AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(4): 2992–3001.
DOI:10.3934/math.2020193
Received: 22 January 2020
Accepted: 13 March 2020
Published: 20 March 2020

Research article

A generalization of Kruyswijk-Olson theorem on Davenport constant in commutative semigroups

Guoqing Wang*

Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, P. R. China

* **Correspondence:** Email: gqwang1979@aliyun.com

Abstract: Let $\mathcal{S}$ be a finite commutative semigroup written additively. An element e of $\mathcal{S}$ is said to be idempotent if $e + e = e$. The Erdős-Burgess constant of the semigroup $\mathcal{S}$ is defined as the smallest positive integer ℓ such that any $\mathcal{S}$-valued sequence T of length ℓ must contain one or more terms with the sum being an idempotent of $\mathcal{S}$. If the semigroup $\mathcal{S}$ is a finite abelian group, the Erdős-Burgess constant reduces to the well-known Davenport constant in Combinatorial Number Theory. In this paper, we determine the value of the Erdős-Burgess constant for a direct sum of two finite cyclic semigroups in some cases, which generalizes the classical Kruyswijk-Olson Theorem on Davenport constant of finite abelian groups in the setting of commutative semigroups.

Keywords: Kruyswijk-Olson theorem; Davenport constant; zero-sum; Erdős-Burgess constant; cyclic semigroups; combinatorial number theory
Mathematics Subject Classification: 11B75, 11A05

1. Introduction

Let G be a finite abelian group written additively. The Davenport constant of G, denoted $\mathrm{D}(G)$, is defined as the smallest positive integer ℓ such that every sequence of terms from G of length at least ℓ must contain one or more terms with the sum being the identity element of G. This invariant was popularized by H. Davenport in the 1960's, notably for its link with algebraic number theory (as reported in [21]), and has been investigated extensively in the past over 50 years. This combinatorial invariant was found with applications in other areas, including Factorization Theory of Algebra (see [5, 12, 13]), Classical Number Theory, Graph Theory, and Coding Theory. For example, the Davenport constant has been applied by W.R. Alford, A. Granville and C. Pomerance [1] to prove that there are infinitely many Carmichael numbers, by N. Alon [2] to prove the existence of regular subgraphs, and by L.E. Marchan, O. Ordaz, I. Santos and W.A. Schmid [19] to establish a link between variant Davenport constants and problems of linear codes. What is more important, a lot of researches were motivated by the Davenport constant together with the celebrated EGZ Theorem obtained by P. Erdős, A. Ginzburg and A. Ziv [9] in 1961 on additive properties of sequences in groups, which have been developed into a branch, called zero-sum theory (see [11] for a survey), in Combinatorial Number Theory.

As a consequence of the Fundamental Theorem for finite abelian groups, any nontrivial finite abelian group can be written as the direct sum $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ of cyclic groups $\mathbb{Z}_{n_1}, \ldots, \mathbb{Z}_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$. D. Kruyswijk [7] and J.E. Olson [22] independently proved the crucial inequality

$$\mathrm{D}(G) \geq 1 + \sum_{i=1}^{r} (n_i - 1).$$

On the other hand, P. Van Emde Boas and D. Kruyswijk [8] and R. Meshulam [20] proved that

$$\mathrm{D}(G) \leq n_r + n_r \log(\frac{|G|}{n_r}).$$

A lot of efforts were also made to find the precise value of Davenport constant of finite abelian groups. However, up to date, besides for the groups of types given in Theorem A (proved independently by D. Kruyswijk (as reported in [7]) and by J.E. Olson [22]) and Theorem B as below, the precise value of this constant was known only for groups of specific forms such as $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2d}$ (see [7]), or $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$ (see [3]), etc. Even to determine the precise value of $\mathrm{D}(G)$ in the case when G is a direct sum of three finite cyclic groups remains open for over 50 years (see [11], Conjecture 3.5). Note that the conclusion $\mathrm{D}(\mathbb{Z}_n) = n$ follows by a simple application of the pigeonhole principle.

Theorem A. (Kruyswijk-Olson Theorem) $\mathrm{D}(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}) = n_1 + n_2 - 1$ *where* $n_1 \mid n_2$.

Theorem B. (J.E. Olson [21]) $\mathrm{D}(\mathbb{Z}_{p^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_r}}) = 1 + \sum\limits_{i=1}^{r} (p^{\alpha_i} - 1)$ *where* p *is prime and* $r, \alpha_1, \ldots, \alpha_r$ *are positive integers.*

For the progress about $\mathrm{D}(G)$ the reader may consult [10, 14, 18, 23, 24]. Recently, the Davenport constant was generalized in the setting of commutative semigroups (see [6, 25, 26, 28, 29]). Although the above Kruyswijk-Olson Theorem is the first classical result on the value of Davenport constant, it has not yet been generalized into semigroups.

Another motivation of this manuscript comes from the following question (see [4, 15]) posed by P. Erdős to D.A. Burgess:

"Let $\mathcal{S}$ be a finite nonempty semigroup of order n. A sequence of terms from $\mathcal{S}$ of length n must contain one or more terms whose product, in some order, is idempotent?"

Burgess [4] in 1969 gave an answer to this question in the case when $\mathcal{S}$ is commutative or contains only one idempotent. This question was completely affirmed by D.W.H. Gillam, T.E. Hall and N.H. Williams [15], and was extended to infinite semigroups by the author [27] in 2019. Naturally, one combinatorial invariant was aroused by the Erdős' question with respect to commutative semigroups (for noncommutative semigroup there is also a similar invariant).

Definition. ([27], Definition 4.1) *For any commutative semigroup $\mathcal{S}$ written additively, define the* **Erdős-Burgess constant** *of $\mathcal{S}$, denoted* $\mathrm{I}(\mathcal{S})$, *to be the least* $\ell \in \mathbb{N} \cup \{\infty\}$ *such that every sequence T of terms from $\mathcal{S}$ and of length at least ℓ must contain one or more terms with sum being an idempotent.*

Note that if the commutative semigroup $\mathcal{S}$ is finite, Gillam-Hall-Williams Theorem definitely tells us that the Erdős-Burgess constant of $\mathcal{S}$ exists, i.e., $\mathrm{I}(\mathcal{S}) \in \mathbb{N}$ is finite. In particular, when the semigroup

$\mathcal{S}$ happens to be a finite abelian group, the Erdős-Burgess constant reduces to the Davenport constant, because the identity element is the unique idempotent in a group.

Therefore, in this manuscript by studying the Erdős-Burgess constant for the direct sum of two finite cyclic semigroups, we extend the Kruyswijk-Olson Theorem into commutative semigroups. Our main result is as follows.

Theorem 1.1. *For any positive integers k_1, k_2, n_1, n_2, let $\mathcal{S} = \mathrm{C}_{k_1;n_1} \oplus \mathrm{C}_{k_2;n_2}$. Then*

$$\mathrm{I}(\mathcal{S}) \leq \max\left(\left(\left\lceil \frac{k_1}{n_1} \right\rceil - 1\right)n_1,\ \left(\left\lceil \frac{k_2}{n_2} \right\rceil - 1\right)n_2\right) + \gcd(n_1,\ n_2) + \mathrm{lcm}(n_1,\ n_2) - 1.$$

Moreover, the equality holds whenever one of the following conditions holds.

(i) $n_1 \mid n_2$ or $n_2 \mid n_1$;

(ii) there exists some $\epsilon \in \{1, 2\}$ such that $(\left\lceil \frac{k_\epsilon}{n_\epsilon} \right\rceil - 1)n_\epsilon = \max\left((\left\lceil \frac{k_1}{n_1} \right\rceil - 1)n_1,\ (\left\lceil \frac{k_2}{n_2} \right\rceil - 1)n_2\right)$ and $\frac{n_{3-\epsilon}}{\gcd(n_1,\ n_2)}$ divides $\left\lceil \frac{k_\epsilon}{n_\epsilon} \right\rceil - 1$.

2. Notation and terminologies

For integers $a, b \in \mathbb{Z}$, we set $[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$. For a real number x, we denote by $\lfloor x \rfloor$ the largest integer that is less than or equal to x, and by $\lceil x \rceil$ the smallest integer that is greater than or equal to x.

Let $\mathcal{S}$ be a commutative semigroup written additively, where the operation of $\mathcal{S}$ is denoted as $+$. For any positive integer m and any element $a \in \mathcal{S}$, we denote by ma the sum $\underbrace{a + \cdots + a}_{m}$. An element e of $\mathcal{S}$ is said to be idempotent if $e + e = e$. A cyclic semigroup is a semigroup generated by a single element x, denoted $\langle x \rangle$, consisting of all elements which can be represented as mx for some positive integer m. If the cyclic semigroup $\langle x \rangle$ is infinite then $\langle x \rangle$ is isomorphic to the semigroup of $\mathbb{N}$ with addition (see [16], Proposition 5.8), and if $\langle x \rangle$ is finite then the least integer $k > 0$ such that $kx = tx$ for some positive integer $t \neq k$ is called the *index* of x, then the least integer $n > 0$ such that $(k + n)x = kx$ is called the *period* of x. We denote a finite cyclic semigroup of index k and period n by $C_{k;n}$.

• *Note that if $k = 1$ the semigroup $C_{k;n}$ reduces to a cyclic group of order n which is isomorphic to the additive group $\mathbb{Z}_n$ of integers modulo n.*

We also need to introduce notation and terminologies on sequences over semigroups and follow the notation of A. Geroldinger, D.J. Grynkiewicz and others used for sequences over groups (cf. [[17], Chapter 10] or [[13], Chapter 5]). Let $\mathcal{F}(\mathcal{S})$ be the free commutative monoid, multiplicatively written, with basis $\mathcal{S}$. We denote multiplication in $\mathcal{F}(\mathcal{S})$ by the boldsymbol $\cdot$ and we use brackets for all exponentiation in $\mathcal{F}(\mathcal{S})$. By $T \in \mathcal{F}(\mathcal{S})$, we mean T is a sequence of terms from $\mathcal{S}$ which is unordered, repetition of terms allowed. Say $T = a_1 a_2 \cdot \ldots \cdot a_\ell$ where $a_i \in \mathcal{S}$ for $i \in [1, \ell]$. The sequence T can be also denoted as $T = \underset{a \in \mathcal{S}}{\bullet} a^{[\mathrm{v}_a(T)]}$, where $\mathrm{v}_a(T)$ is a nonnegative integer and means that the element a occurs $\mathrm{v}_a(T)$ times in the sequence T. By $|T|$ we denote the length of the sequence, i.e., $|T| = \sum_{a \in \mathcal{S}} \mathrm{v}_a(T) = \ell$. By ε we denote the *empty sequence* in $\mathcal{S}$ with $|\varepsilon| = 0$. We call T' a subsequence of T if $\mathrm{v}_a(T') \leq \mathrm{v}_a(T)$ for each element $a \in \mathcal{S}$, denoted by $T' \mid T$, moreover, we write $T'' = T \cdot T'^{[-1]}$ to mean the unique subsequence of T with $T' \cdot T'' = T$. We call T' a *proper* subsequence of T provided that $T' \mid T$ and $T' \neq T$. In particular, the empty sequence ε is a proper subsequence of every nonempty sequence. Let $\sigma(T) = a_1 + \cdots + a_\ell$ be the sum of all terms from T. We call T a *zero-sum* sequence provided that $\mathcal{S}$ is a monoid and $\sigma(T) = 0_{\mathcal{S}}$. In particular, if $\mathcal{S}$ is a monoid, we allow $T = \varepsilon$ to be empty and adopt the convention that $\sigma(\varepsilon) = 0_{\mathcal{S}}$. We say the sequence T is

- an *idempotent-sum sequence* if $\sigma(T)$ is an idempotent;
- an *idempotent-sum free sequence* if T contains no nonempty idempotent-sum subsequence.

It is worth remarking that when the commutative semigroup $\mathcal{S}$ is an abelian group, the notion *zero-sum sequence* and *idempotent-sum sequence* make no difference.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two commutative semigroups written additively with additions $+_{\mathcal{S}_1}$ and $+_{\mathcal{S}_2}$ respectively. The direct sum of $\mathcal{S}_1$ and $\mathcal{S}_2$, denoted $\mathcal{S}_1 \oplus \mathcal{S}_2$, is the commutative semigroup whose underlying set is the Cartesian product of the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ and whose binary operation + is given by

$$(a_1, a_2) + (b_1, b_2) = (a_1 +_{\mathcal{S}_1} b_1,\ \ a_2 +_{\mathcal{S}_2} b_2) \text{ where } a_1, b_1 \in \mathcal{S}_1; a_2, b_2 \in \mathcal{S}_2.$$

Let $\mathcal{S} = \mathrm{C}_{k_1;n_1} \oplus \mathrm{C}_{k_2;n_2}$, where the finite cyclic semigroup $\mathrm{C}_{k_i;n_i}$ is generated by g_i for each $i \in \{1, 2\}$. For any element a of $\mathcal{S}$ and each $i \in \{1, 2\}$, let $a(i)$ be the i-th component of a, i.e., $a = (a(1),\ a(2))$, and let $\mathrm{ind}_{g_i}(a(i))$ be the least positive integer t_i such that $t_i g_i = a(i)$. Let

$$G_{\mathcal{S}} = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$$

be the direct sum of two additive groups of integers modulo n_1 and n_2, which is the largest group contained in $\mathcal{S}$. Define a map $\psi : \mathcal{S} \to G_{\mathcal{S}}$ given by

$$\psi(a) \mapsto \left(\overline{\mathrm{ind}_{g_1}(a(1))},\quad \overline{\mathrm{ind}_{g_2}(a(2))}\right) \in G_{\mathcal{S}}$$

for any element $a \in \mathcal{S}$, where $\overline{\mathrm{ind}_{g_i}(a(i))}$ denotes the congruence class of the integer $\mathrm{ind}_{g_i}(a(i))$ modulo n_i. We extend ψ to the map $\Psi : \mathcal{F}(\mathcal{S}) \to \mathcal{F}(G_{\mathcal{S}})$ given by

$$\Psi : T \mapsto \mathop{\bullet}_{a|T} \psi(a)$$

for any sequence $T \in \mathcal{F}(\mathcal{S})$.

3. Proof of Theorem 1.1

We begin this section with two necessary lemmas.

Lemma 3.1. *([16], Chapter I, Lemma 5.7, Proposition 5.8, Corollary 5.9) Let $\mathcal{S} = C_{k;n}$ be a finite cyclic semigroup generated by the element x. Then $\mathcal{S} = \{x, \ldots, kx, (k+1)x, \ldots, (k+n-1)x\}$ with*

$$ix + jx = \begin{cases} (i+j)x, & \text{if } i+j \le k+n-1; \\ tx, & \text{if } i+j \ge k+n, \text{ where } k \le t \le k+n-1 \text{ and } t \equiv i+j \pmod{n}. \end{cases}$$

Moreover, there exists a unique idempotent, ℓx, in the cyclic semigroup $\langle x \rangle$, where

$$\ell \in [k, k+n-1] \text{ and } \ell \equiv 0 \pmod{n}.$$

By Lemma 3.1, it is easy to derive the following.

Lemma 3.2. *Let $\mathcal{S} = \mathrm{C}_{k_1;n_1} \oplus \mathrm{C}_{k_2;n_2}$ where $\mathrm{C}_{k_i;n_i} = \langle g_i \rangle$ for $i = 1, 2$. Then there exists a unique idempotent e in $\mathcal{S}$, where*

$$\mathrm{ind}_{g_i}(e(i)) \in [k_i, k_i + n_i - 1] \text{ and } \mathrm{ind}_{g_i}(e(i)) \equiv 0 \pmod{n_i}$$

for both $i = 1, 2$. In particular, a sequence $W \in \mathcal{F}(\mathcal{S})$ is an idempotent-sum sequence if, and only if, $\sum\limits_{a|W} \mathrm{ind}_{g_i}(a(i)) \geq \left\lceil \frac{k_i}{n_i} \right\rceil n_i$ and $\sum\limits_{a|W} \mathrm{ind}_{g_i}(a(i)) \equiv 0 \pmod{n_i}$ for both $i = 1, 2$.

Note that in Lemma 3.2, the condition that $\sum\limits_{a|W} \mathrm{ind}_{g_i}(a(i)) \equiv 0 \pmod{n_i}$ for both $i = 1, 2$ is equivalent to that $\Psi(W)$ is a zero-sum sequence in the group $G_{\mathcal{S}}$.

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Say $\mathrm{C}_{k_i;n_i} = \langle g_i \rangle$ for each $i \in \{1, 2\}$. Note that $G_{\mathcal{S}} \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \cong \mathbb{Z}_{\gcd(n_1,\ n_2)} \oplus \mathbb{Z}_{\mathrm{lcm}(n_1,\ n_2)}$. By Theorem A, we have that

$$\mathrm{D}(\mathrm{G}_{\mathcal{S}}) = \gcd(n_1,\ n_2) + \mathrm{lcm}(n_1,\ n_2) - 1. \tag{3.1}$$

Since $\mathrm{C}_{k_1;n_1} \oplus \mathrm{C}_{k_2;n_2} \cong \mathrm{C}_{k_2;n_2} \oplus \mathrm{C}_{k_1;n_1}$, we can assume without loss of generality that

$$(\left\lceil \frac{k_2}{n_2} \right\rceil - 1)n_2 = \max\left((\left\lceil \frac{k_1}{n_1} \right\rceil - 1)n_1,\ (\left\lceil \frac{k_2}{n_2} \right\rceil - 1)n_2\right). \tag{3.2}$$

Let $T \in \mathcal{F}(\mathcal{S})$ be an arbitrary sequence of length

$$|T| = (\left\lceil \frac{k_2}{n_2} \right\rceil - 1)n_2 + \gcd(n_1,\ n_2) + \mathrm{lcm}(n_1,\ n_2) - 1. \tag{3.3}$$

Take a nonempty subsequence L of T such that $\Psi(L)$ is a zero-sum sequence over the group $G_{\mathcal{S}}$, i.e.,

$$\sum_{a|L} \mathrm{ind}_{g_i}(a(i)) \equiv 0 \pmod{n_i} \text{ for each } i \in \{1, 2\}, \tag{3.4}$$

with $|L|$ being maximal. By the maximality of $|L|$, we have that $|T \cdot L^{[-1]}| \leq \mathrm{D}(\mathrm{G}_{\mathcal{S}}) - 1$. By (3.1), (3.2) and (3.3), we have that $\sum\limits_{a|L} \mathrm{ind}_{g_i}(a(i)) \geq |L| \geq (\left\lceil \frac{k_2}{n_2} \right\rceil - 1)n_2 + 1 \geq (\left\lceil \frac{k_i}{n_i} \right\rceil - 1)n_i + 1$, and thus $\sum\limits_{a|L} \mathrm{ind}_{g_i}(a(i)) \geq \left\lceil \frac{k_i}{n_i} \right\rceil n_i$ by (3.4), where $i \in \{1, 2\}$. By Lemma 3.2, we have that L is a nonempty idempotent-sum subsequence of T. By (3.2), (3.3) and the arbitrariness of choosing the sequence T, we conclude that

$$\mathrm{I}(\mathcal{S}) \leq \max\left((\left\lceil \frac{k_1}{n_1} \right\rceil - 1)n_1,\ (\left\lceil \frac{k_2}{n_2} \right\rceil - 1)n_2\right) + \gcd(n_1,\ n_2) + \mathrm{lcm}(n_1,\ n_2) - 1. \tag{3.5}$$

Now we assume one of Conditions (i) and (ii) holds. To prove $\mathrm{I}(\mathcal{S}) = \max\left((\left\lceil \frac{k_1}{n_1} \right\rceil - 1)n_1,\ (\left\lceil \frac{k_2}{n_2} \right\rceil - 1)n_2\right) + \gcd(n_1,\ n_2) + \mathrm{lcm}(n_1,\ n_2) - 1$, by (3.2) and (3.5) it suffices to show that there exists an idempotent-sum free sequence of terms from $\mathcal{S}$ with length exactly $(\left\lceil \frac{k_2}{n_2} \right\rceil - 1)n_2 + \gcd(n_1,\ n_2) + \mathrm{lcm}(n_1,\ n_2) - 2$.

Consider the case when Condition (i) $n_1 \mid n_2$ or $n_2 \mid n_1$ holds. Take two elements $\beta, \gamma \in \mathcal{S}$ with

$$\left(\mathrm{ind}_{g_1}(\beta(1)),\ \mathrm{ind}_{g_2}(\beta(2))\right) = (1,\ n_2) \tag{3.6}$$

and

$$\left(\mathrm{ind}_{g_1}(\gamma(1)),\ \mathrm{ind}_{g_2}(\gamma(2))\right) = (n_1,\ 1)\,. \tag{3.7}$$

Let

$$T_1 = \beta^{[n_1-1]} \cdot \gamma^{\left[\left\lceil \frac{k_2}{n_2}\right\rceil n_2-1\right]}.$$

Since $\gcd(n_1,\ n_2) + \mathrm{lcm}(n_1,\ n_2) = n_1 + n_2$, it follows that $|T_1| = (n_1 - 1) + (\left\lceil \frac{k_2}{n_2}\right\rceil n_2 - 1) = (\left\lceil \frac{k_2}{n_2}\right\rceil - 1)n_2 + n_1 + n_2 - 2 = (\left\lceil \frac{k_2}{n_2}\right\rceil - 1)n_2 + \gcd(n_1,\ n_2) + \mathrm{lcm}(n_1,\ n_2) - 2$. We need only to verify that the sequence T_1 is idempotent-sum free. Assume to the contrary that T_1 contains a nonempty idempotent-sum subsequence

$$U = \beta^{[t_1]} \cdot \gamma^{[t_2]} \tag{3.8}$$

where

$$t_1 \le n_1 - 1 \tag{3.9}$$

and

$$t_2 \le \left\lceil \frac{k_2}{n_2}\right\rceil n_2 - 1. \tag{3.10}$$

It follows from (3.6), (3.7), (3.8) and Lemma 3.2 that

$$t_1 + t_2 n_1 = t_1 \mathrm{ind}_{g_1}(\beta(1)) + t_2\ \mathrm{ind}_{g_1}(\gamma(1)) = \sum_{a|U} \mathrm{ind}_{g_1}(a(1)) \equiv 0 \pmod{n_1} \tag{3.11}$$

and

$$t_1 n_2 + t_2 = t_1 \mathrm{ind}_{g_2}(\beta(2)) + t_2\ \mathrm{ind}_{g_2}(\gamma(2)) = \sum_{a|U} \mathrm{ind}_{g_2}(a(2)) \ge \left\lceil \frac{k_2}{n_2}\right\rceil n_2. \tag{3.12}$$

By (3.9) and (3.11), we have $t_1 = 0$, and then combined with (3.10) and (3.12), we derive a contradiction, done.

Now we consider the case when Condition (ii) holds. Combined with (3.2), we assume that

$$\frac{n_1}{\gcd(n_1,\ n_2)} \mid \left\lceil \frac{k_2}{n_2}\right\rceil - 1. \tag{3.13}$$

Let

$$m_1 = \prod_{\substack{p \text{ is a prime divisor of } n_1 \\ \mathrm{pot}_p(n_1)<\mathrm{pot}_p(n_2)}} p^{\mathrm{pot}_p(n_1)} \tag{3.14}$$

and

$$m_2 = \prod_{\substack{p \text{ is a prime divisor of } n_2 \\ \mathrm{pot}_p(n_2)\le\mathrm{pot}_p(n_1)}} p^{\mathrm{pot}_p(n_2)},$$

where $\mathrm{pot}_p(n)$ denotes the largest integer h such that p^h divides n. Note that

$$m_1 m_2 = \gcd(n_1,\ n_2). \tag{3.15}$$

Take $b, c \in \mathcal{S}$ such that

$$\left(\mathrm{ind}_{g_1}(b(1)),\ \mathrm{ind}_{g_2}(b(2))\right) = (m_1,\ 1) \tag{3.16}$$

and

$$\left(\text{ind}_{g_1}(c(1)),\ \text{ind}_{g_2}(c(2))\right) = \left(\frac{n_1}{m_1},\ \frac{n_2}{\gcd(n_1,\ n_2)}\right). \tag{3.17}$$

Take the sequence

$$T_2 = b^{\left[(\left\lceil\frac{k_2}{n_2}\right\rceil-1)n_2+\frac{n_1n_2}{\gcd(n_1,\ n_2)}-1\right]} \cdot c^{[\gcd(n_1,\ n_2)-1]}.$$

We see that $|T_2| = (\left\lceil\frac{k_2}{n_2}\right\rceil - 1)n_2 + \frac{n_1n_2}{\gcd(n_1,\ n_2)} - 1 + \gcd(n_1,\ n_2) - 1 = (\left\lceil\frac{k_2}{n_2}\right\rceil - 1)n_2 + \text{lcm}(n_1,\ n_2) + \gcd(n_1,\ n_2) - 2$. To prove T_2 is idempotent-sum free, we assume to the contrary that T_2 contains a nonempty idempotent-sum subsequence V. Say

$$V = b^{[s]} \cdot c^{[t]} \tag{3.18}$$

with

$$s \le (\left\lceil\frac{k_2}{n_2}\right\rceil - 1)n_2 + \frac{n_1n_2}{\gcd(n_1,\ n_2)} - 1 \tag{3.19}$$

and

$$t \le \gcd(n_1,\ n_2) - 1. \tag{3.20}$$

By Lemma 3.2, (3.16), (3.17) and (3.18), we derive that

$$sm_1 + t\frac{n_1}{m_1} = \sum_{a|V} \text{ind}_{g_1}(a(1)) \equiv 0 \pmod{n_1} \tag{3.21}$$

and

$$s + t\frac{n_2}{\gcd(n_1,\ n_2)} = \sum_{a|V} \text{ind}_{g_2}(a(2)) \equiv 0 \pmod{n_2}, \tag{3.22}$$

and that $s + t\frac{n_2}{\gcd(n_1,\ n_2)} = \sum\limits_{a|V} \text{ind}_{g_2}(a(2)) \ge \left\lceil\frac{k_2}{n_2}\right\rceil n_2$, combined with (3.20), then

$$s > (\left\lceil\frac{k_2}{n_2}\right\rceil - 1)n_2. \tag{3.23}$$

By (3.14), we have $\gcd(m_1,\ \frac{n_1}{m_1}) = 1$, combined with (3.21), we have that

$$\frac{n_1}{m_1} \mid s \tag{3.24}$$

and that $m_1 \mid t$, combined with (3.15), (3.22), then

$$\frac{n_2}{m_2} \mid s. \tag{3.25}$$

Note that $\gcd(\frac{n_1}{m_1},\ \frac{n_2}{m_2}) = 1$. It follows from (3.15), (3.24) and (3.25) that

$$\frac{n_1n_2}{\gcd(n_1,\ n_2)} = \frac{n_1}{m_1}\frac{n_2}{m_2} \mid s. \tag{3.26}$$

By (3.13), we have $\frac{n_1n_2}{\gcd(n_1,\ n_2)} \mid (\left\lceil\frac{k_2}{n_2}\right\rceil - 1)n_2$. Combined with (3.19) and (3.23), we derive a contradiction to (3.26). This proves Theorem 1.1. □

4. Concluding remarks

In Theorem 1.1, by taking $k_1 = k_2 = 1$, both cyclic semigroups $C_{k_1;n_1}$ and $C_{k_2;n_2}$ reduce to $\mathbb{Z}_{n_1}$ and $\mathbb{Z}_{n_2}$ respectively, and thus

$$\mathcal{S} = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \cong \mathbb{Z}_{\gcd(n_1,\ n_2)} \oplus \mathbb{Z}_{\mathrm{lcm}(n_1,\ n_2)}.$$

We also see that $\lceil \frac{k_i}{n_i} \rceil - 1 = 0$ for both $i \in \{1, 2\}$ and Condition (ii) holds. By the conclusion of Theorem 1.1, we have that $D(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}) = I(\mathcal{S}) = \max\left((\lceil \frac{k_1}{n_1} \rceil - 1)n_1,\ (\lceil \frac{k_2}{n_2} \rceil - 1)n_2\right) + \gcd(n_1,\ n_2) + \mathrm{lcm}(n_1,\ n_2) - 1 = \gcd(n_1,\ n_2) + \mathrm{lcm}(n_1,\ n_2) - 1$. That is, Condition (ii) of Theorem 1.1 implies Kruyswijk-Olson Theorem as a consequence (Condition (i) deduce Kruyswijk-Olson Theorem clearly).

For a finite cyclic semigroup $C_{k;n}$, since $C_{k;n} \cong C_{1;1} \oplus C_{k;n}$ and Condition (i) holds for $C_{1;1} \oplus C_{k;n}$, by applying Theorem 1.1, we have that $I(C_{k;n}) = I(C_{1;1} \oplus C_{k;n}) = \lceil \frac{k}{n} \rceil n$.

We remark that there exists some direct sum of two finite cyclic semigroups for which the Erdős-Burgess constant is strictly less than that upper bound $\max\left((\lceil \frac{k_1}{n_1} \rceil - 1)n_1,\ (\lceil \frac{k_2}{n_2} \rceil - 1)n_2\right) + \gcd(n_1,\ n_2) + \mathrm{lcm}(n_1,\ n_2) - 1$ given in Theorem 1.1. For example, by a straightforward case distinction, we can show that any sequence over $C_{1;3} \oplus C_{3;2}$ of length 7 must contain a nonempty idempotent-sum subsequence, i.e., $I(C_{1;3} \oplus C_{3;2})$ is strictly less than that upper bound 8 given as Theorem 1.1. Therefore, we close this paper with the following conjecture.

Conjecture 4.1. *Let* $\mathcal{S} = C_{k_1;n_1} \oplus C_{k_2;n_2}$. *If* $I(\mathcal{S}) = \max\left((\lceil \frac{k_1}{n_1} \rceil - 1)n_1,\ (\lceil \frac{k_2}{n_2} \rceil - 1)n_2\right) + \gcd(n_1,\ n_2) + \mathrm{lcm}(n_1,\ n_2) - 1$ *then one of the following conditions holds.*

(i) $n_1 \mid n_2$ *or* $n_2 \mid n_1$*;*

(ii) there exists some $\epsilon \in \{1, 2\}$ *such that* $(\lceil \frac{k_\epsilon}{n_\epsilon} \rceil - 1)n_\epsilon = \max\left((\lceil \frac{k_1}{n_1} \rceil - 1)n_1,\ (\lceil \frac{k_2}{n_2} \rceil - 1)n_2\right)$ *and* $\frac{n_{3-\epsilon}}{\gcd(n_1,\ n_2)}$ *divides* $\lceil \frac{k_\epsilon}{n_\epsilon} \rceil - 1$.

Acknowledgments

This work is supported by NSFC (grant no. 11971347, 11501561).

Conflict of interest

The author declares no conflicts of interest.

References

1. W. R. Alford, A. Granville, C. Pomerance, *There are infinitely many Carmichael numbers*, Ann. Math., **140** (1994), 703–722.
2. N. Alon, S. Friedland, G. Kalai, *Regular subgraphs of almost regular graphs*, J. Combin. Theory Ser. B, **37** (1984), 79–91.
3. G. Bhowmik, J.-C. Schlage-Puchta, *Davenport's constant for groups of the form* $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$, In: Granville A., Nathanson M.B., Solymosi J. (eds) Additive Combinatorics, CRM Proc. Lecture Notes, 43, pp. 307–326, Am. Math. Soc., 2007.
4. D. A. Burgess, *A problem on semi-groups*, Studia Sci. Math. Hungar., **4** (1969), 9–11.

5. K. Cziszter, M. Domokos, A. Geroldinger, *The Interplay of Invariant Theory with Multiplicative Ideal Theory and with Arithmetic Combinatorics*. In: Chapman S., Fontana M., Geroldinger A., Olberding B. (eds) Multiplicative Ideal Theory and Factorization Theory, Springer Proceedings in Mathematics & Statistics, Springer, Cham, 2016.
6. C. Deng, *Davenport constant for commutative rings*, J. Number Theory, **172** (2017), 321–342.
7. P. van Emde Boas, *A combinatorial problem on finite abelian groups 2*, Report ZW-1969-007, Mathematical Centre, Amsterdam, 1969.
8. P. van Emde Boas, D. Kruyswijk, *A combinatorial problem on finite abelian groups 3*, Report ZW 1969-008, Stichting Math. Centrum, Amsterdam, 1969.
9. P. Erdős, A. Ginzburg, A. Ziv, *Theorem in the additive number theory*, Bull. Res. Council Israel F, **10** (1961), 41–43.
10. W. Gao, *On Davenport's constant of finite abelian groups with rank three,* Discrete Math., **222** (2000), 111–124.
11. W. Gao, A. Geroldinger, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math., **24** (2006), 337–369.
12. A. Geroldinger, *Additive Group Theory and Non-unique Factorizations*. In: A. Geroldinger and I. Ruzsa (Eds.), Combinatorial Number Theory and Additive Group Theory (Advanced Courses in Mathematics-CRM Barcelona), Birkhäuser, Basel, 2009.
13. A. Geroldinger, F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Chapman & Hall/CRC, 2006.
14. A. Geroldinger, M. Liebmann, A. Philipp, *On the Davenport constant and on the structure of extremal sequences*, Period. Math. Hungar., **64** (2012), 213–225.
15. D. W. H. Gillam, T. E. Hall, N. H. Williams, *On finite semigroups and idempotents*, Bull. London Math. Soc., **4** (1972), 143–144.
16. P. A. Grillet, *Commutative Semigroups*, Kluwer Academic Publishers, 2001.
17. D. J. Grynkiewicz, *Structural Additive Theory*, Developments in Mathematics, Springer, Cham, 2013.
18. C. Liu, *On the lower bounds of Davenport constant*, J. Combin. Theory Ser. A, **171** (2020).
19. L. E. Marchan, O. Ordaz, I. Santos, et al. *Multi-wise and constrained fully weighted Davenport constants and interactions*, J. Combin. Theory Ser. A, **135** (2015), 237–267.
20. R. Meshulam, *An uncertainty inequality and zero subsums*, Discrete Math., **84** (1990), 197–200.
21. J. E. Olson, *A Combinatorial Problem on Finite Abelian Groups, I*, J. Number Theory, **1** (1969), 8–10.
22. J. E. Olson, *A combinatorial problem on finite abelian groups II*, J. Number Theory, **1** (1969), 195–199.
23. A. Plagne, W. A. Schmid, *An application of coding theory to estimating Davenport constants*, Des. Codes Cryptogr., **61** (2011), 105–118.
24. W. A. Schmid, *The inverse problem associated to the Davenport constant for $C_2 \oplus C_2 \oplus C_{2n}$ and applications to the arithmetical characterization of class groups*, Electron. J. Comb., **18** (2011).

25. G. Wang, *Davenport constant for semigroups II*, J. Number Theory, **153** (2015), 124–134.
26. G. Wang, *Additively irreducible sequences in commutative semigroups*, J. Combin. Theory Ser. A, **152** (2017), 380–397.
27. G. Wang, *Structure of the largest idempotent-product free sequences in semigroups*, J. Number Theory, **195** (2019), 84–95.
28. H. Wang, L. Zhang, Q. Wang, et al. *Davenport constant of the multiplicative semigroup of the quotient ring* $\frac{F_p[x]}{\langle f(x)\rangle}$, Int. J. Number Theory, **12** (2016), 663–669.
29. L. Zhang, H. Wang, Y. Qu, *A problem of Wang on Davenport constant for the multiplicative semigroup of the quotient ring of* $F_2[x]$, Colloq. Math., **148** (2017), 123–130.

AIMS Press