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Research article

On the dissipative solutions for the inviscid Boussinesq equations

Feng Cheng*

Hubei Key Laboratory of Applied Mathematics, Faculty of Mathematics and Statistics, Hubei University, Wuhan 430062, P. R. China

* **Correspondence:** Email: fengcheng@hubu.edu.cn.

Abstract: In this paper, we study the dissipative solutions for the inviscid Boussinesq equations. It is shown that there is at least one dissipative solution for the inviscid incompressible Boussinesq equations. Moreover, if there is an unique strong solution then the dissipative solutions must coincide with the strong solution.

Keywords: Boussinesq equations; dissipative solution; weak solution
Mathematics Subject Classification: 35A01, 35Q35, 76B03

1. Introduction

We consider the following N-dimensional inviscid Boussinesq equations on the N-dimensional torus $\mathbb{T}^N$,

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = \theta e_N, \\ \partial_t\theta + u\cdot\nabla\theta = 0, \\ \operatorname{div} u = 0, \\ u(0,x) = u_0(x), \quad \theta(0,x) = \theta_0(x), \end{cases} \tag{1.1}$$

where $u(t,x) = (u_1,\dots,u_N)(t,x)$ represents the vector velocity field, $\theta(t,x)$ the scalar temperature, $p(t,x)$ the scalar pressure, and $e_N = (0,\dots,0,1)$ the unit normal vector. The unknown functions are supposed to satisfy the periodic boundary conditions, and we remark that the results we reach below can be extended or adapted to the case of whole space $\mathbb{R}^N$ or to the case of Dirichlet boundary conditions.

The Boussinesq equations are commonly used to model large scale atmospheric and oceanic flows, for example, tornados, cyclones, and hurricanes. It describes the dynamics of fluid influenced by gravitational force,which playes an very important role in the study of Rayleigh-Bérnard convection, see [7, 13–15].

Besides the physical significance, for the inviscid Boussinesq equations, it is supposed that these equations have strong resemblance with the incompressible Euler equations in many aspects. Among them, the challenging open problem of global regularity of the inviscid Boussinesq equations shared the attention with the problem of Euler equations for many years.

Recently, a considerable literature are devoted to the study of the energy dissipation for the incompressible Euler equation. In particular, the existence of weak solutions with kinetic energy strictly decaying (or increasing, which is equivalent since the Euler flow is reversible) over time is investigated in [9, 18]. The Onsager's conjecture [5, 6] ensures that the energy dissipation cannot hold beyond a certain regularity threshold. Thus, it is crucial to consider rather low regularity weak solution of the incompressible Euler system in order to understand energy dissipation and turbulent flow. In this direction, many developments [1–3, 10, 11] are acheived on the existence of energy dissipating flows enjoying some Hölder regularity. For the inviscid Boussinesq equations, there are similar studies on the Hölder continuous weak solution, for references see [19–21].

In this paper, we study the dissipative solutions for the inviscid Boussinesq equations. The concept of dissipative solution is analogous to [12] in which Lions established the existence of dissipative solution for the incompressible Euler system. The dissipative solutions have two advantages, on one hand they exist globally in time for large initial data in the energy space, on the other hand they allow energy dissipation phenomena to occur. In general, they are not shown to be unique. However, they coincide with the unique strong solution when the latter exists. They also have an important application in a wide range of asymptotic problems, see [16, 17].

The paper is organized as follows. In Section 2, we will give some notations and provide a suitable notion of dissipative solution for the inviscid Boussinesq equation. In Section 3, we prove that there exists a dissipative solution.

2. Notations and main theorem

In this section we will give some notations. Throughout the paper, C denotes a generic constant which may vary from line to line.

Let $v = (v_1, \dots, v_N)$ be a vector function. We say that $v \in L^2(\mathbb{T}^N)^N$ which means $v_i \in L^2(\mathbb{T}^N)$ for each $1 \le i \le N$. We denote the L^2 norm of v by $\|v\|_{L^2} = \sqrt{\sum_{1\le i\le d} \|v_i\|_{L^2}^2}$. Let $\rho \in L^2(\mathbb{T}^N)$ be a scalar function. We say the pair $(v, \rho) \in L^2(\mathbb{T}^N)^N \times L^2(\mathbb{T}^N)$ which means $v \in L^2(\mathbb{T}^N)^N, \rho \in L^2(\mathbb{T}^N)$. We denote the L^2 norm of the pair (v, ρ) to be

$$\|(v, \rho)\|_{L^2} = \sqrt{\|v\|_{L^2}^2 + \|\rho\|_{L^2}^2}.$$

Denote $\langle \cdot, \cdot \rangle$ to be the inner product in L^2 either for vector function or scalar function. Since we work in $\mathbb{T}^N$ throughout the paper, we sometimes omit the notation of domain $\mathbb{T}^N$ when no confusion arise. We also remark that the following arguments are valid for arbitrary dimension $N \ge 2$, and it is of course more reasonable to consider the case $N = 2$ or 3.

Analogous to [12], we first expalin how to modulate the basic energy and establish the so-called weak-strong stability inequality. With such stability inequality, we can provide a suitable notion of dissipative solution for the invisicd incompressible Boussinesq equations in any dimension.

Proposition 2.1. *Let (u, θ) be a smooth solution to the incompressible inviscid Boussinesq Eq.* (1.1). *Further, for any $0 < T < +\infty$, consider test functions $(v, \rho) \in C^\infty([0, T] \times \mathbb{T}^N)$ such that* $\operatorname{div} v = 0$ *and* *denote* $v(0,x) = v_0, \rho(0,x) = \rho_0$ *and*

$$A(v,\rho) = \begin{pmatrix} A_1(v,\rho) \\ A_2(v,\rho) \end{pmatrix} = \begin{pmatrix} -\partial_t v - \mathbb{P}(v\cdot\nabla v) + \mathbb{P}(\rho e_N) \\ -\partial_t \rho - v\cdot\nabla\rho \end{pmatrix},$$

where $\mathbb{P}$ *is the projection onto periodic divergence-free vector fields.*

Then, for any $0 < T < +\infty$, *the following stability inequality holds for* $0 < t < T$,

$$\begin{aligned} \|(u-v,\theta-\rho)(t,\cdot)\|_{L^2}^2 \le & \exp\Big[\int_0^t (1+2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty})ds\Big]\|(u_0-v_0,\theta_0-\rho_0)\|_{L^2}^2 \\ & + \int_0^t \Big[\int_{\mathbb{T}^N} A(v,\rho)\cdot\begin{pmatrix} u-v \\ \theta-\rho\end{pmatrix}dx\Big](s)\exp\Big[\int_s^t (1+2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty})d\sigma\Big]ds, \end{aligned} \tag{2.1}$$

where $d(=d(v)) = (\frac{1}{2}(\partial_i v_j + \partial_j v_i))_{ij}$, *and*

$$\|d^-\|_\infty = \|(\sup_{|\xi|=1} -(d\xi,\xi))^+\|_{L^\infty(\mathbb{T}^d)}.$$

Proof. Since (u,θ) is a solution to the Boussinesq Eq. (1.1) and note that v is divergence free, we can write

$$\partial_t(u-v) + u\cdot\nabla(u-v) + (u-v)\cdot\nabla v + \nabla\pi = (\theta-\rho)e_N + A_1(v,\rho), \tag{2.2}$$

and

$$\partial_t(\theta-\rho) + u\cdot\nabla(\theta-\rho) + (u-v)\cdot\nabla\rho = A_2(v,\rho), \tag{2.3}$$

for some scalar function π. Then, multiplying (2.2) by $(u-v)$ and (2.3) by $\theta-\rho$ and integrating over $\mathbb{T}^N$, we obtain

$$\begin{aligned} \frac{d}{dt}\|(u-v)(t,\cdot)\|_{L^2}^2 = & -2\int_{\mathbb{T}^N}(d(u-v),u-v)dx + 2\int_{\mathbb{T}^N}(\theta-\rho)e_N\cdot(u-v)dx \\ & + 2\int_{\mathbb{T}^N} A_1(v,\rho)\cdot(u-v)dx, \end{aligned} \tag{2.4}$$

and

$$\frac{d}{dt}\|(\theta-\rho)(t,\cdot)\|_{L^2}^2 = -2\int_{\mathbb{T}^N}(u-v)\cdot\nabla\rho(\theta-\rho)dx + 2\int_{\mathbb{T}^N} A_2(v,\rho)(\theta-\rho)dx. \tag{2.5}$$

By Cauchy's inequality, we find

$$\Big|\int_{\mathbb{T}^N}(\theta-\rho)e_N\cdot(u-v)dx\Big| \le \|\theta-\rho\|_{L^2}\|(u-v)\|_{L^2} \le \frac{1}{2}\|(u-v,\theta-\rho)\|_{L^2}^2,$$

and

$$\Big|\int_{\mathbb{T}^N}(u-v)\cdot\nabla\rho(\theta-\rho)dx\Big| \le \|\nabla\rho\|_{L^\infty}\|u-v\|_{L^2}\|\theta-\rho\|_{L^2} \le \frac{\|\nabla\rho\|_{L^\infty}}{2}\|(u-v,\theta-\rho)\|_{L^2}^2.$$

Adding (2.4) and (2.5) together, we arrive at

$$\begin{aligned} \frac{d}{dt}\|(u-v,\theta-\rho)(t,\cdot)\|_{L^2}^2 \le & \, 2\|d^-\|_\infty\|u-v\|_{L^2}^2 + (1+\|\nabla\rho\|_{L^\infty})\|(u-v,\theta-\rho)\|_{L^2}^2 \\ & + 2\int_{\mathbb{T}^N} A(v,\rho)\cdot\begin{pmatrix} u-v \\ \theta-\rho\end{pmatrix}dx \\ \le & \,(1+2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty})\|(u-v,\theta-\rho)\|_{L^2}^2 \\ & + 2\int_{\mathbb{T}^N} A(v,\rho)\cdot\begin{pmatrix} u-v \\ \theta-\rho\end{pmatrix}dx. \end{aligned}$$

By Grownwall's inequality, we deduce

$$\begin{aligned} \|(u-v,\theta-\rho)(t,\cdot)\|_{L^2}^2 \le \exp\Big[\int_0^t (1+2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty})ds\Big]\|(u_0-v_0,\theta_0-\rho_0)\|_{L^2}^2 \\ + \int_0^t \Big[\int_{\mathbb{T}^N} A(v,\rho)\cdot\begin{pmatrix} u-v \\ \theta-\rho \end{pmatrix} dx\Big](s)\exp\Big[\int_s^t (1+2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty})d\sigma\Big]ds. \end{aligned} \tag{2.6}$$

Then (2.1) is proved. □

By analogy with Lions' dissipative solution to the incompressible Euler system [[12], Section 4.4], we provide the definition of dissipative solution for the incompressible inviscid Boussinesq equation.

Definition 2.1. *Let $N = 2$ or 3. For $\forall\ 0 < T < +\infty$, $u \in L^\infty([0,T];L^2)^N \cap C([0,T];L^2-w)$, $\theta \in L^\infty([0,T];L^2) \cap C([0,T];L^2-w)$. Then (u,θ) is called a dissipative solution of the inviscid incompressible Boussinesq Eq.* (1.1) *if $u(0,x) = u_0(x), \theta(0,x) = \theta_0(x)$ and $\operatorname{div} u = 0$ in $\mathcal{D}'([0,T]\times\mathbb{T}^N)$ and* (2.1) *holds for all $v \in C([0,T];L^2)^N, \rho \in C([0,T];L^2)$ such that $d \in L^1(0,T;L^\infty), \nabla\rho \in L^1(0,T;L^\infty), A(v,\rho) \in L^1(0,T;L^2)$ and $\operatorname{div} v = 0$ in $\mathcal{D}'([0,T]\times\mathbb{T}^N)$.*

Remark 2.1. *The inequality* (2.1) *is called the weak-strong stability inequality. If we take $v = 0, \rho = 0$, then* (2.1) *reduces to*

$$\|(u,\theta)(t,\cdot)\|_{L^2}^2 \le e^t\|(u_0,\theta_0)\|_{L^2}^2,$$

which is the formal energy inequality for the inviscid Boussinesq Eq. (1.1)*. Furthermore, if (v,ρ) is the unique strong solution of the inviscid Boussinesq Eq.* (1.1) *with the same initial data, then from* (2.1) *and the Grönwall's inequality of intergral form we immediately have $u = v, \theta = \rho$ for any dissipative solution (u,θ) satisfying the Definition 2.1. And that property is also called the weak-strong uniqueness.*

3. Existence of dissipative solution

As previously mentioned, dissipative solutions define actual solutions in the sense that they coincide with the unique strong solution when the latter exists. The following theorem asserts their existence.

Theorem 3.1. *There exists at least one dissipative solution to the inviscid Boussinesq Eq.* (1.1).

Proof. We first consider the following viscous Boussinesq equations

$$\begin{cases} \partial_t u_\nu + u_\nu\cdot\nabla u_\nu + \nabla p_\nu - \nu\Delta u_\nu = \theta_\nu e_d, \\ \partial_t\theta_\nu + u_\nu\cdot\nabla\theta_\nu - \nu\Delta\theta_\nu = 0, \\ \operatorname{div} u_\nu = 0, \\ u_\nu(0,x) = u_0, \quad \theta_\nu(0,x) = \theta_0. \end{cases} \tag{3.1}$$

By standard energy method [4,8], there is a Leray-Hopf weak solution (u_ν,θ_ν) to the viscous Boussinesq equations satisfying the energy inequalities

$$\|\theta_\nu(t,\cdot)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla\theta_\nu(s,\cdot)\|_{L^2}^2 ds \le \|\theta_0\|_{L^2}^2, \tag{3.2}$$

and

$$\|u_\nu(t,\cdot)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u_\nu(s,\cdot)\|_{L^2}^2 ds \le C\|u_0\|_{L^2}^2 + Ct^2\|\theta_0\|_{L^2}^2, \tag{3.3}$$

where $C > 0$ is a constant independent of ν.

Then we consider (v,ρ) as in Definition 2.1, noting that for all $0 < T < +\infty$ (v,ρ) can be taken arbitrary smooth on $[0,T]\times\mathbb{T}^N$. Then we can write

$$\partial_t(u_\nu - v) - \nu\Delta u_\nu + u_\nu\cdot\nabla(u_\nu - v) + (u_\nu - v)\cdot\nabla v + \nabla\pi_\nu = (\theta_\nu - \rho)e_N + A_1(v,\rho), \tag{3.4}$$

and

$$\partial_t(\theta_\nu - \rho) - \nu\Delta\theta_\nu + u_\nu\cdot\nabla(\theta_\nu - \rho) + (u_\nu - v)\cdot\nabla\rho = A_2(v,\rho). \tag{3.5}$$

Then we multiply both sides of (3.4) by $u_\nu - v$ and integrate over $\mathbb{T}^N$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(u_\nu - v)(t,\cdot)\|_{L^2}^2 - \nu\langle\Delta u_\nu, u_\nu - v\rangle + \langle(u_\nu - v)\cdot\nabla v, u_\nu - v\rangle\\ = \langle(\theta_\nu - \rho)e_N, u_\nu - v\rangle + \langle A_1(v,\rho), u_\nu - v\rangle.\end{aligned} \tag{3.6}$$

Similarly, multiplying both sides of (3.5) by $\theta_\nu - \rho$ and integrating over $\mathbb{T}^N$, we get

$$\frac{1}{2}\frac{d}{dt}\|(\theta_\nu - \rho)(t,\cdot)\|_{L^2}^2 - \nu\langle\Delta\theta_\nu, \theta_\nu - \rho\rangle + \langle(u_\nu - v)\cdot\nabla\rho, \theta_\nu - \rho\rangle = \langle A_2(v,\rho), \theta_\nu - \rho\rangle. \tag{3.7}$$

Summing up (3.6) with (3.7), by use of Hölder inequality we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|(u_\nu - v, \theta_\nu - \rho)(t,\cdot)\|_{L^2}^2 \le \nu\|\nabla u_\nu\|_{L^2}\|\nabla v\|_{L^2} + \nu\|\nabla\theta_\nu\|_{L^2}\|\nabla\rho\|_{L^2}\\ &+ \|d^-\|_\infty\|(u_\nu - v)\|_{L^2}^2 + (1 + \|\nabla\rho\|_{L^\infty})\|u_\nu - v\|_{L^2}\|\theta_\nu - \rho\|_{L^2}\\ &+ \int_{\mathbb{T}^N} A(v,\rho)\cdot\begin{pmatrix} u_\nu - v\\ \theta_\nu - \rho\end{pmatrix}dx.\end{aligned}$$

By Cauchy's inequality, we have

$$\begin{aligned}\frac{d}{dt}&\|(u_\nu - v, \theta_\nu - \rho)(t,\cdot)\|_{L^2}^2 \le 2\nu\|\nabla u_\nu\|_{L^2}\|\nabla v\|_{L^2} + 2\nu\|\nabla\theta_\nu\|_{L^2}\|\nabla\rho\|_{L^2}\\ &+ (1 + 2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty})\|(u_\nu - v, \theta_\nu - \rho)\|_{L^2}^2\\ &+ 2\int_{\mathbb{T}^N} A(v,\rho)\cdot\begin{pmatrix} u_\nu - v\\ \theta_\nu - \rho\end{pmatrix}dx.\end{aligned}$$

By Gröwnwall's inequality, we obtain

$$\begin{aligned}\|(u_\nu - v, \theta_\nu - \rho)(t,\cdot)\|_{L^2}^2 &\le \exp\Big[\int_0^t (1 + 2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty})ds\Big]\\ &\quad\times\int_0^t (2\nu\|\nabla u_\nu\|_{L^2}\|\nabla v\|_{L^2} + 2\nu\|\nabla\theta_\nu\|_{L^2}\|\nabla\rho\|_{L^2})ds\\ &+ \exp\Big[\int_0^t (1 + 2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty})ds\Big]\|(u_0 - v_0, \theta_0 - \rho_0)\|_{L^2}^2\\ &+ \int_0^t \Big[\int_{\mathbb{T}^N} A(v,\rho)\cdot\begin{pmatrix} u_\nu - v\\ \theta_\nu - \rho\end{pmatrix}dx\Big](s)\exp\Big[\int_s^t (1 + 2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty})d\sigma\Big]ds.\end{aligned} \tag{3.8}$$

Noting that from (3.2) and (3.3), we have for all $0 < T < \infty$

$$\nu \int_0^T \|\nabla u_\nu(s,\cdot)\|_{L^2}^2 ds + \nu \int_0^T \|\nabla \theta_\nu(s,\cdot)\|_{L^2}^2 ds \le C_T,$$

where C_T depends on the initial data (u_0, θ_0) and T but not on ν. Thus we have

$$\begin{aligned}\int_0^t \nu\|\nabla u_\nu\|_{L^2}\|\nabla v\|_{L^2} ds \le& \sqrt{\nu}\Big(\nu\int_0^t \|\nabla u_\nu\|_{L^2}^2 ds\Big)^{1/2}\|\nabla v\|_{L^2(dxds)} \\ \le& C_T\sqrt{\nu}\|\nabla v\|_{L^2(dxds)},\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\int_0^t \nu\|\nabla \theta_\nu\|_{L^2}\|\nabla \rho\|_{L^2} ds \le& \sqrt{\nu}\Big(\nu\int_0^t \|\nabla \theta_\nu\|_{L^2}^2 ds\Big)^{1/2}\|\nabla \rho\|_{L^2(dxds)} \\ \le& C_T\sqrt{\nu}\|\nabla \rho\|_{L^2(dxds)}.\end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we then get

$$\begin{aligned}\|(u_\nu - v, \theta_\nu - \rho)(t,\cdot)\|_{L^2}^2 \le& \exp\Big[\int_0^t (1 + 2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty})ds\Big] \\ &\times C_T\sqrt{\nu}(\|\nabla v\|_{L^2(dxds)} + \|\nabla\rho\|_{L^2(dxds)}) \\ &+ \exp\Big[\int_0^t (1 + 2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty})ds\Big]\|(u_0 - v_0, \theta_0 - \rho_0)\|_{L^2}^2 \\ &+ \int_0^t \Big[\int_{\mathbb{T}^N} A(v,\rho)\cdot\begin{pmatrix} u_\nu - v \\ \theta_\nu - \rho\end{pmatrix} dx\Big](s)\exp\Big[\int_s^t (1 + 2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty})d\sigma\Big]ds.\end{aligned} \tag{3.11}$$

Then we need to pass to limit in the above inequality. When $N = 2$, by use of Sobolev embedding, we have

$$\begin{aligned}&\partial_t u_\nu \in L^2([0,T]; H^{-1}) + L^\infty([0,T]; L^2), \\ &\partial_t \theta_\nu \in L^2([0,T]; H^{-1}).\end{aligned}$$

It is possible to show that (u_ν, θ_ν) converges to $(u, \theta) \in C([0,T]; w - L^2)$. When $N \ge 3$, we recall that since (u_ν, θ_ν) is a weak solution to the viscous Boussinesq Eq. (3.1), then one can easily have that for $\forall\ 0 < T < +\infty$ and for all $\nu > 0$,

$$u_\nu \in L^\infty([0,T]; L^2(\mathbb{T}^N)), \quad \theta_\nu \in L^\infty([0,T], L^2(\mathbb{T}^N)).$$

Note that we can not use the $L^2([0,T]; H^1)$-bound for (u_ν, θ_ν), because we will take limit as ν goes to 0. however, it follows from (3.2) and (3.3) that

$$\sqrt{\nu}\nabla u_\nu \in L^2([0,T], L^2(\mathbb{T}^N)), \quad \sqrt{\nu}\nabla\theta_\nu \in L^2([0,T], L^2(\mathbb{T}^N)).$$

Furthermore, noticing that $\partial_t u_\nu = -\mathbb{P}(\nabla\cdot(u_\nu \otimes u_\nu)) + \mathbb{P}(\nu\Delta u_\nu) + \mathbb{P}(\theta_\nu e_N)$, thus $\partial_t u_\nu$ is bounded in $L^2([0,T]; H^{-1}(\mathbb{T}^N)) + L^\infty([0,T]; W^{-(1+\lambda),1}(\mathbb{T}^N)) + L^\infty([0,T]; L^2(\mathbb{T}^N))$ for all $\lambda > 0$. In a similar way, it follows from $\partial_t\theta_\nu = -\nabla\cdot(u_\nu\theta_\nu) + \nu\Delta\theta_\nu$ that $\partial_t\theta_\nu$ is also bounded in $L^2([0,T]; H^{-1}(\mathbb{T}^N)) + L^\infty([0,T]; W^{-(1+\lambda),1}(\mathbb{T}^N))$ for all $\lambda > 0$. It is also possible to show (see [[12]

Appendix C]) that (u_ν, θ_ν) converges to $(u, \theta) \in C([0,T]; w - L^2)$ weakly in L^2 uniformly in $t \in [0,T]$ for all $T \in (0,\infty)$. And $\operatorname{div} u = 0$ in $\mathcal{D}'([0,T] \times \mathbb{T}^N)$, $u|_{t=0} = u_0, \theta|_{t=0} = \theta_0$. Then by the weak lower semi-continuity of the norms and letting $\nu \to 0^+$ in (3.11), we obtain that, for every $0 < t < T$,

$$\|(u - v, \theta - \rho)(t,\cdot)\|_{L^2}^2 \le \exp\Big[\int_0^t (1 + 2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty}) ds\Big] \|(u_0 - v_0, \theta_0 - \rho_0)\|_{L^2}^2 + \int_0^t \Big[\int_{\mathbb{T}^N} A(v,\rho) \cdot \begin{pmatrix} u - v \\ \theta - \rho \end{pmatrix} dx\Big](s) \exp\Big[\int_s^t (1 + 2\|d^-\|_\infty + \|\nabla\rho\|_{L^\infty}) d\sigma\Big] ds.$$

The Theorem 3.1 is then proved. □

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Conflict of interest

The author declares that there is no conflict of interest.

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