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Research article

Distinguished subspaces in topological sequence spaces theory

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Abstract: In this paper, we study R_λ-semiconservative FK-spaces for Riesz-method defined by the Riesz matrix (R) and give some characterizations. We show that if ℓ_A is ℓ-replaceable, then A can not be R_λ-semiconservative and also if X_A is R_λ-conull FK-space then it must be R_λ-semiconservative space. In addition, we determine a new $r(\lambda)$ and $rb(\lambda)$ type duality of a sequence space X containing φ. The paper aims to develop some new subspaces which each one has its own value on topological sequence spaces theory. These subspaces are called as $R_\lambda S$; $R_\lambda W$; $R_\lambda F^+$; and $R_\lambda B^+$ for a locally convex FK-space X containing φ. The subspaces mentioned in the work requires some serious studies and they arose independently from the literature which was done at the recent stage of the development of summability through functional analysis.

Keywords: topological sequence space; multiplier spaces; semiconcervative FK-spaces
Mathematics Subject Classification: 46A45, 46A20

1. Introduction

Let ω denote the space of all real or complex valued sequences. An FK-space is a locally convex vector subspaces of ω which is also a Fréchet space (complete linear metric) with continuous coordinates. A BK-space is a normed FK-space [1]. Some articles about BK-space and FK-space are studied in [1–20]. The definition of semiconservative FK-spaces and some properties of these spaces were given by Snyder and Wilansky in [9]. Ince [10] continued to work on Cesáro semiconservative FK-spaces and gave some characterizations. The main purpose of this paper is to introduce the concept of Reisz semiconservative FK-spaces, which contains the space strictly increasing sequence of positive integers lambda. We have proved several interesting conclusions about this concept in section 3. The results in this article are new, accurate and interesting. In addition, our work is extension of the works in [6, 9, 10, 12, 16]. The work brings innovations and information that attracts the attention of the mathematics reader.

2. Preliminaries

Let F be an infinite subset of $\mathbb{N}$ and F as the range of a strictly increasing sequence of positive integers, say $F = \{\lambda(n)\}_{n=1}^{\infty}$. The Cesáro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \qquad n = 1, 2, \cdots,$$

where $\{x_k\}$ is a sequence of real or complex numbers [11]. The Riesz submethod is defined as the following; Let (q_k) be a positive sequence of real numbers.

$$R_n^\lambda(f;x) = \frac{1}{Q_{\lambda(n)}} \sum_{k=0}^{\lambda(n)} q_k s_k(f;x),$$

where

$$s_n(f;x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t) D_n(t) dt,$$

and

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin(\frac{t}{2})}.$$

Also,

$$Q_{\lambda(n)} = q_0 + q_1 + \cdots + q_{\lambda(n)} \neq 0, \qquad (n \geq 0).$$

In case $\lambda(n) = n$, the method $R_n^\lambda(f;x)$ give us classical known Riesz mean. Provided that $q_n = 1$ for all $(n \geq 0)$ Riesz mean yields

$$\sigma_n^\lambda(f;x) = \frac{1}{\lambda(n)+1} \sum_{k=0}^{\lambda(n)} s_k(f;x).$$

In addition to this, if $\lambda(n) = n$ for $\sigma_n^\lambda(f;x)$, then it coincides with Cesáro method C_1 [12]. Let $q = (q_k)$ and (Q_n) be given $q_0 > 0$, $q_k \geq 0$ $(\forall k \in \mathbb{N})$, $Q_n = \sum_{k=1}^{n} q_k$ $(n \in \mathbb{N})$. The matrix $R = q_{nk}$ defined by

$$(q_{nk}) = \begin{cases} \frac{q_k}{Q_n} & , \ k \leq n \\ 0 & , \ otherwise \end{cases}$$

is called a Riesz matrix.

In this paper, The Riesz submethod is symbolized by $R_n^\lambda(f;x)$ or, in short, R_λ. The sequences space

$$\begin{aligned} cs &= \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} x_n \ \text{convergent} \right\}, \\ bs &= \left\{ x = (x_n) \in \omega : \sup_k \left| \sum_{n=1}^{k} x_n \right| < \infty \right\}, \\ c_0 &= \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} x_n \ \text{convergent to zero} \right\} \end{aligned}$$

are FK space with the classical norm. Note that $(cs)_R = rs(\text{or r}), (bs)_R = rb, (c_0)_R = r_0$ are FK-spaces with the norms $\|x\|_{rb} = \sup_n \left|\frac{1}{Q_n}\sum_{k=1}^{n}\sum_{j=1}^{k} q_j x_j\right|$ and $\|x\|_{r_o} = \sup_n \frac{1}{Q_n}\left|\sum_{k=1}^{n} q_j x_j\right|$ respectively [15]. Also,

$$rs(\lambda) = \left\{x \in \omega : \lim_n \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} q_j x_j \ \text{ exists}\right\},$$

$rb(\lambda) = \left\{x \in \omega : \sup_n \left|\frac{1}{Q_{\lambda(n)}}\sum_{k=1}^{\lambda(n)}\sum_{j=1}^{k} q_j x_j\right| < \infty\right\}$, $r_0(\lambda) = \left\{x \in \omega : \lim_n \left|\frac{1}{Q_{\lambda(n)}}\sum_{j=1}^{\lambda(n)} q_j x_j\right| = 0\right\}$ are FK-spaces with the norms $\|x\|_{rb(\lambda)} = \sup_n \left|\frac{1}{Q_{\lambda(n)}}\sum_{k=1}^{\lambda(n)}\sum_{j=1}^{k} q_j x_j\right|, \|x\|_{r_o(\lambda)} = \sup_n \frac{1}{Q_{\lambda(n)}}\left|\sum_{k=1}^{\lambda(n)} q_j x_j\right|$, respectively. Throughout the paper, δ denotes sequences of the form $(1, 1, \ldots, 1 \ldots)$. Let $\varphi = span\{\delta^k : k \in \mathbb{N}\}$ and $\varphi_1 = \varphi \cup \{\delta\}$. The topological dual of X is denoted by X'. Let (X, τ) be a K-space with $\varphi \subset X$ and dual space X', and let $x = (x_k) \in X$ be arbitrarily given. Define the n^{th} section of x to be sequence $x^{[n]} = \sum_{k=1}^{n} x_k \delta^k = (x_1, x_2, \ldots, x_n, 0, \ldots)$, where δ^k denotes the sequence having 1 in the j-th position and 0's elsewhere [13, 14]. Also, $r^{[n]}x = \frac{1}{Q_n}\sum_{k=1}^{n} q_k x_k \delta^k$ is called the n^{th} Riesz section of x [15]. This here r is the set $\{r^n : n \in \mathbb{N}\}$. We define the following properties:
x has AK if $x^{[n]} \to x$ in (X, τ),
x has SAK if $x^{[n]} \to x$ in $(X, \sigma(X, X'))$,
x has FAK if $\sum_k x_k f(\delta^k)$ converges for all $f \in X'$,
x has AB if $\{x^{[n]} : n \in \mathbb{N}\}$ is bounded in (X, τ),
x has AD if $X = \overline{\varphi}$ (closed of φ),
x has $rK(\lambda)(riesz\ \ sectional\ \ convergence)$ if $\frac{1}{Q_{\lambda(n)}}\sum_{k=1}^{\lambda(n)} q_k x^{(k)} \to x \ \ , n \to \infty$ [15]. Then, some duals of a subset X are defined to be

$$\begin{aligned}
X^f &= \left\{\{f(\delta^k)\} : f \in X'\right\}, \\
X^Y &= \{x : yx = (y_k x_k) \in Y \ \text{ for every } \ y \in X\} = (X \to Y), \\
X^\beta &= \{x : yx = (y_k x_k) \in cs \ \text{ for every } \ y \in X\} \\
&= \left\{x : \sum_{k=1}^{\infty} x_k y_k \ \text{ exists for every } \ y \in X\right\}, \\
X^r &= \{x : yx = (y_k x_k) \in rs \ \text{ for every } \ y \in X\} \\
&= \left\{x : \lim_n \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j y_j \ \text{ exists for every } \ y \in X\right\}, \\
X^{rb} &= \{x : yx = (y_k x_k) \in rb \ \text{ for every } \ y \in X\} \\
&= \left\{x : \sup_n \frac{1}{Q_n} \left|\sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j y_j\right| < \infty \ \text{ for every } \ y \in X\right\}, \\
X^{r(\lambda)} &= \{x : yx = (y_k x_k) \in rs(\lambda) \ \text{ for every } \ y \in X\} \\
&= \left\{x : \lim_n \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} q_j x_j y_j \ \text{ exists for every } \ y \in X\right\},
\end{aligned}$$

$$\begin{aligned} X^{rb(\lambda)} &= \{x : yx = (y_k x_k) \in rb(\lambda) \text{ for every } y \in X\} \\ &= \left\{x : \sup_n \frac{1}{Q_{\lambda(n)}} \left| \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} q_j x_j y_j \right| < \infty \text{ for every } y \in X \right\}. \end{aligned}$$

By taking advantage of [1], we can easily see the following lemma:

Lemma 2.1. *Let X, X_1 be sets of sequences. Then for $k = f, \beta, r, rb, r(\lambda), rb(\lambda)$*
(1) $X \subset X^{kk}$,
(2) $X^{kkk} = X^k$,
(3) if $X \subset X_1$ then $X_1^k \subset X^k$ holds.

Let $A = (a_{ij})$ be an infinite matrix. The matrix A may be considered as a linear transformation of sequence by the formula $y = Ax$, where $y_i = \sum_{j=1}^{\infty} a_{ij} x_j$. For an FK-space (λ, u) we consider the summability domain $\lambda_A = \{x \in \omega \ : Ax \in \lambda\}$. Then λ_A is an FK-space under the semi-norms $p_i = |x_i|, \ (1, 2, \dots)$. A conservative matrix A, and the corresponding matrix method, is called conull if $\chi(A) = 0$, where $\chi(A) = \lim_A \delta - \sum_k \lim_A \delta^k$ [1].

Recall that, given a matrix A with $\ell_A \supset \varphi$ is called ℓ-replaceable if there is a matrix $B = (b_{nk})$ with $\ell_B = \ell_A$ and $\sum_{k=1}^{\infty} b_{nk} = 1$ for all $k \in \mathbb{N}$ [16].

In addition an FK-space X is called semiconservative if $X^f \subset cs$, this means that $X \supset \varphi$ and $\sum_{j=1}^{\infty} f(\delta^j)$ is convergent for each $f \in X'$ [9].

3. Main results

Firstly, we have defined the notations of R_λ-semiconservative FK-space in this section. Then, we investigate the properties of these spaces and we also give the relationship between ℓ-reblaceable and R_λ-semiconservative FK-space. Note that it is accepted $Q_n \to \infty, (n \to \infty)$ in this paper.

Definition 3.1. *An FK-space X is called R_λ-semiconservative if $X^f \subset rs(\lambda)$. It is obvious that $X^f \subset rs(\lambda)$ if and only if $\left\{\frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} q_k f(\delta^k)\right\}$ is convergent for each $f \in X'$ equivalently*

$$\lim_n \left\{ \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} q_j f(\delta^j) \right\}$$

exists.

Definition 3.2. *An FK-space containing φ_1 is called R_λ-conull if*

$$f(\delta) = \lim_n \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} q_j f(\delta^j),$$

for all, $f \in X'$.

Lemma 3.3. *Let X be an FK-space containing φ. Then*
(1)$X^\beta \subset X^{r(\lambda)} \subset X^{rb(\lambda)} \subset X^f$,
(2)If X is a $rK(\lambda)$ space then $X^f = X^{r(\lambda)}$,
(3)If X is an AD space then $X^{r(\lambda)} = X^{rb(\lambda)}$.

Proof. (2) Let $u \in X^{r(\lambda)}$ and define

$$f(x) = \lim_n \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} q_j x_j u_j$$

for $x \in X$. Then $f \in X'$; by the Banach-Steinhaus Theorem 1.0.4 of [2]. Also $f(\delta^p) = \lim_n \frac{1}{Q_{\lambda(n)}}(\lambda(n) - (p-1))q_p u_p = u_p, (p < \lambda(n))$ so $u \in X^f$. Thus $X^{r(\lambda)} \subset X^f$. Now we show that $X^f \subset X^{r(\lambda)}$. Let $u \in X^f$. Since X is $rK(\lambda)$ space

$$f(x) = \lim_n \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} q_j x_j f(\delta^j) = \lim_n \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} q_j x_j u_j,$$

for $x \in X$, then $u \in X^{r(\lambda)}$. Hence $X^f = X^{r(\lambda)}$.
(3) Let $u \in X^{rb(\lambda)}$ and define $f_n(x) = \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} q_j x_j u_j$ for $x \in X$. Then $\{f_n\}$ is pointwise bounded, hence equicontinuous by [2, Theorem 7.0.2]. Since $\lim_n f(\delta^p) = u_p$ $(p < \lambda(n))$ then $\varphi \subset \{x : lim_n f_n(x) \ \ exists\}$. Hence $\{x : lim_n f_n(x) \ \ exists\}$ is closed subspace of X by the Convergence Lemma [2, Theorem 1.0.5, 7.0.3]. Since X is an AD space then $X = \{x : lim_n f_n(x) \ \ exists\} = \bar{\varphi}$ and then $\lim_n f_n(x)$ exists for all $x \in X$. Thus $u \in X^{r(\lambda)}$. The opposite inclusion is trivial.
(1) $\bar{\varphi} \subset X$ by the hypothesis. Since $\bar{\varphi}$ is AD space, then $X^{rb(\lambda)} \subset (\bar{\varphi})^{rb(\lambda)} = (\bar{\varphi})^{r(\lambda)} = (\bar{\varphi})^f = X^f$ by (2), (3) and [2, Theorem 7.2.4]. □

Theorem 3.4. *If a matrix A is ℓ-replaceable then ℓ_A is not R_λ-semiconservative FK-space.*

Proof. If A is ℓ-replaceable then there is $f \in \ell_A'$ such that $f(\delta^j) = 1$ for all $j \in \mathbb{N}$ [16]. Hence

$$\lim_n \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} q_j f(\delta^j)$$

does not exist, so ℓ_A is not R_λ-semiconservative space. □

Theorem 3.5. *If X_A is R_λ-conull FK-space then it is R_λ-semiconservative space.*

Proof. Suppose that X_A is R_λ-conull. Then

$$f(\delta) = \lim_n \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} q_j f(\delta^j),$$

for all $f \in X_A'$. Hence $X_A^f \subset rs(\lambda)$. □

Theorem 3.6. *(1) A closed subspace, containing φ, of a R_λ-semiconservatif space is a R_λ-semiconservative space.*
(2) An FK-space that contains a R_λ-semiconservative space must be a R_λ-semiconservative space.
(3) A countable intersection of R_λ-semiconservative space is a R_λ-semiconservative space.

The proof is easily obtained from elementary properties of FK-spaces (see [2]).

Theorem 3.7. *If $z^{r(\lambda)}$ is a R_λ-semiconservative space then $z \in rs(\lambda)$.*

Proof. Let $z^{r(\lambda)}$ be a R_λ-semiconservative space. Then $z^{r(\lambda)f} \subset rs(\lambda)$. Since $z^{r(\lambda)}$ is a $rK(\lambda)$ space, we have $z^{r(\lambda)f} = z^{r(\lambda)r(\lambda)}$. Since $\{z\} \subset z^{r(\lambda)r(\lambda)} \subset rs(\lambda)$, we get $z \in rs(\lambda)$. □

Now we give study a new the subspaces which are called $R_\lambda S, R_\lambda W, R_\lambda F^+$ and $R_\lambda B^+$.

Definition 3.8. *Let X be an FK-space containing φ. Then, the following definitions hold.*

$$\begin{aligned} R_\lambda S &= R_\lambda S(X) \\ &= \{x \in X : \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} q_k x^{(k)} \to x \ \ in \ \ X\} \\ &= \{x \in X : \lim_n \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} q_j x_j \delta^j = x \ \} \\ &= \{x \in X : x \ \ has \ \ rK(\lambda) \ \ \ in \ \ X\}, \end{aligned}$$

$$\begin{aligned} R_\lambda W &= R_\lambda W(X) \\ &= \{x \in X : \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} q_k x^{(k)} \to x \ \ (weakly) \ \ in \ \ X\} \\ &= \{x \in X : \lim_n \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} q_j x_j f(\delta^j) = f(x) \ \ for \ \ all \ \ f \in X'\} \\ &= \{x \in X : x \ \ has \ \ SrK(\lambda) \ \ in \ \ X\}, \end{aligned}$$

$$\begin{aligned} R_\lambda F^+ &= R_\lambda F^+(X) \\ &= \{x \in X : \left(\frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} q_k x^{(k)}\right) \ \ is \ \ weakly \ \ Cauchy \ \ in \ \ X\} \\ &= \{x \in X : \{x_n f(\delta^n)\} \in rs(\lambda) \ \ for \ \ all \ \ f \in X'\} \\ &= \{x \in X : x \ \ has \ \ FrK(\lambda) \ \ \ in \ \ X\}, \end{aligned}$$

$$\begin{aligned} R_\lambda B^+ &= R_\lambda B^+(X) \\ &= \{x \in X : \left(\frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} q_k x^{(k)}\right) \ \ is \ \ bounded \ \ in \ \ X\} \\ &= \{x \in X : \{x_n f(\delta^n)\} \in rb(\lambda) \ \ for \ \ all \ \ f \in X'\} \\ &= \{x \in X : x \ \ has \ \ rB(\lambda) \ \ \ in \ \ X\}. \end{aligned}$$

Also, $R_\lambda F = R_\lambda F^+ \cap X$ and $R_\lambda B = R_\lambda B^+ \cap X$.

Definition 3.9. *Sequence sets of above definitions show that:*

1. $X_{rK(\lambda)} = R_\lambda S = \{x \in X : x \ \ has \ \ rK(\lambda)\} \subset X$

2. $X_{SrK(\lambda)} = R_\lambda W = \{x \in X : x \;\; has \;\; SrK(\lambda)\} \subset X$
3. $X_{FrK(\lambda)} = R_\lambda F = \{x \in X : x \;\; has \;\; FrK(\lambda)\} \subset X$
4. $X_{rB(\lambda)} = R_\lambda B = \{x \in X : x \;\; has \;\; rB(\lambda)\} \subset X$

Corollary 3.1. *By definition 3.8 we obtain from following results:*

1. X *has* $FrK(\lambda)$ *iff* $X \subset R_\lambda F$ *,i.e.,* $X = R_\lambda F$,
2. X *has* $rB(\lambda)$ *iff* $X \subset R_\lambda B$ *,i.e.,* $X = R_\lambda B$.

Theorem 3.10. *Let X be an FK-space containing φ and $z \in \omega$. Then $z \in R_\lambda F^+$ if and only if $Y = z^{-1}X = \{x : zx = \{z_n x_n\} \in X\}$ is a R_λ-semiconservative FK-space.*

Proof. Let $(z^{-1}X)$ be an R_λ-semiconservative space. Hence $f \in (z^{-1}X)'$. Then $f(x) = \alpha x + g(zx)$, $\alpha \in \varphi, g \in Y'$, by [2, Theorem 4.4.10] and $f(\delta^n) = \alpha_n + g(z\delta^n) = \alpha_n + g(z_n\delta^n) = \alpha_n + z_n g(\delta^n)$. Hence, since $\alpha \in \varphi \subset rs(\lambda)$ then $\{f(\delta^n)\} \in rs(\lambda)$ if and only if $\{z_n g(\delta^n)\} \in rs(\lambda)$, i. e., $z \in R_\lambda F^+$. □

An FK-space is called bounded convex R_λ-semiconservative if it is R_λ-semiconservative space and includes δ.

Theorem 3.11. *Let X be an FK-space containing φ and $z \in \omega$. Then $z \in R_\lambda F$ if and only if $z^{-1}X$ is bounded convex R_λ-semiconservative FK-space.*

Proof. Let $z \in R_\lambda F$. Since $R_\lambda F = R_\lambda F^+ \cap X$ then $z \in X$ so $\delta \in z^{-1}X$ and since $z \in R_\lambda F^+$, $z^{-1}X$ is R_λ-semiconservative FK-space by Theorem 3.10. Thus $z^{-1}X \in X$ is bounded convex R_λ-semiconservative FK-space. Contrary, let $z^{-1}X \in X$ is bounded convex R_λ-semiconservative FK-space. Then $z^{-1}X$ is R_λ-semiconservative FK-space and $\delta \in z^{-1}X$ so $z \in X$. Since $z \in R_\lambda F^+$ by Theorem 3.10 and $z \in X$, we get the result $z \in R_\lambda F$. □

Theorem 3.12. *Let X be an FK-space containing φ. Then*
$\varphi \subset R_\lambda S \subset R_\lambda W \subset R_\lambda F \subset R_\lambda B \subset X$ *and* $\varphi \subset R_\lambda S \subset R_\lambda W \subset \bar{\varphi}$.

Proof. First conclusion is obvious by Definition 3.8. We show that the inclusion $R_\lambda W \subset \bar{\varphi}$. Let $f \in X'$ and $f = 0$ on φ. The definition of $R_\lambda W$ shows that $f = 0$ on $R_\lambda W$. Thus, the Hanh-Banach theorem gives the result. □

Theorem 3.13. *The subspaces $E = R_\lambda S, R_\lambda W, R_\lambda F, R_\lambda B, R_\lambda F^+$ and $R_\lambda B^+$ of X are monotone, i. e., if $X \subset Y$ then $E(X) \subset E(Y)$.*

Proof. The proof is obtained from elementary properties of FK-spaces (see [2]) and Definition 3.8. □

Theorem 3.14. *Let X be an FK-space containing φ. Then,*
(1) $R_\lambda B^+ = (X^f)^{rb(\lambda)}$.
(2) $R_\lambda B^+$ is the same for all FK-spaces Y between $\bar{\varphi}$ and X; i. e. ,$\bar{\varphi} \subset Y \subset X$ implies $R_\lambda B^+(Y) = R_\lambda B^+(X)$. Here the closure of φ is calculated in X.

Proof. (1) By Definition 3.8, $z \in R_\lambda B^+$ if and only if $zu \in rb(\lambda)$ for each $u \in X^f$. Hence $R_\lambda B^+ \subset (X^f)^{rb(\lambda)}$ holds. The converse inclusion is trivial. This is precisely the claim.
(2) By Theorem 3.13, we have $R_\lambda B^+(\bar{\varphi}) \subset R_\lambda B^+(Y) \subset R_\lambda B^+(X)$. By Theorem 3.14 (1), the first and last are equal. □

Theorem 3.15. *Let X be an FK-space containing φ. Then*
(1) $R_\lambda F^+ = (X^f)^{r(\lambda)}$.
(2) $R_\lambda F^+$ *is the same for all FK-spaces Y between $\bar{\varphi}$ and X; i. e.* $,\bar{\varphi} \subset Y \subset X$ *implies* $R_\lambda F^+(Y) = R_\lambda F^+(X)$. *Here the closure of φ is calculated in X.*

Proof. This can be proved as in Theorem 3.14, with $rs(\lambda)$ instead of $rb(\lambda)$. □

Theorem 3.16. *Let X be an FK-space in which $\bar{\varphi}$ has $rK(\lambda)$. Then*
(1) $R_\lambda F^+ = (\bar{\varphi})^{r(\lambda)r(\lambda)}$.
(2) X has $FrK(\lambda)$ if and only if $X \subset (\bar{\varphi})^{r(\lambda)r(\lambda)}$.
(3) If the inclusion $R_\lambda B \supset \bar{\varphi}$ *holds,* $R_\lambda S = R_\lambda W = \bar{\varphi}$.

Proof. (1) It is obvious that $R_\lambda F^+ = (X^f)^{r(\lambda)}$. Since $X^f = (\bar{\varphi})^f$ by [2], we have $(X^f)^{r(\lambda)} = (\bar{\varphi}^f)^{r(\lambda)}$. Thus by Lemma 3.3 the result follows.
(2) Firstly, suppose that X has $FrK(\lambda)$. X has $rB(\lambda)$ since $R_\lambda F \subset R_\lambda B$. If $\bar{\varphi}$ has $rK(\lambda)$ then $X \subset R_\lambda F^+$. Hence $X \subset (\bar{\varphi})^{r(\lambda)r(\lambda)}$. Sufficiency is given by Theorem 3.16 (1).
(3) By Theorem 3.12, $\varphi \subset R_\lambda S \subset R_\lambda W \subset \bar{\varphi} \subset R_\lambda B$. Firstly, suppose that X has $R_\lambda B$. Define $f_n : X \to X$ by $x \to f_n(x) = x - \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} q_k x^{(k)}$. Then $\{f_n\}$ is pointwise bounded, hence equicontinuous by [2]. Since $f_n \to 0$ on φ then also $f_n \to 0$ on $\bar{\varphi}$ by [2]. By $\bar{\varphi}$ has $rK(\lambda)$, the proof is complete. □

Theorem 3.17. *Let X be an FK-space containing φ. Then X has*
(1) $rB(\lambda)$ property if and only if $X^f \subset X^{rb(\lambda)}$.
(2) $rF(\lambda)$ property if and only if $X^f \subset X^{r(\lambda)}$.

Proof. Necessity; Let X be $rB(\lambda)$ property. Then $X \subset R_\lambda B^+ = (X^f)^{rb(\lambda)}$ and $X^{rb(\lambda)} \supset (X^f)^{rb(\lambda)rb(\lambda)} \supset X^f$. Sufficiency is clear. The proof of (2) is similar to that of (1). □

Theorem 3.18. *Let Y be a R_λ-semiconservative FK-space and Z an AD-space. Suppose that for an FK-space X,* $X \supset Y.Z$. *Then* $X \supset Z$*,where* $Y.Z = \{y.z : y \in Y, z \in Z\}$.

Proof. Let $z \in Z$. Then, since $X \supset Y.Z$, $z^{-1}.X \supset Y$. Thus, since Y is R_λ-semiconservative space then $z^{-1}.X$ is R_λ-semiconservative space by Theorem 3.6 and so $z \in R_\lambda F^+$ by Theorem 3.10. Hence $Z \subset R_\lambda F^+ = (X^f)^{r(\lambda)}$. Thus $X^f \subset X^{fr(\lambda)r(\lambda)} \subset Z^{r(\lambda)} \subset Z^f$ and so $X \supset Z$ by Theorem 8.6.1 of [2]. □

Theorem 3.19. *Let X be an FK-space containing φ. The following statements are equivalent:*
(1) X has $FrK(\lambda)$,
(2) $X \subset (R_\lambda S)^{r(\lambda)r(\lambda)}$,
(3) $X \subset (R_\lambda W)^{r(\lambda)r(\lambda)}$,
(4) $X \subset (R_\lambda F)^{r(\lambda)r(\lambda)}$,
(5) $X^{r(\lambda)} = (R_\lambda S)^{r(\lambda)}$,
(6) $X^{r(\lambda)} = (R_\lambda F)^{r(\lambda)}$.

Proof. Since $R_\lambda S \subset R_\lambda W \subset R_\lambda F \subset X$, it is trivial that (2) ⇒ (3) and (3) ⇒ (4). If (4) is true, then

$$X^f \subset (R_\lambda F)^{r(\lambda)} = (X^{fr(\lambda)})^{r(\lambda)} \subset X^{r(\lambda)}$$

so (1) is true by Lemma 3.3. If (1) holds, then Theorem 3.16 implies that $\bar{\varphi} = R_\lambda S$ which means (2) holds. The equivalence of (5), (6) with others is clear. □

Theorem 3.20. *Let X be an FK-space containing φ. The following are equivalent:*
(1) X has $SrK(\lambda)$,
(2) X has $rK(\lambda)$,
(3) $X^{r(\lambda)} = X'$.

Proof. By Theorem 3.12, it is clear (2) implies (1). Conversely if X has $SrK(\lambda)$ it must have AD for $R_\lambda W \subset \bar{\varphi}$ by Theorem 3.12. It also has $rB(\lambda)$ since $R_\lambda W \subset R_\lambda B$. Thus X has $rK(\lambda)$ by Theorem 3.16, this proves that (1) and (2) are equivalent. Assume that (3) holds. Let $f \in X'$, then there exists $u \in X^{r(\lambda)}$ such that

$$f(x) = \lim_n \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} u_j x_j q_j$$

for $x \in X$. Since $u_j = f(\delta^j)$, it follows that each $x \in R_\lambda W$ which shows that (3) implies (1). Let X has $rK(\lambda)$, then by Theorem 3.12 it has $SrK(\lambda)$. So, by Definition 3.8, for all $f \in X'$ there is

$$f(x) = \lim_n \frac{1}{Q_{\lambda(n)}} \sum_{k=1}^{\lambda(n)} \sum_{j=1}^{k} u_j x_j q_j$$

such that $u \in X^{r(\lambda)}$ which is $u_j = f(\delta^j), (\forall x \in X)$. This shows that (2) implies (3). □

Theorem 3.21. *Let X be an FK-space containing φ. The following are equivalent:*
(1) $R_\lambda W$ is closed in X,
(2) $\bar{\varphi} \subset R_\lambda B$,
(3) $\bar{\varphi} \subset R_\lambda F$,
(4) $\bar{\varphi} = R_\lambda W$,
(5) $\bar{\varphi} = R_\lambda S$,
(6) $R_\lambda S$ is closed in X.

Proof. (2) ⇒ (5): By Theorem 3.16, $\bar{\varphi}$ has $rK(\lambda)$, i.e., $\bar{\varphi} \subset R_\lambda S$. The opposite inclusion is Theorem 3.12. Note that (5) implies (4), (4) implies (3) and (3) implies (2) because

$$R_\lambda S \subset R_\lambda W \subset \bar{\varphi}, R_\lambda W \subset R_\lambda F \subset R_\lambda B;$$

(1)⇒ (4) and (6)⇒ (5) since $\varphi \subset R_\lambda S \subset R_\lambda W \subset \bar{\varphi}$. Finally (4) implies (1) and (5) implies (6). □

4. Conclusion

The main motivation of this article is to develop some distinguished subspaces in the theory of topological sequence spaces. These subspaces are called as $R_\lambda S, R_\lambda W, R_\lambda F^+$ and $R_\lambda B^+$ for a locally convex FK-space X containing φ. Moreover, we study R_λ-semiconservative FK-spaces for Riesz method defined by the Riesz matrix (R) and give some characterizations. In addition, we determine a new $r(\lambda)$ and $rb(\lambda)$ type duality of a sequence space X containing φ and we examine monotone of the distinguished subspaces. Finally, we prove some theorems related to the f-, $r(\lambda)$- and $rb(\lambda)$- duality of a sequence spaces X. Our main results give information that holds the mathematics reader's attention.

Authors' contributions

Authors contributed to each part of this work equally, and they read and approved the final manuscript.

Conflict of interest

The authors declare no conflict of interest.

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