \sum_{i=0}^{m} h_i(x,t)\right| \leq \frac{\mu^{n+1}}{1-\mu}\, \|h_0(x,t)\|\,.$$

Similarly we can show the inequality

$$\left|u(x,t) - \sum_{i=0}^{m} u_i(x,t)\right| \leq \frac{\mu^{n+1}}{1-\mu}\, \|u_0(x,t)\|\,.$$

This complete the proof.

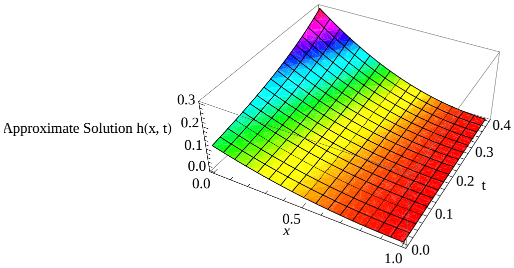
□

4. Numerical simulations and discussions

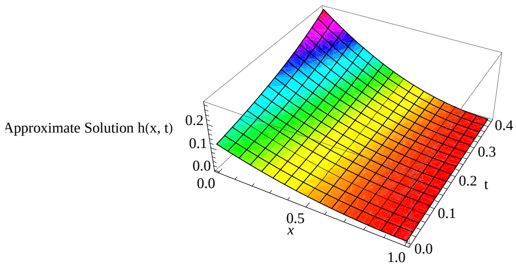
In this section, comparison of RPSM and MHATM are made in a systematic fashion through different graphical representation and tabulated data.

4.1. Comparison of the approximate solution obtained by RPSM and MHATM method regarding to the exact solution

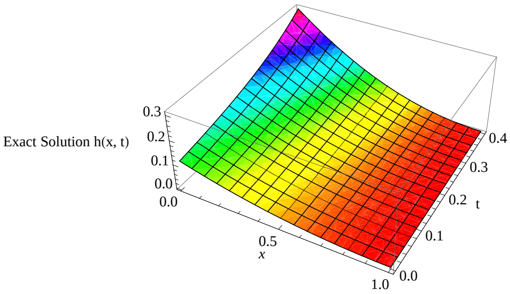
The geometrical behaviour of the obtained solutions of Eq (1.4) are compare by depicted through 3D Figures 1–5 of the 5$^{\text{th}}$ order MHATM, 5$^{\text{th}}$ order RPSM and the exact solution represented by the Eq (2.2). The scenario of subfigures reveals that their surface graphic and profile are almost the same even if for different values of α. Figure 6 explore the comparison of the approximate solution received by RPSM and MHATM method with consideration to exact solutions at time instance $t = 0.5$ when $\lambda = \beta = 1$. Figures indicate that a high level of accuracy has been attained between the exact solution and the solutions obtained by MHATM and RPSM.



(a) By MHATM

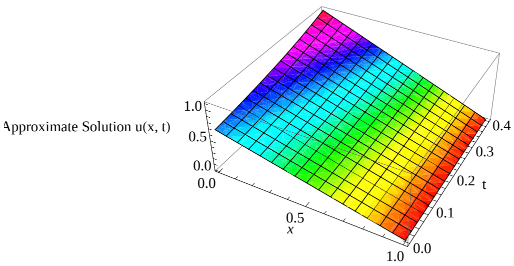


(b) By RPSM

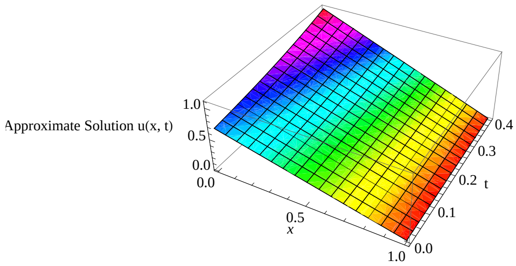


(c) Exact solution

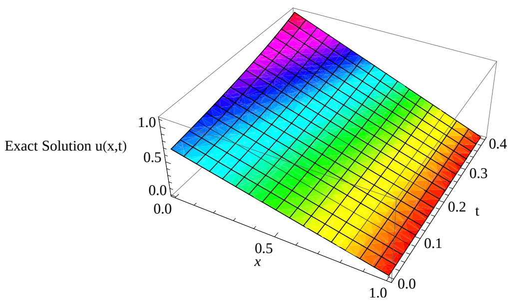
Figure 1. The surfaces show (a) the numerical approximate solution of $h(x,t)$ by MHATM, (b) the numerical approximate solution of $h(x,t)$ by RPSM and (c) the exact solution of $h(x,t)$ when $\lambda = 1, \beta = 1$.



(a) By MHATM

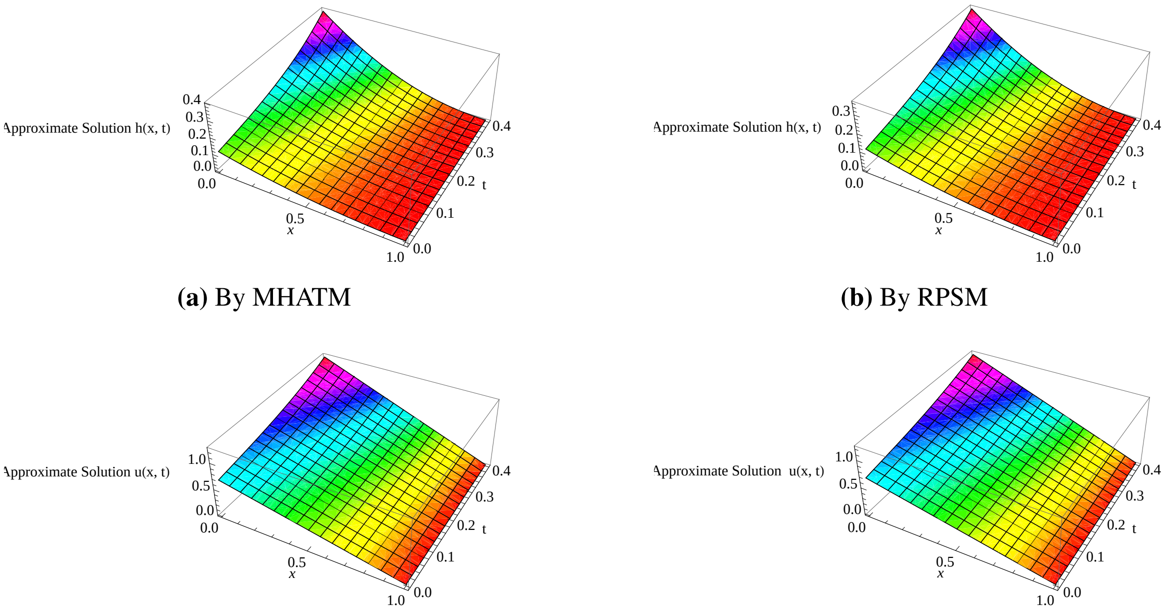


(b) By RPSM



(c) Exact solution

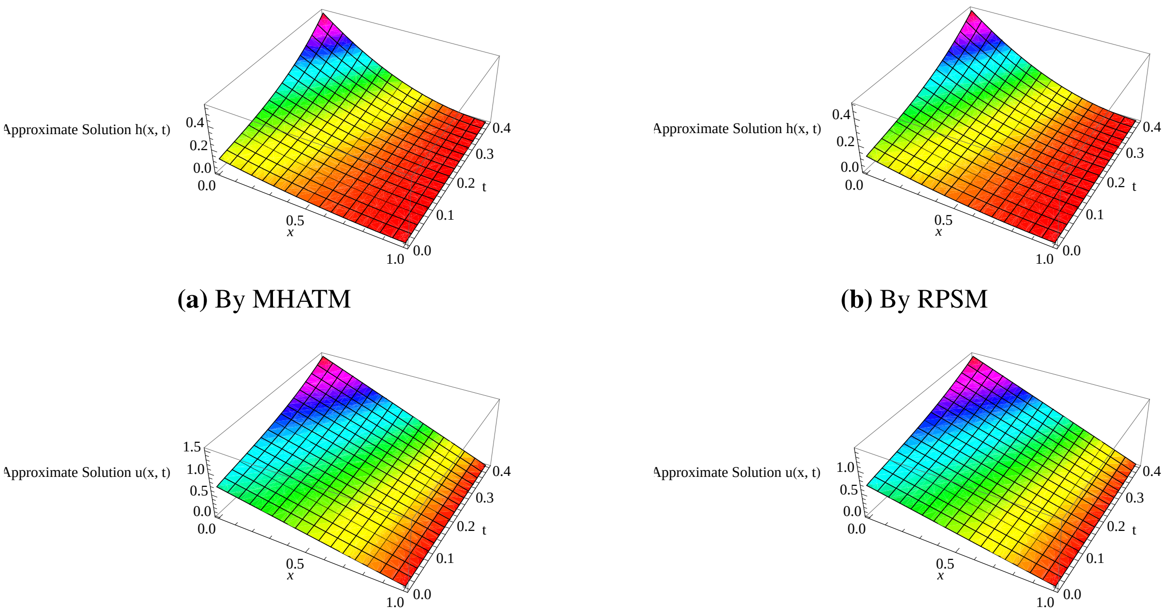
Figure 2. The surfaces show (a) the numerical approximate solution of $u(x,t)$ by MHATM, (b) the numerical approximate solution of $u(x,t)$ by RPSM and (c) the exact solution of $u(x,t)$ when $\lambda = 1, \beta = 1$.



(**a**) By MHATM (**b**) By RPSM

(**c**) By MHATM (**d**) By RPSM

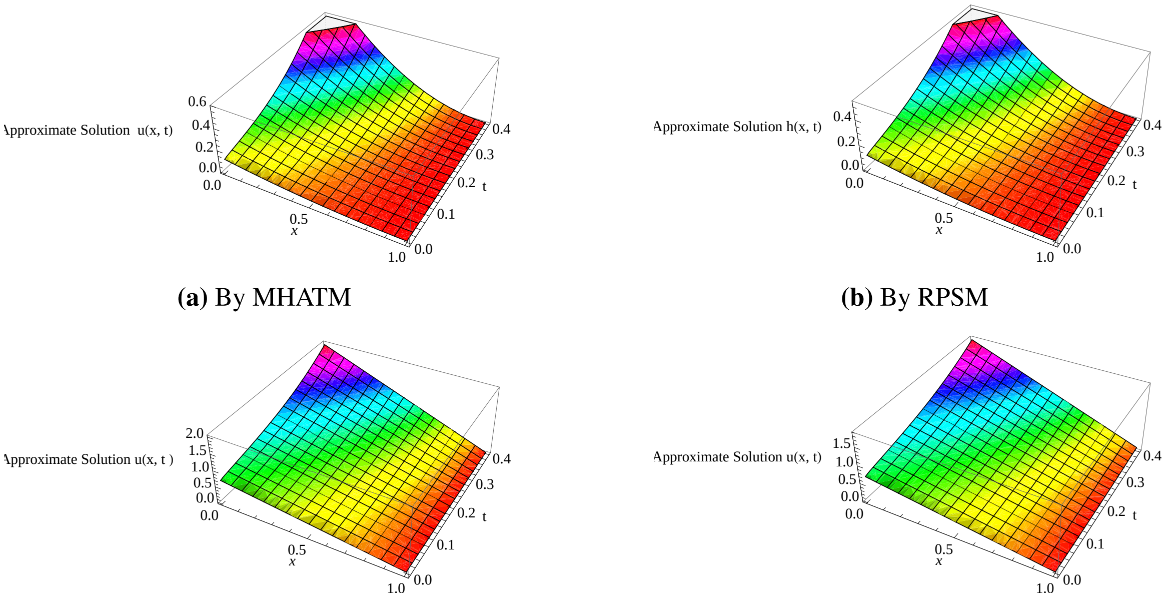
Figure 3. The surfaces show (a) the numerical approximate solution of $h(x, t)$ by MHATM, (b) the numerical approximate solution of $h(x, t)$ by RPSM, (c) the numerical approximate solution of $u(x, t)$ by MHATM and (d) the numerical approximate solution of $u(x, t)$ by RPSM when $\lambda = 0.9, \beta = 0.9$.



(**a**) By MHATM (**b**) By RPSM

(**c**) By MHATM (**d**) By RPSM

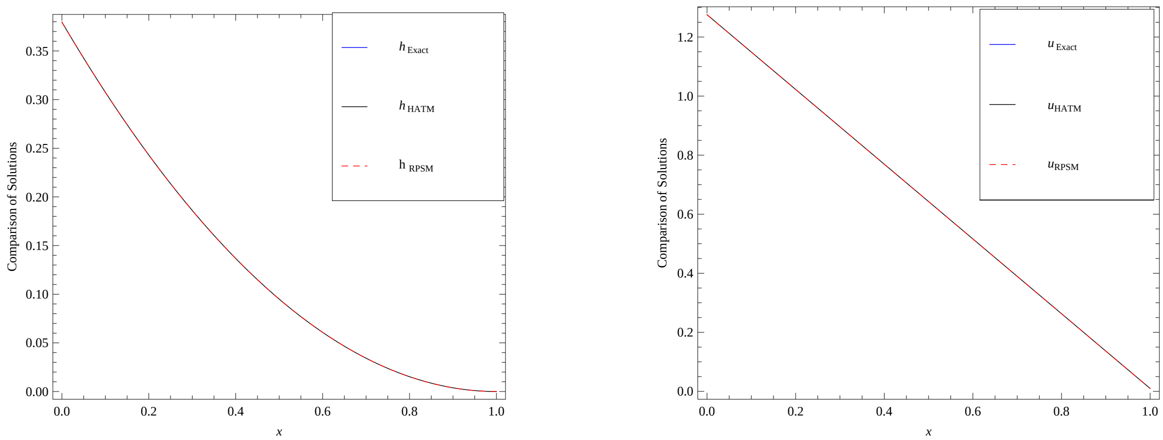
Figure 4. The surfaces show (a) the numerical approximate solution of $h(x, t)$ by MHATM, (b) the numerical approximate solution of $h(x, t)$ by RPSM, (c) the numerical approximate solution of $u(x, t)$ by MHATM and (d) the numerical approximate solution of $u(x, t)$ by RPSM when $\lambda = 0.8, \beta = 0.8$.



(a) By MHATM **(b)** By RPSM

(c) By MHATM **(d)** By RPSM

Figure 5. The surfaces show (a) the numerical approximate solution of $h(x,t)$ by MHATM, (b) the numerical approximate solution of $h(x,t)$ by RPSM, (c) the numerical approximate solution of $u(x,t)$ by MHATM and (d) the numerical approximate solution of $u(x,t)$ by RPSM when $\lambda = 0.7, \beta = 0.7$.

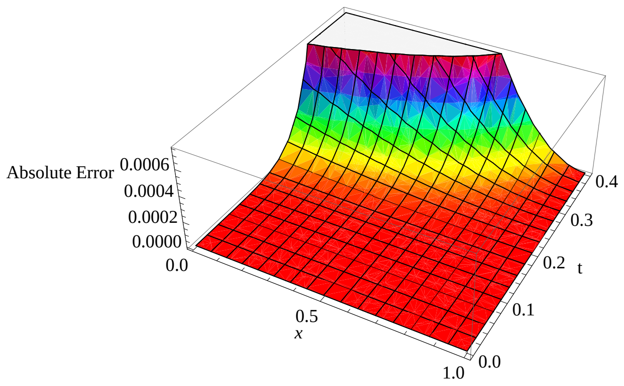


(a) **(b)**

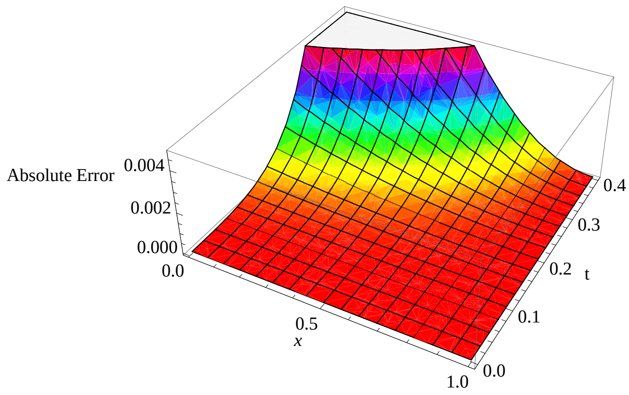
Figure 6. The surfaces show (a) comparison of five term MHATM solution and five term RPSM solution with regard to exact solution of $h(x,t)$ at time instance $t = 0.5$ and (b) comparison of five term MHATM solution and five term RPSM solution with regard to exact solution of $u(x,t)$ at time instance $t = 0.5$.

4.2. *Comparison of absolute error for RPSM and MHATM solutions*

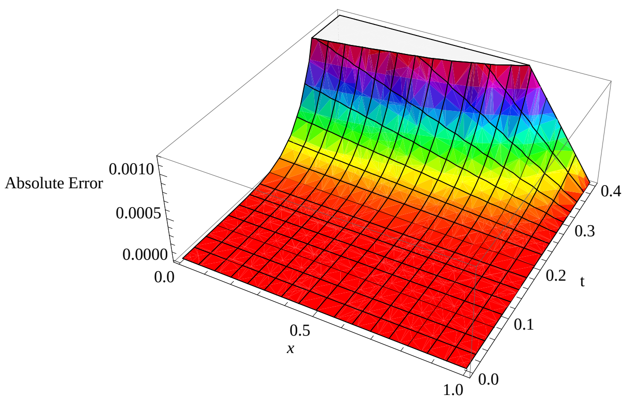
Figure 7 indicating the numerical simulations for comparison of the absolute error for RPSM and MHATM solutions. Even if both the present methods are reliable and efficient, Figure 6 guarantee plausibility to consider MHATM give more accurate than RPSM solutions for fractional SWEs.



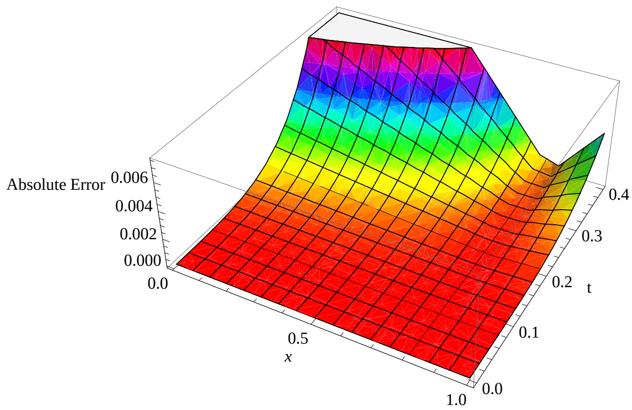
(a) By MHATM



(b) By RPSM



(c) By MHATM



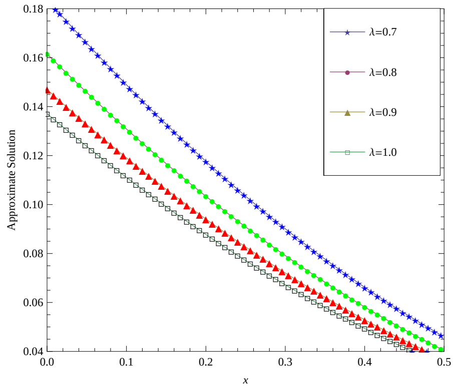
(d) By RPSM

Figure 7. The surfaces show (a) absolute error $E_5(h) = |h(x,t) - h_5(x,t)|$ by MHATM, (b) absolute error $E_5(h) = |h(x,t) - h_5(x,t)|$ by RPSM, (c) absolute error $E_5(u) = |u(x,t) - u_5(x,t)|$ by MHATM and (d) absolute error $E_5(u) = |u(x,t) - u_5(x,t)|$ by RPSM.

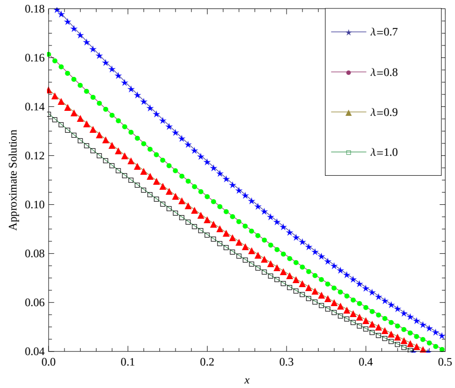
4.3. *Comparison of approximate solution for different values of λ and β*

Figure 8 shows the comparison of the approximate analytical solutions acheived by RPSM and MHATM for $\lambda = 0.7, \lambda = 0.8, \lambda = 0.9$ and $\lambda = 1$. Also for the $\beta = 0.7, \beta = 0.8, \beta = 0.9$ and $\beta = 1$.

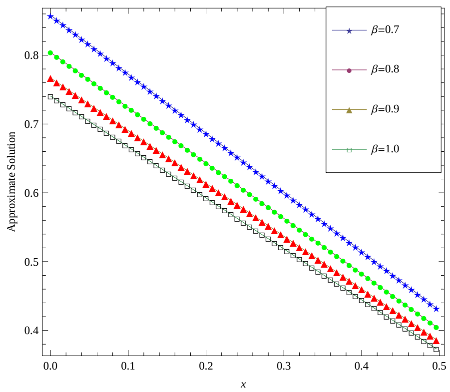
The comparison of results between proposed methods RPSM and MHATM at different points of x and t using the parameters $c = \frac{1}{2}, k = -1, b = 9$ and $\hbar = -1$ presented in Table 1.



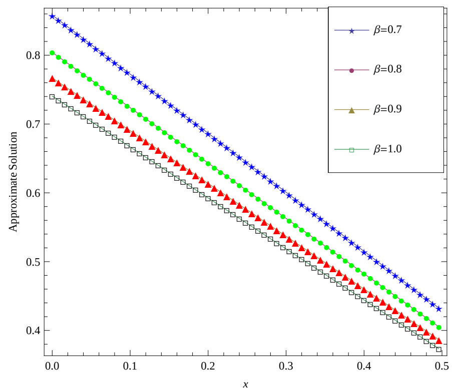
(a) By MHATM



(b) By RPSM



(c) By MHATM



(d) By RPSM

Figure 8. The surfaces show (a) plot of $h_5(x,t)$ versus time t for different values of λ using MHATM, (b) plot of $h_5(x,t)$ versus time t for different values of λ using RPSM, (c) plot of $u_5(x,t)$ versus time t for different values of β using MHATM and (d) Plot of $u_5(x,t)$ versus time t for different values of β using RPSM when $\hbar = -1$.

Table 1. The comparison of the absolute error in the solution of fractional SWEs using five term approximation for MHATM and RPSM at different points of x and t with $c = \frac{1}{2}, k = -1, b = 9$ and $\hbar = -1$ for $\lambda = 1, \beta = 1$.

(x,t)	$\lvert h_{exact} - h_{HATM}\rvert$	$\lvert u_{exact} - u_{HATM}\rvert$	$\lvert h_{exact} - h_{RPSM}\rvert$	$\lvert u_{exact} - u_{RPSM}\rvert$
(0.1,0.1)	7.11111×10^{-7}	6.66667×10^{-7}	8.27052×10^{-5}	9.15580×10^{-5}
(0.1,0.2)	5.22000×10^{-5}	4.80000×10^{-5}	9.40010×10^{-4}	1.02467×10^{-4}
(0.1,0.3)	6.96269×10^{-4}	6.24857×10^{-4}	4.66383×10^{-3}	4.95146×10^{-3}
(0.2,0.1)	5.61866×10^{-7}	5.92593×10^{-7}	6.53415×10^{-5}	7.97169×10^{-5}
(0.2,0.2)	4.12444×10^{-5}	4.26667×10^{-5}	7.42537×10^{-4}	8.83779×10^{-4}
(0.2,0.3)	5.50139×10^{-4}	5.55429×10^{-4}	3.68358×10^{-3}	4.26263×10^{-3}
(0.3,0.1)	4.30178×10^{-7}	5.18519×10^{-7}	5.00213×10^{-5}	6.78757×10^{-5}
(0.3,0.2)	3.15778×10^{-5}	3.73333×10^{-5}	5.68320×10^{-4}	7.42887×10^{-4}
(0.3,0.3)	4.21200×10^{-4}	4.86000×10^{-4}	2.88188×10^{-3}	3.57380×10^{-3}

4.4. Optimal values of $\hbar$ in MHATM

At the mth-order of approximation, the exact square residual error are:

$$\Delta_m^h = \int_0^1 \int_0^1 \left(N\left[\sum_{i=0}^{m} h_i(x,t) \right] \right)^2 dx\, dt$$

and

$$\Delta_m^u = \int_0^1 \int_0^1 \left(N\left[\sum_{i=0}^{m} u_i(x,t) \right] \right)^2 dx\, dt \tag{4.1}$$

where $N[h(x,t)] = \frac{\partial^\lambda h}{\partial t^\lambda} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x}$ and $N[u(x,t)] = \frac{\partial^\beta u}{\partial t^\beta} + u\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x}$.

Next, for the convenience point of view, we also introduced the averaged residual error defined by [47]

$$E_m^h = \frac{1}{k_1^2} \sum_{j=1}^{k_1} \sum_{l=1}^{k_1} \left(N\left[\sum_{i=0}^{m} h_i(j\Delta x, l\Delta t) \right] \right)^2$$

and

$$E_m^u = \frac{1}{k_1^2} \sum_{j=1}^{k_1} \sum_{l=1}^{k_1} \left(N\left[\sum_{i=0}^{m} u_i(j\Delta x, l\Delta t) \right] \right)^2 \tag{4.2}$$

where $\Delta x = \frac{1}{40k_1}, \Delta t = \frac{1}{40k_2}, k_1 = k_2 = 5$ for SWEs. The optimal value of $\hbar$ can be achieved by means of minimizing the so called averaged residual error E_m defined by (4.2), Equivalent to the nonlinear algebraic equations $\frac{\partial E_m^h}{\partial \hbar} = 0$ and $\frac{\partial E_m^u}{\partial \hbar} = 0$.

Tables 2 and 3 display the comparison of the averaged residual error for the optimal value of $\hbar$ with a different order of approximation. Also, the accuracy and validity of the MHATM technique can be demonstrated using the averaged residual error.

Table 2. Optimal value of $\hbar$ for $h(x,t)$.

Order of approx.	Optimal value of $\hbar$ for $\lambda = 1$	Optimal value of $\hbar$ for $\lambda = 0.9$	value of E_m for $\lambda = 1$	value of E_m for $\lambda = 0.9$
1	-1.0472	-0.722061	2.95623×10^{-9}	1.78708×10^{-7}
2	-0.99085	-0.989655	5.28515×10^{-10}	6.73821×10^{-10}
3	-0.98990	-0.827603	6.32161×10^{-10}	3.20375×10^{-7}

Table 3. Optimal value of $\hbar$ for $u(x, t)$.

Order of approx.	Optimal value of $\hbar$ for $\beta = 1$	Optimal value of $\hbar$ for $\beta = 0.9$	value of E_m for $\beta = 1$	value of E_m for $\beta = 0.9$
1	-1.03090	-0.710627	1.20212×10^{-8}	1.83569×10^{-6}
2	-1.02271	-1.04991	1.21163×10^{-11}	2.45324×10^{-6}
3	-1.06235	-1.03767	1.13253×10^{-14}	2.48388×10^{-6}

5. Conclusions

In this work, we have fruitfully applied MHATM and RPSM for solving time-fractional coupled SWEs. There are various features assumed for this equation are summarized as follows:
(i) The key procedure of the new adaption in MHATM has decomposed the non-linear term $N(u)$ into the sum of homotopy polynomial P_m, which helps for obtaining the rapid convergent of the series solution.
(ii) Further, the fractional coupled SWEs have been solved by using two independent analytic methods such as MHATM and RPSM.
(iii) We compare these two methods and show that the results of the MHATM method are in excellent agreement with results of the RPSM method and the obtained numerical solutions are present graphically which approves the validity of the MHATM and RPSM.
(iv) From the obtained results, it can be noted that, although both the featured techniques are reliable and efficient to handle the different nonlinear problems appearing in science and engineering, MHATM provides highly accurate numerical solution of fractional SWEs, in comparison with RPSM. The paper is concluded by observing that, MHATM is more efficient and accurate for solving the fractional coupled SWEs.

Acknowledgments

The author K.S. Nisar thanks to Deanship of Scientific research (DSR), Prince Sattam bin Abdulaziz University, Saudi Arabia for providing facilities and support.

Conflict of interest

The authors declare no conflict of interest in this paper.

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