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Research article

Refinements of Huygens- and Wilker- type inequalities

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Abstract: In this paper we give some refinements and sharpness of the Huygens- and Wilker- type inequalities, and show a proof of the second conjecture by Chen and Chueng in [10].

Keywords: circular functions; hyperbolic functions; refinements and sharpness of the Huygens- and Wilker- type inequalities; proof of the second conjecture by Chen and Chueng
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1. Introduction

For $0 < x < \pi/2$, the famous Huygens inequality (see [1, 2]) and Wilker inequality (see [3]) are known as follows:

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3, \tag{1.1}$$

and

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{1.2}$$

Yang and Chu [4], Chu et al [5], Sun, Yang and Chu [6] studied some combinations of inequalities (1.1) and (1.2) with parameters and came to some good conclusions.

Neuman established the following three inequalities in [7]:

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 3, \; x > 0, \tag{1.3}$$

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 2\frac{x}{\sin x} + \frac{x}{\tan x} > 3, \; 0 < x < \frac{\pi}{2}, \tag{1.4}$$

and

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}>2\frac{x}{\sinh x}+\frac{x}{\tanh x}>3,\ x>0. \tag{1.5}$$

Mortici improved inequality (1.3) in [2] as follows.

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}>3+\frac{3}{20}x^4-\frac{3}{56}x^6,\ x>0. \tag{1.6}$$

The Wilker inequality was refined by some researchers. Sumner et al. [8] affirmed the truth of the Problems presented by Wilker [3] and obtained a further results as follows:

Theorem 1.1. *If* $0<x<\pi/2$, *then*

$$\frac{16}{\pi^4}x^3\tan x<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2<\frac{8}{45}x^3\tan x. \tag{1.7}$$

Furthermore, $16/\pi^4$ *and* $8/45$ *are the best constants in (1.7).*

In 2012, Neuman proposed us two inequality chains in [7] as:

$$2\frac{\sin x}{x}+\frac{\tan x}{x}>\frac{\sin x}{x}+2\frac{\tan(x/2)}{x/2}>2\frac{x}{\sin x}+\frac{x}{\tan x}>3, \tag{1.8}$$

and

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}$$

$$>\frac{\sin x}{x}+\left[\frac{\tan(x/2)}{x/2}\right]^2>\frac{x/2}{\sin(x/2)}+\left[\frac{x/2}{\tan(x/2)}\right]^2>2. \tag{1.9}$$

The hyperbolic counterparts of the right two inequalities of (1.9) were also given in [7] as follows:

$$\frac{\sinh x}{x}+\left[\frac{\tanh(x/2)}{x/2}\right]^2>\frac{x}{\sinh x}+\left[\frac{x/2}{\tanh(x/2)}\right]^2>2. \tag{1.10}$$

Then in 2014, Jiang et al. [9] established some new Huygens- and Wilker- type inequalities described as the following Theorems 1.2 – 1.5:

Theorem 1.2. *Let* $0<|x|<\pi/2$. *Then*

$$3+\frac{1}{60}x^3\sin x<2\frac{x}{\sin x}+\frac{x}{\tan x}<3+\frac{8\pi-24}{\pi^3}x^3\sin x, \tag{1.11}$$

$1/60$ *and* $(8\pi-24)/\pi^3$ *are the best constants in (1.11).*

Theorem 1.3. *Let* $0<|x|<\pi/2$. *Then*

$$2+\frac{17}{720}x^3\sin x<\frac{x}{\sin x}+\left[\frac{x/2}{\tan(x/2)}\right]^2<2+\frac{\pi^2+8\pi-32}{2\pi^3}x^3\sin x, \tag{1.12}$$

the constants $17/720$ *and* $(\pi^2+8\pi-32)/(2\pi^3)$ *in (1.12) are the best possible.*

Theorem 1.4. *Let* $x > 0$. *Then*

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 3 + \frac{3}{20}x^3 \tanh x, \tag{1.13}$$

the constant $3/20$ *in (1.13) is the best possible.*

Theorem 1.5. *Let* $x > 0$. *Then*

$$\frac{\sinh x}{x} + \left[\frac{\tanh(x/2)}{x/2}\right]^2 > 2 + \frac{23}{720}x^3 \tanh x, \tag{1.14}$$

the constant $23/720$ *in (1.14) is the best possible.*

Lately, Chen and Chueng posed three conjectures for Wilker- type inequalities in [10]. The aim of this paper is to give some refinements and sharpness of the above Huygens- and Wilker- type inequalities, and to show a proof of the second conjecture by Chen and Chueng in [10].

2. Lemmas

In order to prove the main conclusions of this paper, we need the following lemmas.

Lemma 2.1 ([13])**.** *Let* $0 < |x| < \pi$. *Then*

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1}.$$

Lemma 2.2 ([11, 12])**.** *Let* $0 < |x| < \pi$. *Then*

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)|B_{2n}|}{(2n)!}x^{2n-1}.$$

Lemma 2.3 ([12, 14–18])**.** *For all integers* $n \geq 1$, *let* B_{2n} *be the even-indexed Bernoulli numbers. Then the double inequality*

$$\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{-2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{1-2n}} \tag{2.1}$$

holds.

3. Main results and their proofs

Now let us state and prove some results about the refinements and sharpness of the Huygens- and Wilker- type inequalities.

Theorem 3.1. *For* $x > 0$, *we have*

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} < 3 + \frac{3}{20}x^3 \sinh x, \tag{3.1}$$

and $3/20$ *is the best constant in (3.1).*

Proof. We know that (3.1) is equivalent to

$$3x\cosh x + \frac{3}{40}x^4\sinh 2x > \sinh 2x + \sinh x.$$

Let

$$F_1(x) \equiv 3x\cosh x + \frac{3}{40}x^4\sinh 2x - (\sinh 2x + \sinh x), \ \ x > 0.$$

Then by the expansions of power series of hyperbolic sine and hyperbolic cosine functions we can get

$$\begin{aligned} F_1(x) &= 3x\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} + \frac{3}{40}x^4\sum_{n=0}^{\infty}\frac{(2x)^{2n+1}}{(2n+1)!} - \left[\sum_{n=0}^{\infty}\frac{(2x)^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\right] \\ &= \sum_{n=3}^{\infty}\frac{a_n}{10(2n+1)!}x^{2n+1}, \end{aligned}$$

where

$$a_n = \left(12n^4 - 12n^3 - 3n^2 + 3n - 160\right)2^{2n-3} + 20(3n+1) > 0$$

for $n \geq 3$. So $F_1(x) > 0$ holds for all $x \in (0,\infty)$, which leads to (3.1). At the same time,

$$\lim_{x\to 0^+}\frac{2\frac{\sinh x}{x} + \frac{\tanh x}{x} - 3}{x^3\sinh x} = \frac{3}{20},$$

the proof of Theorem 3.1 is completed.

Theorem 3.2. *For $x > 0$, we have*

$$\frac{\sinh x}{x} + \left[\frac{\tanh(x/2)}{x/2}\right]^2 < 2 + \frac{23}{720}x^3\sinh x, \tag{3.2}$$

and the constant $23/720$ is the best possible in (3.2).

Proof. If $x = 2t$, then inequality (3.2) is equivalent to

$$\frac{\sinh 2t}{2t} + \left(\frac{\sinh t}{t\cosh t}\right)^2 < 2 + \frac{23}{720}(2t)^3\sinh 2t,$$

or

$$180t\sinh 2t\cosh^2 t + 360\sinh^2 t < 720t^2\cosh^2 t + 92t^5\sinh 2t\cosh^2 t.$$

Let

$$\begin{aligned} F_2(t) &\equiv 720t^2\cosh^2 t + 92t^5\sinh 2t\cosh^2 t - \left(180t\sinh 2t\cosh^2 t + 360\sinh^2 t\right) \\ &= 360t^2\cosh 2t - 180\cosh 2t + 46t^5\sinh 2t + 23t^5\sinh 4t - 90t\sinh 2t \\ &\quad -45t\sinh 4t + 360t^2 + 180. \end{aligned}$$

Then from the expansions of power series of hyperbolic sine and hyperbolic cosine functions, we can obtain

$$F_2(t) = \sum_{n=3}^{\infty}\frac{2^{2n-3}b_n}{(2n+2)!}t^{2n+2},$$

where

$$\begin{aligned} b_n &= (n+1)\left(92n^4 - 92n^3 - 23n^2 + 23n - 2880\right)2^{2n} + 16\left(92n^5 - 115n^3 + 720n^2 + 923n - 180\right) \\ &> 0 \end{aligned}$$

for $n \geq 3$. So $F_2(t) > 0$ holds for all $t \in (0, \infty)$, which leads to (3.2). At the same time,

$$\lim_{x\to 0^+} \frac{\frac{\sinh x}{x} + \left[\frac{\tanh(x/2)}{x/2}\right]^2 - 2}{x^3 \sinh x} = \frac{23}{720},$$

the proof of Theorem 3.2 is completed now.

Theorem 3.3. *For $x > 0$, we have*

$$\frac{x}{\sinh x} + \left[\frac{x/2}{\tanh(x/2)}\right]^2 < 2 + \frac{119}{5040} x^3 \sinh x, \tag{3.3}$$

and the constant $119/5040$ is the best possible in (3.3).

Proof. For $x = 2t$, inequality (3.3) is equivalent to

$$720t \sinh t + 720t^2 \cosh^3 t < 1440 \sinh^2 t \cosh t + 136t^3 \sinh 2t \sinh^2 t \cosh t,$$

we can let

$$\begin{aligned} F_3(t) &= 1440 \sinh^2 t \cosh t + 136t^3 \sinh 2t \sinh^2 t \cosh t - \left(720t \sinh t + 720t^2 \cosh^3 t\right) \\ &= 360 \cosh 3t - 360 \cosh t - 180t^2 \cosh 3t - 17t^3 \sinh 3t + 17t^3 \sinh 5t \\ &\quad -720t \sinh t - 540t^2 \cosh t - 34t^3 \sinh t. \end{aligned}$$

By the expansions of power series of hyperbolic sine and hyperbolic cosine functions, we can obtain

$$F_3(t) = \sum_{n=3}^{\infty} \frac{4c_n}{(2n+2)!} t^{2n+2},$$

where

$$\begin{aligned} c_n &= 17n(2n+1)(n+1)5^{2n-1} - \left(34n^3 + 591n^2 + 827n - 2160\right)3^{2n-1} \\ &\quad -2\left(34n^3 + 321n^2 + 602n + 360\right). \end{aligned}$$

We can calculate

$$\begin{aligned} c_3 &= 2856\,960 > 0, \\ c_4 &= 211\,092\,480 > 0, \\ c_5 &= 10\,543\,656\,960 > 0, \\ c_6 &= 447\,656\,297\,472 > 0. \end{aligned}$$

For $n \geq 7$, from binomial expansion, we can get

$$\begin{aligned}
c_n &= 17n(2n+1)(n+1)(3+2)^{2n-1} - \left(34n^3 + 591n^2 + 827n - 2160\right)3^{2n-1} \\
&\quad -2\left(34n^3 + 321n^2 + 602n + 360\right) \\
&= 17n(2n+1)(n+1)\left[3^{2n-1} + (2n-1)3^{2n-2}\cdot 2 + \cdots + (2n-1)3\cdot 2^{2n-2} + 2^{2n-1}\right] \\
&\quad -\left(34n^3 + 591n^2 + 827n - 2160\right)3^{2n-1} - 2\left(34n^3 + 321n^2 + 602n + 360\right) \\
&> 17n(2n+1)(n+1)\left[3^{2n-1} + (2n-1)3^{2n-2}\cdot 2\right] - \left(34n^3 + 591n^2 + 827n - 2160\right)3^{2n-1} \\
&\quad +\left[(2n-1)3\cdot 2^{2n-2} + 2^{2n-1}\right] - 2\left(34n^3 + 321n^2 + 602n + 360\right) \\
&= 2\left(68n^4 + 68n^3 - 827n^2 - 1232n + 3240\right)3^{2n-2} \\
&\quad +\frac{1}{4}\left[2^{2n}(6n-1) - 8\left(34n^3 + 321n^2 + 602n + 360\right)\right].
\end{aligned}$$

Obviously, $68n^4 + 68n^3 - 827n^2 - 1232n + 3240 > 0$ for $n \geq 7$. So we can complete the proof of $c_n > 0$ for $n \geq 7$ when proving

$$2^{2n} > \frac{8\left(34n^3 + 321n^2 + 602n + 360\right)}{6n-1}. \tag{3.4}$$

By mathematical induction we can prove the inequality (3.4). First, The inequality (3.4) is obviously true for $n = 7$. Let's assume that (3.4) holds for $n = m$, that is,

$$2^{2m} > \frac{8\left(34m^3 + 321m^2 + 602m + 360\right)}{6m-1}$$

holds. In the following we shall prove that (3.4) holds for $n = m + 1$. Since

$$2^{2m+2} = 4\cdot 2^{2m} > 4\cdot\frac{8\left(34m^3 + 321m^2 + 602m + 360\right)}{6m-1},$$

we can complete the proof of (3.4) as long as

$$4\left[\frac{8\left(34m^3 + 321m^2 + 602m + 360\right)}{6m-1}\right] > \frac{8\left(34(m+1)^3 + 321(m+1)^2 + 602(m+1) + 360\right)}{6(m+1)-1},$$

that is,

$$\frac{A}{B} =: \frac{4\left(34m^3 + 321m^2 + 602m + 360\right)}{6m-1} > \frac{34m^3 + 423m^2 + 1346m + 1317}{6m+5} := \frac{C}{D}.$$

In fact,

$$AD - BC = 3\left(204m^4 + 1960m^3 + 4405m^2 + 4708m + 2839\right) > 0.$$

So $c_n > 0$ holds for all $n \geq 3$ and $F_3(t) > 0$ holds for all $t \in (0, \infty)$, which leads to (3.3). At the same time,

$$\lim_{x\to 0^+}\frac{\frac{x}{\sinh x} + \left[\frac{x/2}{\tanh(x/2)}\right]^2 - 2}{x^3\sinh x} = \frac{119}{5040},$$

the proof of Theorem 3.3 is completed.

At the end of this section, via the Lemma 2.1 and Lemma 2.2 we can obtain the following result:

$$\begin{aligned}2\frac{x}{\sin x}+\frac{x}{\tan x} &= 2\left[1+\sum_{n=1}^{\infty}\frac{2(2^{2n-1}-1)|B_{2n}|}{(2n)!}x^{2n}\right]+x\left[\frac{1}{x}-\sum_{n=1}^{\infty}\frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1}\right]\\ &= 3+\sum_{n=2}^{\infty}\frac{2^{2n}-4}{(2n)!}|B_{2n}|x^{2n}\\ &> 3+\frac{1}{60}x^4+\frac{1}{504}x^6+\frac{1}{4800}x^8+\ldots+\frac{2^{2n}-4}{(2n)!}|B_{2n}|x^{2n}.\end{aligned}$$

Motivated by the result above, we can establish the following double inequality:

Theorem 3.4. *For* $0<x<\pi/2$*, we have*

$$3+\frac{1}{60}x^4+\frac{1}{504}x^5\sin x<2\frac{x}{\sin x}+\frac{x}{\tan x}<3+\frac{1}{60}x^4+\frac{960(\pi-3)-\pi^4}{30\pi^6}x^5\sin x, \tag{3.5}$$

and the constants $1/504$ *and* $(960(\pi-3)-\pi^4)/(30\pi^6)$ *are the best possible in (3.5).*

Proof. Let

$$F_4(x)=\frac{2\frac{x}{\sin x}+\frac{x}{\tan x}-3-\frac{1}{60}x^4}{x^5\sin x}=\frac{2}{x^4\sin^2 x}+\frac{\cos x}{x^4\sin^2 x}-\frac{3}{x^5\sin x}-\frac{1}{60x\sin x}.$$

By Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}\frac{1}{\sin^2 x} &= -(\cot x)'=\frac{1}{x^2}+\sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)|B_{2n}|}{(2n)!}x^{2n-2},\\ \frac{\cos x}{\sin^2 x} &= -\left(\frac{1}{\sin x}\right)'=\frac{1}{x^2}-\sum_{n=1}^{\infty}\frac{2(2n-1)(2^{2n-1}-1)|B_{2n}|}{(2n)!}x^{2n-2},\end{aligned}$$

then

$$\begin{aligned}F_4(x) &= \frac{2}{x^4}\left[\frac{1}{x^2}+\sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)|B_{2n}|}{(2n)!}x^{2n-2}\right]\\ &\quad+\frac{1}{x^4}\left[\frac{1}{x^2}-\sum_{n=1}^{\infty}\frac{2(2n-1)(2^{2n-1}-1)|B_{2n}|}{(2n)!}x^{2n-2}\right]\\ &\quad-\frac{3}{x^5}\left[\frac{1}{x}+\sum_{n=1}^{\infty}\frac{2(2^{2n-1}-1)|B_{2n}|}{(2n)!}x^{2n-1}\right]-\frac{1}{60x}\left[\frac{1}{x}+\sum_{n=1}^{\infty}\frac{2(2^{2n-1}-1)|B_{2n}|}{(2n)!}x^{2n-1}\right]\\ &= \sum_{n=1}^{\infty}\frac{2n\cdot 2^{2n+4}+4n+12}{(2n+4)!}|B_{2n+4}|x^{2n-2}-\sum_{n=1}^{\infty}\frac{2(2^{2n-1}-1)}{60\cdot(2n)!}|B_{2n}|x^{2n-2}\\ &= \sum_{n=1}^{\infty}u_n x^{2n-2}\end{aligned}$$

with $u_1 = 1/504$ and for $n \geq 2$,

$$u_n = \frac{2n \cdot 2^{2n+4} + 4n + 12}{(2n+4)!}|B_{2n+4}| - \frac{2(2^{2n-1}-1)}{60 \cdot (2n)!}|B_{2n}|.$$

By Lemma 2.3 we obtain

$$\begin{aligned} u_n &> \frac{2n \cdot 2^{2n+4} + 4n + 12}{(2n+4)!} \frac{2(2n+4)!}{(2^{2n+4}-1)\pi^{2n+4}} - \frac{2(2^{2n-1}-1)}{60 \cdot (2n)!} \frac{2(2n)!}{(2^{2n-2}-2)\pi^{2n}} \\ &= \frac{(4n-\pi^2)2^{4n+4} - (120n - 129\pi^2 - 24)2^{2n} - 2\left(8n + 24 + \pi^2\right)}{30\pi^{2n+4}(2^{2n+4}-1)(2^{2n}-2)} \\ &: = \frac{p(n)}{15\pi^{2n+4}(2^{2n+4}-1)(2^{2n}-2)}, \end{aligned}$$

where

$$p(n) = 16(4n-\pi^2)2^{4n} - (120n - 129\pi^2 - 24)2^{2n} - 2\left(8n + 24 + \pi^2\right)$$

for $n \geq 2$. Since

$$p(2) = 29\,232 - 2034\pi^2 > 0, \; p(3) = 764\,832 - 57\,282\pi^2 > 0,$$

we can obtain that $p(n) > 0$ for $n \geq 2$ when proving

$$2^{2n} > \frac{2\left(8n + 24 + \pi^2\right)}{16(4n-\pi^2)2^{2n} - (120n - 129\pi^2 - 24)} \tag{3.6}$$

holds for all $n \geq 4$. First, The inequality (3.6) is obviously true for $n = 4$. Let's assume that (3.6) holds for $n = m$, that is,

$$2^{2m} > \frac{2\left(8m + 24 + \pi^2\right)}{16(4m-\pi^2)2^{2m} - (120m - 129\pi^2 - 24)}$$

holds. In the following we shall prove that (3.6) holds for $n = m + 1$. Since

$$2^{2(m+1)} = 4 \cdot 2^{2m} > 4 \cdot \frac{2\left(8m + 24 + \pi^2\right)}{16(4m-\pi^2)2^{2m} - (120m - 129\pi^2 - 24)},$$

we can complete the proof of (3.6) as long as

$$\begin{aligned} &4 \cdot \frac{2\left(8m + 24 + \pi^2\right)}{16(4m-\pi^2)2^{2m} - (120m - 129\pi^2 - 24)} \\ &> \frac{2\left(8(m+1) + 24 + \pi^2\right)}{16(4(m+1)-\pi^2)2^{2m+2} - (120(m+1) - 129\pi^2 - 24)} \end{aligned}$$

or

$$2^{2m} > \frac{2880m^2 - \left(2736\pi^2 - 10\,944\right)m - \left(7848\pi^2 + 387\pi^4 - 9984\right)}{7680m^2 + (30\,720 - 960\pi^2)\,m - (4608\pi^2 + 240\pi^4 - 24\,576)} \tag{3.7}$$

holds for $m \geq 4$.

We find that (3.7) is obviously true for $m = \overline{4, 10}$. We continue to prove (3.7) holds for $m \geq 10$ by mathematical induction. Let's say (3.7) is true when $m = n$, that is,

$$2^{2n} > \frac{2880n^2 - \left(2736\pi^2 - 10\,944\right)n - \left(7848\pi^2 + 387\pi^4 - 9984\right)}{7680n^2 + (30\,720 - 960\pi^2)\,n - (4608\pi^2 + 240\pi^4 - 24\,576)}.$$

Since

$$2^{2(n+1)} = 4 \cdot 2^{2n} > 4 \cdot \frac{2880n^2 - \left(2736\pi^2 - 10\,944\right)n - \left(7848\pi^2 + 387\pi^4 - 9984\right)}{7680n^2 + (30\,720 - 960\pi^2)\,n - (4608\pi^2 + 240\pi^4 - 24\,576)},$$

we complete the proof of (3.7) as long as

$$\begin{aligned} \frac{a}{b} \quad : &= \frac{4\left[2880n^2 - \left(2736\pi^2 - 10\,944\right)n - \left(7848\pi^2 + 387\pi^4 - 9984\right)\right]}{7680n^2 + (30\,720 - 960\pi^2)\,n - (4608\pi^2 + 240\pi^4 - 24\,576)} \\ &> \frac{2880\,(n+1)^2 - \left(2736\pi^2 - 10\,944\right)(n+1) - \left(7848\pi^2 + 387\pi^4 - 9984\right)}{7680\,(n+1)^2 + (30\,720 - 960\pi^2)\,(n+1) - (4608\pi^2 + 240\pi^4 - 24\,576)} := \frac{c}{d}. \end{aligned}$$

In fact,

$$\begin{aligned} & ad - bc \\ = \; & 66\,355\,200\,(n-10)^4 + \left(3304\,488\,960 - 71\,331\,840\pi^2\right)(n-10)^3 \\ & + \left(61\,604\,167\,680 - 2796\,871\,680\pi^2 - 3110\,400\pi^4\right)(n-10)^2 \\ & + \left(3084\,480\pi^6 - 59\,844\,096\pi^4 - 36\,463\,509\,504\pi^2 + 509\,612\,654\,592\right)(n-10) \\ & + 42\,674\,688\pi^6 - 253\,228\,032\pi^4 - 158\,109\,253\,632\pi^2 + 278\,640\pi^8 + 1578\,576\,642\,048 \\ > \; & 0 \end{aligned}$$

holds for all $n \geq 10$.

So $u_n > 0$ for all $n \geq 1$ and $F(x)$ is increasing for $0 < x < \pi/2$. In addition to the conclusion $\lim_{x\to 0^+} F_4(x) = 1/504$ and $\lim_{x\to(\pi/2)^-} F_4(x) = (960(\pi - 3) - \pi^4)/(30\pi^6)$, the proof of Theorem 3.4 is completed.

4. Proof of of the second conjecture by Chen and Chueng

Chen and Chueng posed three conjectures in paper [10] and now we give the rigorous proof of the second conjecture described as Theorem 4.1.

Theorem 4.1. *For $0 < x < \pi/2$ and $n \geq 1$,*

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < 2 + \sum_{k=2}^{n} \frac{(k-1)2^{2k+1}|B_{2k}|}{(2k)!} x^{2k} + \frac{n \cdot 2^{2n+3}|B_{2n+2}|}{(2n+2)!} x^{2n+1} \tan x$$

holds.

Proof. Let

$$
\begin{aligned}
F_5(x) &= 2+\sum_{k=2}^{n}\frac{(k-1)2^{2k+1}|B_{2k}|}{(2k)!}x^{2k}+\frac{n\cdot 2^{2n+3}|B_{2n+2}|}{(2n+2)!}x^{2n+1}\tan x-\left[\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}\right]\\
&= \frac{n\cdot 2^{2n+3}|B_{2n+2}|}{(2n+2)!}x^{2n+1}\tan x-\sum_{k=n+1}^{\infty}\frac{(k-1)2^{2k+1}|B_{2k}|}{(2k)!}x^{2k}\\
&= \frac{n\cdot 2^{2n+3}|B_{2n+2}|}{(2n+2)!}x^{2n+1}\sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}x^{2k-1}-\sum_{k=n+1}^{\infty}\frac{(k-1)2^{2k+1}|B_{2k}|}{(2k)!}x^{2k}\\
&= \frac{n\cdot 2^{2n+3}|B_{2n+2}|}{(2n+2)!}\sum_{k=n+1}^{\infty}\frac{2^{2k-2n}(2^{2k-2n}-1)|B_{2k-2n}|}{(2k-2n)!}x^{2k}-\sum_{k=n+1}^{\infty}\frac{(k-1)2^{2k+1}|B_{2k}|}{(2k)!}x^{2k}\\
&= v_{n+1}x^{2n+2}+\sum_{k=n+2}^{\infty}v_k x^{2k},
\end{aligned}
$$

where

$$v_{n+1}=\frac{4n\cdot 2^{2n+3}|B_{2n+2}|}{(2n+2)!},$$

and

$$v_k=\frac{n\cdot 2^{2n+3}|B_{2n+2}|}{(2n+2)!}\frac{2^{2k-2n}(2^{2k-2n}-1)|B_{2k-2n}|}{(2k-2n)!}-\frac{(k-1)2^{2k+1}|B_{2k}|}{(2k)!}$$

for all $n\geq 3$ and $k\geq n+2$.

By the Lemma 2.3 we have

$$
\begin{aligned}
v_k &> \frac{n\cdot 2^{2n+3}|B_{2n+2}|}{(2n+2)!}\frac{2^{2k-2n}(2^{2k-2n}-1)}{(2k-2n)!}\frac{2(2k-2n)!}{(2\pi)^{2k-2n}}\cdot\frac{1}{1-2^{-2k+2n}}\\
&\quad -\frac{(k-1)2^{2k+1}}{(2k)!}\frac{2(2k)!}{(2\pi)^{2k}}\cdot\frac{1}{1-2^{1-2k}}\\
&= \frac{2^{2k+2}}{\pi^{2k}}\left[\frac{n\cdot 2^{2n+2}|B_{2n+2}|}{(2n+2)!}\cdot\frac{\pi^{2n}}{2^{2n}}-\frac{2(k-1)}{2^{2k}-2}\right]\\
&> \frac{2^{2k+2}}{\pi^{2k}}\left[\frac{n\cdot 2^{2n+2}}{(2n+2)!}\cdot\frac{\pi^{2n}}{2^{2n}}\cdot\frac{2(2n+2)!}{(2\pi)^{2n+2}}\frac{1}{1-2^{-2n-2}}-\frac{2(k-1)}{2^{2k}-2}\right]\\
&= \frac{2^{2k+2}}{\pi^{2k}}\left[\frac{8n}{\pi^2(2^{2n+2}-1)}-\frac{2(k-1)}{2^{2k}-2}\right]\\
&> \frac{2^{2k+2}}{\pi^{2k}}\left[\frac{8n}{\pi^2(2^{2n+2}-1)}-\frac{2(n+1)}{2^{2n+4}-2}\right]\\
&= \frac{2^{2k+2}}{\pi^{2k}}\left[\frac{\left[(108-8\pi^2)n-8\pi^2\right]2^{2n}+2(n+1)\pi^2-16n}{\pi^2(2^{2n+2}-1)(^{2n+4}-2)}\right]\\
&> 0
\end{aligned}
$$

for all $n\geq 3$ and $k\geq n+2$.

So the proof of Theorem 4.1 is completed.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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