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Research article

Integral inequalities of Hermite-Hadamard type for exponentially subadditive functions

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Abstract: In this paper, we introduce a new class of functions, which is called exponentially subadditive functions. We establish Hermite-Hadamard inequalities via exponentially subadditive functions. We also give some related inequalities according with Hermite-Hadamard inequalities. Results obtained in this paper can be viewed as generalization of previously known results.

Keywords: subadditive function; exponentially subadditive function; Hermite-Hadamard inequalities
Mathematics Subject Classification: 26A51, 26D10, 26D15

1. Introduction

Additivity, subadditivity and superadditivity plays an important role both in measure theory and in different fields of mathematics. Especially, the subadditive principle is a powerful and effective approach for solving numerous problems that arises not only in pure and applied mathematics but also in mathematical physics and other applied sciences. Subadditivity occurs in the thermodynamic chatacteristics of non-ideal solutions and blends like the excess molar volume and heat of mixing or excess enthalpy. An other interesting aspect of subadditivity is its close relationship with inequalities. Inequalities and subadditive functions can be seen in electrical network, ergodic theory and dynamic systems, quantum relative entropy, perturbations of repulsive and equilibrium theory, see, for example [2, 6, 12, 18, 19]. Here, we mention the results of [3, 5, 10, 17] and the references therein.

Definition 1.1. *A function* $f : [0, \infty) \subset \mathbb{R} \to \mathbb{R}$ *is said to be subadditive if for every* $x, y \in [0, \infty)$,

$$f(x + y) \leq f(x) + f(y).$$

If equality holds, f *is called additive. If the inequality reversed,* f *is called superadditive.*

Definition 1.2. *A real valued function* f *is convex on the positive real line* $[0, \infty)$, *if*

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

holds for all $x, y \in [0, \infty)$ *and* $t \in [0, 1]$.

Remark 1.3. *If* f *is convex and subadditive on* $[0, \infty)$ *and if* $f(0) = 0$*, then* f *is additive on* $[0, \infty)$.

Definition 1.4. *[23] The function* $f : [0, b] \to \mathbb{R}$, $b > 0$ *is said to be starshaped if we have* $f(tx) \le tf(x)$ *for all* $x \in [0, b]$ *and* $t \in [0, 1]$.

Remark 1.5. *If a subadditive function* $f : A \subset [0, \infty) \to \mathbb{R}$ *is also starshaped, then* f *is a convex function.*

The concept of convexity is one of the most important research area in many branches of pure and applied mathematics. It has a key role in many fields of applications, especially in optimization theory and the theory of inequalities. A useful inequality for convex functions is given as follows:
Let f is a convex function on the interval $I = [u, v]$ of real numbers with $u < v$, then

$$f\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v f(x)\,dx \le \frac{f(u)+f(v)}{2}, \quad u, v \in I. \tag{1.1}$$

This double inequality is well known in the literature as Hermite-Hadamard integral inequality for convex functions (see, e.g., [4, 15]). Recently, Hermite-Hadamard integral inequality for convex functions has received renewed attention by many researchers and as gradually a remarkable of generalizations, extensions and refinements in various directions have been found, see [1, 7–9, 11, 13, 14, 16, 20–22] and the references included there.

2. Main results

In this section, first we give definition of exponential subadditive functions. Then, we establish Hermite-Hadamard type integral inequalities and related inequalities for exponentially subadditive functions.

Definition 2.1. *Let* $\alpha \in \mathbb{R}$. *A function* $f : I \subset [0, \infty) \to \mathbb{R}$ *is said to be exponentially subadditive function, if*

$$f(x+y) \le \frac{f(x)}{e^{\alpha x}} + \frac{f(y)}{e^{\alpha y}}$$

for all $x, y \in I$.

Proposition 2.2. *Let* $I \subset [0, \infty)$ *be a real interval and* $f : I \to \mathbb{R}$ *is a function, then;*

1. *If* $\alpha \ge 0$ *and* f *is an exponentially subadditive function, then* f *is a subadditive function.*
2. *If* $\alpha \le 0$ *and* f *is a subadditive function, then* f *is an exponentially subadditive function.*

Proof. 1. For $\alpha \ge 0$, we have $e^{\alpha x} \ge 1$, $x \in [0, \infty)$. If f is an exponentially subadditive function, then

$$f(x+y) \le \frac{f(x)}{e^{\alpha x}} + \frac{f(y)}{e^{\alpha y}} \le f(x) + f(y)$$

for all $x, y \in [0, \infty)$. So, f is a subadditive function.

2. For $\alpha \leq 0$, we have $e^{\alpha x} \leq 1$, $x \in [0, \infty)$. If f is a subadditive function, then

$$f(x+y) \leq f(x) + f(y) \leq \frac{f(x)}{e^{\alpha x}} + \frac{f(y)}{e^{\alpha y}},$$

for all $x, y \in [0, \infty)$. So, f is an exponentially subadditive function.

□

Remark 2.3. *For $\alpha = 0$, the definition of exponential subadditive function reduces to the definition of subadditive function.*

Example 2.4. *Let $\alpha \geq 1$ and $f : [1, \infty) \to \mathbb{R}$, $f(x) = c$ for $c < 0$, then f is an exponentially subadditive function. For every $x, y \geq 1$, $e^{\alpha(x+y)} \geq e^{\alpha x} + e^{\alpha y}$. So, one has $ce^{\alpha(x+y)} \leq c\left(e^{\alpha x} + e^{\alpha y}\right)$. Thus,*

$$f(x+y) = c \leq c\left(\frac{e^{\alpha x} + e^{\alpha y}}{e^{\alpha(x+y)}}\right) = c\left(\frac{1}{e^{\alpha x}} + \frac{1}{e^{\alpha y}}\right) = \frac{f(x)}{e^{\alpha x}} + \frac{f(y)}{e^{\alpha y}}.$$

Example 2.5. *Let $\alpha < 0$ and $f(x) = \sqrt{x}$ for $x \in [0, \infty)$. For every $x, y \geq 0$, we have*

$$f(x+y) = \sqrt{x+y} \leq \sqrt{x} + \sqrt{y} = f(x) + f(y).$$

So, f is a subadditive function. From Proposition 2.2, f is also an exponentially subadditive function.

Theorem 2.6. *Let $f : [u, v] \subset [0, \infty) \to \mathbb{R}$ be a continuous exponentially subadditive function, then*

$$\frac{1}{2} f(u+v) \leq \frac{1}{v-u} \int_u^v \frac{f(x)}{e^{\alpha x}} dx \leq \frac{1}{u} \int_0^u \frac{f(x)}{e^{2\alpha x}} dx + \frac{1}{v} \int_0^v \frac{f(x)}{e^{2\alpha x}} dx.$$

Proof. Let $x \in [u, v]$ and $x = tu + (1-t)v$ (*or* $x = (1-t)u + tv$) for $t \in [0, 1]$. By using exponential subadditivity of the function f, we have

$$f(tu + (1-t)v) \leq \frac{f(tu)}{e^{\alpha tu}} + \frac{f((1-t)v)}{e^{\alpha(1-t)v}} \tag{2.1}$$

and

$$f((1-t)u + tv) \leq \frac{f((1-t)u)}{e^{\alpha(1-t)u}} + \frac{f(tv)}{e^{\alpha tv}}. \tag{2.2}$$

Adding (2.1) and (2.2), and by using the fact that f is an exponentially subadditive function, we have

$$\begin{aligned} f(u+v) &= f(tu + (1-t)v + (1-t)u + tv) \\ &\leq \frac{f(tu + (1-t)v)}{e^{\alpha(tu+(1-t)v)}} + \frac{f((1-t)u + tv)}{e^{\alpha((1-t)u+tv)}} \\ &\leq \frac{f(tu)}{e^{\alpha(2tu+(1-t)v)}} + \frac{f((1-t)v)}{e^{\alpha(tu+2(1-t)v)}} + \frac{f((1-t)u)}{e^{\alpha(2(1-t)u+tv)}} + \frac{f(tv)}{e^{\alpha((1-t)u+2tv)}}. \end{aligned} \tag{2.3}$$

Integrating both sides of (2.3) with respect to t on $[0, 1]$, it follows that

$$f(u+v) \leq \int_0^1 \frac{f(tu + (1-t)v)}{e^{\alpha(tu+(1-t)v)}} dt + \int_0^1 \frac{f((1-t)u + tv)}{e^{\alpha((1-t)u+tv)}} dt$$

$$\leq \quad 2\int_0^1 \frac{f(tu)}{e^{\alpha(2tu+(1-t)v)}}dt + 2\int_0^1 \frac{f(tv)}{e^{\alpha((1-t)u+2tv)}}dt.$$

By using the change of variable technique, we obtain that

$$\frac{1}{2}f(u+v) \leq \frac{1}{v-u}\int_u^v \frac{f(x)}{e^{\alpha x}}dx \leq \frac{1}{u}\int_0^u \frac{f(x)}{e^{2\alpha x}}dx + \frac{1}{v}\int_0^v \frac{f(x)}{e^{2\alpha x}}dx,$$

which completes the proof of the theorem. □

Remark 2.7. *Note that, for $\alpha = 0$, Theorem 2.6 reduces to Theorem 2 in [21].*

Remark 2.8. *Under the conditions of Theorem 2.6, if $f(tx) \leq tf(x)$ and $\alpha = 0$, then Theorem 2.6 becomes to Corollary 1 in [21].*

Theorem 2.9. *Let $f, g : [u, v] \subset [0, \infty) \to \mathbb{R}$ be two continuous exponentially subadditive functions, then*

$$\begin{aligned}
& \frac{2}{v-u}\int_u^v \frac{f(x)g(x)}{e^{2\alpha x}}dx \\
\leq \quad & \frac{2}{u}\int_0^u \frac{f(x)g(x)}{e^{2\alpha x}}dx + \frac{2}{v}\int_0^v \frac{f(x)g(x)}{e^{2\alpha x}}dx + 2\int_0^1 \left[\frac{f(tu)g((1-t)v) + f((1-t)v)g(tu)}{e^{\alpha(tu+(1-t)v)}}\right]dt \\
\leq \quad & \frac{1}{u}\int_0^u \left[\frac{(f(x))^2 + (g(x))^2}{e^{2\alpha x}}\right]dx + \frac{1}{v}\int_0^v \left[\frac{(f(x))^2 + (g(x))^2}{e^{2\alpha x}}\right]dx \\
& +2\int_0^1 \frac{f(tu)f((1-t)v) + g(tu)g((1-t)v)}{e^{\alpha(tu+(1-t)v)}}dt.
\end{aligned}$$

Proof. Since f and g are exponentially subadditive functions, we have

$$f(tu+(1-t)v) \leq \frac{f(tu)}{e^{\alpha tu}} + \frac{f((1-t)v)}{e^{\alpha(1-t)v}}, \tag{2.4}$$

$$g(tu+(1-t)v) \leq \frac{g(tu)}{e^{\alpha tu}} + \frac{g((1-t)v)}{e^{\alpha(1-t)v}}. \tag{2.5}$$

Multiplying the inequalities (2.4) and (2.5) , we get

$$\begin{aligned}
& f(tu+(1-t)v)\,g(tu+(1-t)v) \\
\leq \quad & \left(\frac{f(tu)}{e^{\alpha tu}} + \frac{f((1-t)v)}{e^{\alpha(1-t)v}}\right)\left(\frac{g(tu)}{e^{\alpha tu}} + \frac{g((1-t)v)}{e^{\alpha(1-t)v}}\right) \\
= \quad & \frac{f(tu)g(tu)}{e^{2\alpha tu}} + \frac{f(tu)g((1-t)v)}{e^{\alpha(tu+(1-t)v)}} + \frac{f((1-t)v)g(tu)}{e^{\alpha(tu+(1-t)v)}} + \frac{f((1-t)v)g((1-t)v)}{e^{2\alpha(1-t)v}} \\
\leq \quad & \frac{1}{2}\left[\left(\frac{f(tu)}{e^{\alpha tu}} + \frac{f((1-t)v)}{e^{\alpha(1-t)v}}\right)^2 + \left(\frac{g(tu)}{e^{\alpha tu}} + \frac{g((1-t)v)}{e^{\alpha(1-t)v}}\right)^2\right].
\end{aligned}$$

Taking integral with respect to t on $[0, 1]$ and changing the variables of integration, we get the desired result. □

Remark 2.10. *Note that, for $\alpha = 0$, Theorem 2.9 coincides with Theorem 3 in [21].*

Remark 2.11. *Under the conditions of Theorem 2.9, if $f(tx) \leq tf(x)$ and $\alpha = 0$, then Theorem 2.9 reduces to Corollary 2 in [21].*

Theorem 2.12. *Let $f, g : [u, v] \subset [0, \infty) \to \mathbb{R}$ be two continuous exponentially subadditive functions, then*

$$\begin{aligned}
&\frac{1}{2} f(u+v)\, g(u+v) \qquad (2.6)\\
\leq\ & \frac{1}{v-u}\int_u^v \frac{f(x)\, g(x)}{e^{2\alpha x}} dx\\
&+\frac{1}{e^{\alpha(u+v)}}\int_0^1\left[\frac{f(tu)\, g((1-t)u)}{e^{\alpha u}}+\frac{f(tv)\, g((1-t)v)}{e^{\alpha v}}+\frac{f(tu)\, g(tv)+f(tv)\, g(tu)}{e^{\alpha(u+v)}}\right]dt.
\end{aligned}$$

Proof. By using exponential subadditivity of the functions f and g, we can write

$$f(u+v) \leq \frac{f(tu+(1-t)v)}{e^{\alpha(tu+(1-t)v)}}+\frac{f((1-t)u+tv)}{e^{\alpha((1-t)u+tv)}}, \qquad (2.7)$$

$$g(u+v) \leq \frac{g(tu+(1-t)v)}{e^{\alpha(tu+(1-t)v)}}+\frac{g((1-t)u+tv)}{e^{\alpha((1-t)u+tv)}}. \qquad (2.8)$$

Multiplying the inequalities (2.7) and (2.8), we have

$$\begin{aligned}
&f(u+v)\, g(u+v)\\
\leq\ & \frac{f(tu+(1-t)v)\, g(tu+(1-t)v)}{e^{2\alpha(tu+(1-t)v)}}+\frac{f((1-t)u+tv)\, g((1-t)u+tv)}{e^{2\alpha((1-t)u+tv)}}\\
&+\frac{f(tu+(1-t)v)\, g((1-t)u+tv)+f((1-t)u+tv)\, g(tu+(1-t)v)}{e^{\alpha(u+v)}}\\
\leq\ & \frac{f(tu+(1-t)v)\, g(tu+(1-t)v)}{e^{2\alpha(tu+(1-t)v)}}+\frac{f((1-t)u+tv)\, g((1-t)u+tv)}{e^{2\alpha((1-t)u+tv)}}\\
&+\frac{1}{e^{\alpha(u+v)}}\left[\left(\frac{f(tu)}{e^{\alpha tu}}+\frac{f((1-t)v)}{e^{\alpha(1-t)v}}\right)\left(\frac{g((1-t)u)}{e^{\alpha(1-t)u}}+\frac{g(tv)}{e^{\alpha tv}}\right)\right.\\
&\left.+\left(\frac{f((1-t)u)}{e^{\alpha(1-t)u}}+\frac{f(tv)}{e^{\alpha tv}}\right)\left(\frac{g(tu)}{e^{\alpha tu}}+\frac{g((1-t)v)}{e^{\alpha(1-t)v}}\right)\right]\\
=\ & \frac{f(tu+(1-t)v)\, g(tu+(1-t)v)}{e^{2\alpha(tu+(1-t)v)}}+\frac{f((1-t)u+tv)\, g((1-t)u+tv)}{e^{2\alpha((1-t)u+tv)}}\\
&+\frac{1}{e^{\alpha(u+v)}}\left[\frac{f(tu)\, g((1-t)u)}{e^{\alpha u}}+\frac{f(tu)\, g(tv)}{e^{\alpha t(u+v)}}+\frac{f((1-t)v)\, g((1-t)u)}{e^{\alpha(1-t)(u+v)}}+\frac{f((1-t)v)\, g(tv)}{e^{\alpha v}}\right.\\
&\left.+\frac{f((1-t)u)\, g(tu)}{e^{\alpha u}}+\frac{f((1-t)u)\, g((1-t)v)}{e^{\alpha t(u+v)}}+\frac{f(tv)\, g(tu)}{e^{\alpha(1-t)(u+v)}}+\frac{f(tv)\, g((1-t)v)}{e^{\alpha v}}\right].
\end{aligned}$$

Integrating with respect to t on $[0, 1]$, we have

$$f(u+v)\, g(u+v)$$

$$\leq \frac{2}{v-u}\int_u^v \frac{f(x)g(x)}{e^{2\alpha x}}dx$$
$$+\frac{2}{e^{\alpha(u+v)}}\int_0^1\left[\left(\frac{f(tu)g((1-t)u)}{e^{\alpha u}}+\frac{f(tv)g((1-t)v)}{e^{\alpha v}}\right)+\left(\frac{f(tu)g(tv)}{e^{\alpha(u+v)}}+\frac{f(tv)g(tu)}{e^{\alpha(u+v)}}\right)\right]dt.$$

which gives us the required result. □

Remark 2.13. *Note that, for $\alpha = 0$, (2.6) coincides with (2.10) of Theorem 4 in [21].*

Theorem 2.14. *Let $f, g : [u, v] \subset [0, \infty) \to \mathbb{R}$ be two continuous exponentially subadditive functions, then*

$$\frac{1}{v-u}\int_u^v f(x)g(x)dx \tag{2.9}$$
$$\leq \frac{1}{u}\int_0^u \frac{f(x)g(x)}{e^{2\alpha x}}dx+\frac{1}{v}\int_0^v \frac{f(x)g(x)}{e^{2\alpha x}}dx+\frac{1}{2}\int_0^1\left[\frac{f(tu)g((1-t)v)}{e^{\alpha(tu+(1-t)v)}}+\frac{f(tv)g((1-t)u)}{e^{\alpha((1-t)u+tv)}}\right]dt.$$

Proof. Since f and g are exponentially subadditive functions, we can write

$$f(tu+(1-t)v) \leq \frac{f(tu)}{e^{\alpha tu}}+\frac{f((1-t)v)}{e^{\alpha(1-t)v}},$$
$$g(tu+(1-t)v) \leq \frac{g(tu)}{e^{\alpha tu}}+\frac{g((1-t)v)}{e^{\alpha(1-t)v}},$$

and

$$f((1-t)u+tv) \leq \frac{f((1-t)u)}{e^{\alpha(1-t)u}}+\frac{f(tv)}{e^{\alpha tv}},$$
$$g((1-t)u+tv) \leq \frac{g((1-t)u)}{e^{\alpha(1-t)u}}+\frac{g(tv)}{e^{\alpha tv}}.$$

Multiplying the above inequalities, we get

$$f(tu+(1-t)v)g(tu+(1-t)v)+f((1-t)u+tv)g((1-t)u+tv)$$
$$\leq \frac{f(tu)g(tu)}{e^{2\alpha tu}}+\frac{f(tu)g((1-t)v)}{e^{\alpha(tu+(1-t)v)}}+\frac{f((1-t)v)g(tu)}{e^{\alpha(tu+(1-t)v)}}+\frac{f((1-t)v)g((1-t)v)}{e^{2\alpha(1-t)v}}$$
$$+\frac{f((1-t)u)g((1-t)u)}{e^{2\alpha(1-t)u}}+\frac{f((1-t)u)g(tv)}{e^{\alpha((1-t)u+tv)}}+\frac{f(tv)g((1-t)u)}{e^{\alpha((1-t)u+tv)}}+\frac{f(tv)g(tv)}{e^{2\alpha tv}}.$$

Taking the integral with respect to t on $[0, 1]$ and changing the variables of integration, we get the desired result. □

Remark 2.15. *Note that, for $\alpha = 0$, (2.9) coincides with (2.11) of Theorem 4 in [21].*

Remark 2.16. *If one takes $f(tx) \leq tf(x)$, and $\alpha = 0$, the combination of Theorem 2.12 and Theorem 2.14 gives Corollary 3 in [21].*

3. Conclusion

In this paper, we have introduced and studied a new class of subadditive functions, which is called exponentially subadditive function. We have established Hermite-Hadamard type integral inequalities and related inequalities. By selecting specific values of parameters quite interesting results can be obtained. The idea can be extended for more diversified classes for subadditive functions.

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Conflict of interest

The author declares no conflict of interest in this paper.

References

1. N. Alp, M. Z. Sarıkaya, M. Kunt, et al. *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, J. King Saud Univ. Sci., **30** (2018), 193–203.
2. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Springer-Verlag, Berlin, 1975.
3. F. M. Dannan, *Submultiplicative and subadditive functions and integral inequalities of Bellman-Bihari type*, J. Math. Anal. Appl., **120** (1986), 631–646.
4. S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.
5. E. Hille, R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., **31** (1957), 163–179.
6. V. Hutson, *The stability under perturbations of repulsive sets*, J. Differ. Equations, **76** (1988), 77–90.
7. İ. İşcan, S. Turhan, S. Maden, *Some Hermite-Hadamard-Fejer type inequalities for harmonically convex functions via fractional integral*, New Trends Math. Sci., **4** (2016), 1–10.
8. M. Kadakal, İ. İşcan, *Inequalities of Hermite-Hadamard and Bullen type for AH-convex functions*, Universal J. Math. Appl., **2** (2019), 152–158.
9. M. Kunt, İ. İşcan, *Fractional Hermite-Hadamard-Fejer type inequalities for GA-convex functions*, Turkish J. Inequal., **2** (2018), 1–20.
10. J. Matkowski, T. Swiatkowski, *On subadditive functions*, P. Am. Math. Soc., **119** (1993), 187–197.
11. M. U. Awan, M. A. Noor, K. I. Noor, *Hermite-Hadamard inequalities for exponentially convex functions*, Appl. Math. Inform. Sci., **12** (2018), 405–409.
12. V. I. Oseledec, *A multiplicative ergodic theorem*, Trans. Moscow Math. Soc., **19** (1968), 197–231.
13. S. Özcan, İ. İşcan, *Some new Hermite–Hadamard type inequalities for s-convex functions and their applications*, J. Inequal. Appl., **2019** (2019), 1–11.
14. B. G. Pachpatte, *On some inequalities for convex functions*, RGMIA Res. Rep. Coll., **6** (2003), 1–9.
15. C. E. M. Pearce, J. Pecaric, *Inequalities for differentiable mappings with application to special means and quadrature formulae*, Appl. Math. Lett., **13** (2000), 51–55.

16. F. Qi, T. Y. Zhang, B. Y. Xi, *Hermite-Hadamard type integral inequalities for functions whose first derivatives are convex*, Ukrainian Math. J., **67** (2015), 625–640.

17. R. A. Rosenbaum, *Sub-additive functions*, Duke Math. J., **17** (1950), 227–247.

18. D. Ruelle, *Ergodic theory of differentiable dynamic systems*, Publ. Math. Paris, **50** (1979), 27–58.

19. M. B. Ruskai, *Inequalities for quantum entropy: A review with conditions for equality*, J. Math. Phys., **43** (2002), 4358–4375.

20. M. Z. Sarikaya, H. Budak, *Generalized Hermite-Hadamard type integral inequalities for fractional integrals*, Filomat, **30** (2016), 1315–1326.

21. M. Z. Sarikaya, M. A. Ali, *Hermite-Hadamard type inequalities and related inequalities for subadditive functions*, 2019, Available from: `https://www.researchgate.net/publication/337022560`.

22. E. Set, M. E. Özdemir, M. Z. Sarıkaya, et al. *Hermite-Hadamard type inequalities for* (α, m)*-convex functions via fractional integrals*, Moroccan J. Pure Appl. Anal., **3** (2017), 15–21.

23. G. H. Toader, *On generalization of the convexity*, Mathematica, **30** (1988), 83–87.

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