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Research article

New weighted generalizations for differentiable exponentially convex mapping with application

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Abstract: The main aim of the present paper is to present a novel approach base on the exponentially convex function to broaden the utilization of celebrated Hermite-Hadamard type inequality. The proposed technique presents an auxiliary result of constructing the set of base functions and gives deformation equations in a simple form. The auxiliary result in the convexity has provided a convenient way of establishing the convergence region of several novel results. The strategy is not limited to the small parameter, such as in the classical method. The numerical examples obtained by the proposed approach indicate that the approach is easy to implement and computationally very attractive. The implementation of this numerical scheme clearly exhibits its effectiveness, reliability, and easiness regarding the applications in error estimates for weighted mean, the integral formula, rth moments of a continuous random variable, application to weighted special means and in developing the variants by extraordinary choices of n and θ as well as its better approximation.

Keywords: convex function; exponentially convex function; Hermite-Hadamard inequality; rth moments; weighted means
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1. Introduction

The classical convexity and concavity of functions are two fundamental notions in mathematics, they have widely applications in many branches of mathematics and physics [1–30]. The origin theory of convex functions is generally attributed to Jensen [31]. The well-known book [32] played an indispensable role in the the theory of convex functions.

The significance of inequalities is increasing day by day in the real world because of their fertile applications in our life and used to solve many complex problems in all areas of science and technology [33–40]. Integral inequalities have numerous applications in number theory, combinatorics, orthogonal polynomials, hypergeometric functions, quantum theory, linear programming, optimization theory, mechanics and in the theory of relativity [41–48]. This subject has received considerable attention from researchers [49–54] and hence it is assumed as an incorporative subject between mathematics, statistics, economics, and physics [55–60].

One of the most well known and considerably used inequalities for convex function is the Hermite-Hadamard inequality, which can be stated as follows.

Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval, $Y : \mathcal{I} \to \mathbb{R}$ be a convex function. Then the double inequality

$$Y\Big(\frac{\rho_1+\rho_2}{2}\Big) \leq \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} Y(\varrho)d\varrho \leq \frac{Y(\rho_1)+Y(\rho_2)}{2} \tag{1.1}$$

holds for all $\rho_1, \rho_2 \in \mathcal{I}$ with $\rho_1 \neq \rho_2$. If Y is concave on the interval $\mathcal{I}$, then the reversed inequality (1.1) holds.

The Hermite-Hadamard inequality (1.1) has wide applications in the study of functional analysis (geometry of Banach spaces) and in the field of non-linear analysis [61]. Interestingly, both sides of the above integral inequality (1.1) can characterize the convex functions.

Closely related to the convex (concave) functions, we have the concept of exponentially convex (concave) functions. The exponentially convex (concave) functions can be considered as a noteworthy extension of the convex functions and have potential applications in information theory, big data analysis, machine learning, and statistics [62, 63]. Bernstein [64] and Antczak [65] introduced these exponentially convex functions implicitly and discuss their role in mathematical programming. Dragomir and Gomm [66] and Rashid et al. [67] established novel outcomes for these exponentially convex functions.

Now we recall the concept of exponentially convex functions, which is mainly due to Awan et al. [68].

Definition 1.1. ([68]) Let $\theta \in \mathbb{R}$. Then a real-valued function $Y : [0,\infty) \to \mathbb{R}$ is said to be θ-exponentially convex if

$$Y(\tau\rho_1 + (1-\tau)\rho_2) \leq \tau e^{\theta\rho_1} Y(\rho_1) + (1-\tau)e^{\theta\rho_2} Y(\rho_2) \tag{1.2}$$

for all $\rho_1, \rho_2 \in [0,\infty)$ and $\tau \in [0,1]$. Inequality (1.2) will hold in the reverse direction if Y is concave.

For example, the mapping $Y : \mathbb{R} \to \mathbb{R}$, defined by $Y(\upsilon) = -\upsilon^2$ is a concave function, thus this mapping is an exponentially convex for all $\theta > 0$. Exponentially convex functions are employed for statistical analysis, recurrent neural networks, and experimental designs. The exponentially convex functions are highly useful due to their dominant features.

Recall the concept of exponentially quasi-convex function, introduced by Nie et al. [69].

Definition 1.2. ([69]) Let $\theta \in \mathbb{R}$. Then a mapping $Y : [0,\infty) \subseteq \mathbb{R} \to \mathbb{R}$ is said to be θ-exponentially quasi-convex if

$$Y(\tau\rho_1 + (1-\tau)\rho_2) \leq \max\{e^{\theta\rho_1}Y(\rho_1), e^{\theta\rho_2}Y(\rho_2)\}$$

for all $\rho_1, \rho_2 \in [0,\infty)$ and $\tau \in [0,1]$.

Kirmaci [70], and Pearce and Pečarič [71] established the new inequalities involving the convex functions as follows.

Theorem 1.3. ([70]) Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval, $\rho_1, \rho_1 \in \mathcal{I}$ with $\rho_1 < \rho_2$, and $Y : \mathcal{I} \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ (where and in what follows $\mathcal{I}^\circ$ denotes the interior of $\mathcal{I}$) such that $Y' \in L([\rho_1, \rho_2])$ and $|Y'|$ is convex on $[\rho_1, \rho_2]$. Then

$$\left| Y\left(\frac{\rho_1+\rho_2}{2}\right) - \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} Y(\varrho)d\varrho \right| \leq \frac{(\rho_2-\rho_1)\left(|Y'(\rho_1)| + |Y'(\rho_2)|\right)}{8}. \tag{1.3}$$

Theorem 1.4. ([71]) Let $\lambda \in \mathbb{R}$ with $\lambda \neq 0$, $\mathcal{I} \subseteq \mathbb{R}$ be an interval, $\rho_1, \rho_1 \in \mathcal{I}$ with $\rho_1 < \rho_2$, and $Y : \mathcal{I} \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ such that $Y' \in L([\rho_1, \rho_2])$ and $|Y'|^\lambda$ is convex on $[\rho_1, \rho_2]$. Then

$$\left| Y\left(\frac{\rho_1+\rho_2}{2}\right) - \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} Y(\varrho)d\varrho \right| \leq \frac{(\rho_2-\rho_1)}{4}\left[\frac{|Y'(\rho_1)|^\lambda + |Y'(\rho_2)|}{2}\right]^{\frac{1}{\lambda}}. \tag{1.4}$$

The principal objective of this work is to determine the novel generalizations for weighted variants of (1.3) and (1.4) associated with the class of functions whose derivatives in absolute value at certain powers are exponentially convex with the aid of the auxiliary result. Moreover, an analogous improvement is developed for exponentially quasi-convex functions. Utilizing the obtained consequences, some new bounds for the weighted mean formula, rth moments of a continuous random variable and special bivariate means are established. The repercussions of the Hermite-Hadamard inequalities have depicted the presentations for various existing outcomes. Results obtained by the application of the technique disclose that the suggested scheme is very accurate, flexible, effective and simple to use.

In what follows we use the notations

$$\mathcal{L}(\rho_1, \rho_2, \tau) = \frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2$$

and

$$\mathcal{M}(\rho_1, \rho_2, \tau) = \frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2$$

for $\tau \in [0,1]$ and all $n \in \mathbb{N}$.

2. New generalization for exponentially convex functions

From now onwards, let $\rho_1, \rho_2 \in \mathbb{R}$ with $\rho_1 < \rho_2$ and $\mathcal{I} = [\rho_1, \rho_2]$, unless otherwise specified. The following lemma presented as an auxiliary result which will be helpful for deriving several new results.

Lemma 2.1. Let $n \in \mathbb{N}$, $Y : \mathcal{I} \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ such that $Y' \in L_1([\rho_1, \rho_2])$, and $\mathcal{U} : [\rho_1, \rho_2] \to [0, \infty)$ be differentiable mapping. Then one has

$$\frac{1}{2}\Big[\mathcal{U}(\rho_1)\left[Y(\rho_1) + Y(\rho_2)\right] - \Big\{\mathcal{U}\left(\frac{n\rho_1 + \rho_2}{n+1}\right) - \mathcal{U}\left(\frac{\rho_1 + n\rho_2}{n+1}\right) + \mathcal{U}(\rho_2)\Big\} Y\left(\frac{n\rho_1 + \rho_2}{n+1}\right)$$

$$-\Big\{\mathcal{U}\left(\frac{n\rho_1 + \rho_2}{n+1}\right) - \mathcal{U}\left(\frac{\rho_1 + n\rho_2}{n+1}\right) + \mathcal{U}(\rho_2)\Big\} Y\left(\frac{n\rho_1 + \rho_2}{n+1}\right)\Big] + \frac{\rho_2 - \rho_1}{2(n+1)} \int_0^1 \Big\{\Big[Y\Big(\frac{n+\tau}{n+1}\rho_1$$

$$\frac{1-\tau}{n+1}\rho_2\Big) + Y\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\Big]\Big[\mathcal{U}'\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) + \mathcal{U}'\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\Big]\Big\} d\tau$$

$$= \frac{\rho_2 - \rho_1}{2(n+1)}\Big\{\int_0^1 \Big[\mathcal{U}\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) - \mathcal{U}\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right) + \mathcal{U}(\rho_2)\Big]$$

$$\times\Big[-Y'\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) + Y'\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\Big] d\tau\Big\}. \tag{2.1}$$

Proof. It follows from integration by parts that

$$I_1 = -\int_0^1 \Big[\mathcal{U}\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) - \mathcal{U}\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right) + \mathcal{U}(\rho_2)\Big] Y'\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) d\tau$$

$$= \frac{n+1}{\rho_2 - \rho_1}\Big\{\mathcal{U}\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) - \mathcal{U}\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right) + \mathcal{U}(\rho_2)\Big\} Y\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right)\Big|_0^1$$

$$-\frac{\rho_1 - \rho_2}{n+1}\int_0^1 Y\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right)\Big[\mathcal{U}'\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) + \mathcal{U}'\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\Big] d\tau$$

$$= \frac{n+1}{\rho_2 - \rho_1}\Big[\mathcal{U}(\rho_1)Y(\rho_1) - \Big[\mathcal{U}\left(\frac{n\rho_1 + \rho_2}{n+1}\right) - \mathcal{U}\left(\frac{\rho_1 + n\rho_2}{n+1}\right) + \mathcal{U}(\rho_2)\Big]\Big] Y\left(\frac{n\rho_1 + \rho_2}{n+1}\right)$$

$$+\int_0^1 Y\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right)\Big[\mathcal{U}'\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) + \mathcal{U}'\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\Big] d\tau.$$

Similarly, we have

$$I_2 = \int_0^1 \Big[\mathcal{U}\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) - \mathcal{U}\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right) + \mathcal{U}(\rho_2)\Big] Y'\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right) d\tau$$

$$= \frac{n+1}{\rho_2-\rho_1}\left[\mathcal{U}(\rho_1)Y(\rho_1) - \left[\mathcal{U}\left(\frac{n\rho_1+\rho_2}{n+1}\right) - \mathcal{U}\left(\frac{\rho_1+n\rho_2}{n+1}\right) + \mathcal{U}(\rho_2)\right]\right] Y\left(\frac{n\rho_1+\rho_2}{n+1}\right)$$

$$+ \int_0^1 Y\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\left[\mathcal{U}'\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) + \mathcal{U}'\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\right] d\tau.$$

Adding I_1 and I_2, then multiplying by $\frac{\rho_2-\rho_1}{2(n+1)}$ we get the desired identity (2.1). □

Theorem 2.2. Let $n \in \mathbb{N}$, $\theta \in \mathbb{R}$, $Y : \mathcal{I} \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ such that $|Y'|$ is θ-exponentially convex on $\mathcal{I}$, and $\mathcal{V} : \mathcal{I} \to [0, \infty)$ be a continuous and positive mapping such it is symmetric with respect to $\frac{n\rho_1+\rho_2}{n+1}$. Then

$$\left|\int_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\left(\frac{n\rho_1+\rho_2}{n+1}\right)\int_{\rho_1}^{\rho_2}\mathcal{V}(\varrho)d\varrho\right|$$

$$\leq \frac{\rho_2-\rho_1}{n+1}\left[|e^{\theta\rho_1}Y'(\rho_1)| + |e^{\theta\rho_2}Y'(\rho_2)|\right]\int_0^1 \int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)} \mathcal{V}(\varrho)d\varrho d\tau. \tag{2.2}$$

Proof. Let $\tau \in [\rho_1, \rho_2]$ and $Y(\tau) = \int_{\rho_1}^{\tau}\mathcal{V}(\varrho)d\varrho$. Then it follows from Lemma 2.1 that

$$\frac{\rho_2-\rho_1}{2(n+1)}\int_0^1\left[Y\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) + Y\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\right]\left[\mathcal{V}\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right)\right.$$

$$\left. +\mathcal{V}\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\right]d\tau - Y\left(\frac{n\rho_1+\rho_2}{n+1}\right)\int_{\rho_1}^{\rho_2}\mathcal{V}(\varrho)d\varrho$$

$$= \frac{\rho_2-\rho_1}{2(n+1)}\int_0^1\left\{\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho + \int_{\mathcal{M}(\rho_1,\rho_2,\tau)}^{\rho_2}\mathcal{V}(\varrho)d\varrho\right\}$$

$$\times\left[-Y'\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) + Y'\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\right]d\tau. \tag{2.3}$$

Since $\mathcal{V}(\varrho)$ is symmetric with respect to $\varrho = \frac{n\rho_1+\rho_2}{n+1}$, we have

$$\frac{\rho_2-\rho_1}{2(n+1)}\int_0^1\left[Y\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) + Y\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\right]\left[\mathcal{V}\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right)\right.$$

$$\left. +\mathcal{V}\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\right]d\tau$$

$$= \frac{\rho_2 - \rho_1}{(n+1)} \int_0^1 Y\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) \mathcal{V}\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right) d\tau$$

$$+ \frac{\rho_2 - \rho_1}{(n+1)} \int_0^1 Y\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right) \mathcal{V}\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right) d\tau$$

$$= \int_{\rho_1}^{\frac{n\rho_1+\rho_2}{n+1}} Y(\varrho)\mathcal{V}(\varrho)d\varrho + \int_{\frac{\rho_1+n\rho_2}{n+1}}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho = \int_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho \tag{2.4}$$

and

$$\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)} \mathcal{V}(\varrho)d\varrho = \int_{\mathcal{M}(\rho_1,\rho_2,\tau)}^{\rho_2} \mathcal{V}(\varrho)d\varrho \quad \forall \tau \in [0,1]. \tag{2.5}$$

From (2.3)–(2.5) we clearly see that

$$\left| \int_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\left(\frac{n\rho_1+\rho_2}{n+1}\right) \int_{\rho_1}^{\rho_2} \mathcal{V}(\varrho)d\varrho \right|$$

$$\leq \frac{\rho_2-\rho_1}{n+1} \left\{ \int_0^1 \int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)} \left| Y'\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right)\right| d\tau + \int_0^1 \int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)} \left| Y'\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\right| d\tau \right\}. \tag{2.6}$$

Making use of the exponentially convexity of $|Y'|$ we get

$$\int_0^1 \int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)} \mathcal{V}(\varrho) \left| Y'\left(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\right)\right| d\varrho d\tau + \int_0^1 \int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)} \mathcal{V}(\varrho) \left| Y'\left(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\right)\right| d\varrho d\tau$$

$$\leq \int_0^1 \int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)} \mathcal{V}(\varrho) \left[\frac{n+\tau}{n+1} \left|e^{\theta\rho_1} Y'(\rho_1)\right| + \frac{1-\tau}{n+1} \left|e^{\theta\rho_2} Y'(\rho_2)\right| + \frac{1-\tau}{n+1} \left|e^{\theta\rho_1} Y'(\rho_1) + \frac{n+\tau}{n+1} \left|e^{\theta\rho_2} Y'(\rho_2)\right|\right| \right] d\varrho d\tau$$

$$= \left[\left|e^{\theta\rho_1} Y'(\rho_1)\right| + \left|e^{\theta\rho_2} Y'(\rho_2)\right| \right] \int_0^1 \int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)} \mathcal{V}(\varrho) d\varrho d\tau. \tag{2.7}$$

Therefore, inequality (2.2) follows from (2.6) and (2.7). □

Corollary 2.1. Let $\theta = 0$. Then Theorem 2.2 leads to

$$\left| \int_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\left(\frac{n\rho_1+\rho_2}{n+1}\right) \int_{\rho_1}^{\rho_2} \mathcal{V}(\varrho)d\varrho \right|$$

$$\leq \frac{\rho_2-\rho_1}{n+1}\left[|Y'(\rho_1)|+|Y'(\rho_2)|\right]\int_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau.$$

Corollary 2.2. Let $n=1$. Then Theorem 2.2 reduces to

$$\left|\int_{\rho_1}^{\rho_2}Y(\varrho)\mathcal{V}(\varrho)d\varrho-Y\left(\frac{\rho_1+\rho_2}{2}\right)\int_{\rho_1}^{\rho_2}\mathcal{V}(\varrho)d\varrho\right|$$

$$\leq \frac{\rho_2-\rho_1}{2}\left[|e^{\theta\rho_1}Y'(\rho_1)|+|e^{\theta\rho_2}Y'(\rho_2)|\right]\int_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau.$$

Corollary 2.3. Let $\mathcal{V}(\varrho)=1$. Then then Theorem 2.3 becomes

$$\left|Y\left(\frac{n\rho_1+\rho_2}{n+1}\right)-\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}Y(\varrho)d\varrho\right|$$

$$\leq \frac{\rho_2-\rho_1}{2(n+1)^2}\left[\left|e^{\theta\rho_1}Y'(\rho_1)\right|+\left|e^{\theta\rho_2}Y'(\rho_2)\right|\right].$$

Remark 2.1. Theorem 2.2 leads to the conclusion that
(1) If $n=1$ and $\theta=0$, then we get Theorem 2.2 of [72].
(2) If $n=\mathcal{V}(\varrho)=1$ and $\theta=0$, then we obtain inequality (1.2) of [70]

Theorem 2.3. Taking into consideration the hypothesis of Theorem 2.2 and $\lambda\geq 1$. If $\theta\in\mathbb{R}$ and $|Y'|^\lambda$ is θ-exponentially convex on $\mathcal{I}$, then

$$\left|\int_{\rho_1}^{\rho_2}Y(\varrho)\mathcal{V}(\varrho)d\varrho-Y\left(\frac{n\rho_1+\rho_2}{n+1}\right)\int_{\rho_1}^{\rho_2}\mathcal{V}(\varrho)d\varrho\right|$$

$$\leq \frac{2(\rho_2-\rho_1)}{n+1}\left[\frac{|e^{\theta\rho_1}Y'(\rho_1)|^\lambda+|e^{\theta\rho_2}Y'(\rho_2)|^\lambda}{2}\right]^{\frac{1}{\lambda}}\int_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau \tag{2.8}$$

for all $n\in\mathbb{N}$.

Proof. Continuing inequality (2.6) in the proofs of Theorem 2.2 and using the well-known Hölder integral inequality, one has

$$\left|\int_{\rho_1}^{\rho_2}Y(\varrho)\mathcal{V}(\varrho)d\varrho-Y\left(\frac{n\rho_1+\rho_2}{n+1}\right)\int_{\rho_1}^{\rho_2}\mathcal{V}(\varrho)d\varrho\right|$$

$$\leq \frac{\rho_2-\rho_1}{n+1}\left\{\left(\int\limits_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau\right)^{1-\frac{1}{\lambda}}\left(\int\limits_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)\left|Y'\left(\frac{n+\tau}{n+1}\rho_1+\frac{1-\tau}{n+1}\rho_2\right)\right|^{\lambda}d\varrho d\tau\right)^{\frac{1}{\lambda}}\right.$$

$$\left.+\left(\int\limits_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau\right)^{1-\frac{1}{\lambda}}\left(\int\limits_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)\left|Y'\left(\frac{1-\tau}{n+1}\rho_1+\frac{n+\tau}{n+1}\rho_2\right)\right|^{\lambda}d\varrho d\tau\right)^{\frac{1}{\lambda}}\right\}$$

$$\leq \frac{\rho_2-\rho_1}{n+1}\left(\int\limits_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau\right)^{1-\frac{1}{\lambda}}\left\{\left(\int\limits_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)\left|Y'\left(\frac{n+\tau}{n+1}\rho_1+\frac{1-\tau}{n+1}\rho_2\right)\right|^{\lambda}d\varrho d\tau\right)^{\frac{1}{\lambda}}\right.$$

$$\left.+\left(\int\limits_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)\left|Y'\left(\frac{1-\tau}{n+1}\rho_1+\frac{n+\tau}{n+1}\rho_2\right)\right|^{\lambda}d\varrho d\tau\right)^{\frac{1}{\lambda}}\right\}. \tag{2.9}$$

It follows from the power-mean inequality

$$\mu^a+\nu^a<2^{1-a}(\mu+\nu)^a$$

for $\mu,\nu>0$ and $a<1$ that

$$\left(\int\limits_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)\left|Y'\left(\frac{n+\tau}{n+1}\rho_1+\frac{1-\tau}{n+1}\rho_2\right)\right|^{\lambda}d\varrho d\tau\right)^{\frac{1}{\lambda}} \tag{2.10}$$

$$+\left(\int\limits_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)\left|Y'\left(\frac{1-\tau}{n+1}\rho_1+\frac{n+\tau}{n+1}\rho_2\right)\right|^{\lambda}d\varrho d\tau\right)^{\frac{1}{\lambda}}$$

$$\leq 2^{1-\frac{1}{\lambda}}\left\{\int\limits_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)\left(\left|Y'\left(\frac{n+\tau}{n+1}\rho_1+\frac{1-\tau}{n+1}\rho_2\right)\right|^{\lambda}+\left|Y'\left(\frac{1-\tau}{n+1}\rho_1+\frac{n+\tau}{n+1}\rho_2\right)\right|^{\lambda}\right)d\varrho d\tau\right\}^{\frac{1}{\lambda}}.$$

Since $|Y'|^{\lambda}$ is an θ-exponentially convex on $\mathcal{I}$, we have

$$\left|Y'\Big(\frac{n+\tau}{n+1}\rho_1+\frac{1-\tau}{n+1}\rho_2\Big)\right|^{\lambda}+\left|Y'\Big(\frac{1-\tau}{n+1}\rho_1+\frac{n+\tau}{n+1}\rho_2\Big)\right|$$

$$\leq \frac{n+\tau}{n+1}|e^{\theta\rho_1}Y'(\rho_1)|^q+\frac{1-\tau}{n+1}|e^{\theta\rho_2}Y'(\rho_2)|^q+\frac{1-\tau}{n+1}|e^{\theta\rho_1}Y'(\rho_1)|^q+\frac{n+\tau}{n+1}|e^{\theta\rho_2}Y'(\rho_2)|^q$$

$$=|e^{\theta\rho_1}Y'(\rho_1)|^q+|e^{\theta\rho_2}Y'(\rho_2)|^q. \tag{2.11}$$

Combining (2.9)–(2.11) gives the required inequality (2.8). □

Corollary 2.4. Let $n = 1$. Then Theorem 2.3 reduces to

$$\Big|\int_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\Big(\frac{\rho_1+\rho_2}{2}\Big)\int_{\rho_1}^{\rho_2} \mathcal{V}(\varrho)d\varrho\Big|$$

$$\leq (\rho_2-\rho_1)\Big[\frac{|e^{\theta\rho_1}Y'(\rho_1)|^{\lambda} + |e^{\theta\rho_2}Y'(\rho_2)|^{\lambda}}{2}\Big]^{\frac{1}{\lambda}} \int_0^1 \int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)} \mathcal{V}(\varrho)d\varrho d\tau.$$

Corollary 2.5. Let $\theta = 0$. Then Theorem 2.3 leads to

$$\Big|\int_{\rho_1}^{\rho_2} Y(x)\mathcal{V}(x)dx - Y\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\int_{\rho_1}^{\rho_2} \mathcal{V}(\varrho)d\varrho\Big|$$

$$\leq \frac{2(\rho_2-\rho_1)}{n+1}\Big[\frac{|Y'(\rho_1)|^{\lambda} + |Y'(\rho_2)|^{\lambda}}{2}\Big]^{\frac{1}{\lambda}} \int_0^1 \int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)} \mathcal{V}(\varrho)d\varrho d\tau.$$

Corollary 2.6. Let $\mathcal{V}(\varrho) = 1$. Then Theorem 2.3 becomes

$$\Big|Y\Big(\frac{n\rho_1+\rho_2}{n+1}\Big) - \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} Y(\varrho)d\varrho\Big| \leq \frac{(\rho_2-\rho_1)}{2(n+1)}\Big[\frac{|Y'(\rho_1)|^{\lambda} + |Y'(\rho_2)|^{\lambda}}{2}\Big]^{\frac{1}{\lambda}}.$$

Remark 2.2. From Theorem 2.3 we clearly see that
(1) If $n = 1$ and $\theta = 0$, then we get Theorem 2.4 in [72].
(2) If $\mathcal{V}(\varrho) = n = 1$ and $\theta = 0$, then we get inequality (1.3) in [71].

In the following result, the exponentially convex functions in Theorem 2.3 can be extended to exponentially quasi-convex functions.

Theorem 2.4. Using the hypothesis of Theorem 2.2. If $|Y'|$ is θ-exponentially quasi-convex on $\mathcal{I}$, then

$$\Big|\int_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\int_{\rho_1}^{\rho_2} \mathcal{V}(\varrho)d\varrho\Big| \tag{2.12}$$

$$\leq \frac{(\rho_2-\rho_1)}{n+1}\Big[\max\Big\{|e^{\theta\rho_1}Y'(\rho_1)|, \Big|e^{\theta\left(\frac{n\rho_1+\rho_2}{n+1}\right)}Y'\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\Big|\Big\}$$

$$+\max\Big\{|e^{\theta\rho_2}Y'(\rho_2)|, \Big|e^{\theta\left(\frac{\rho_1+n\rho_2}{n+1}\right)}Y'\Big(\frac{\rho_1+n\rho_2}{n+1}\Big)\Big|\Big\}\Big] \int_0^1 \int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)} \mathcal{V}(\varrho)d\varrho d\tau$$

for all $n \in \mathbb{N}$.

Proof. Using the exponentially quasi-convexity of $|Y'|$ for (2.6) in the proofs of Theorem 2.2, we get

$$\left|Y'\Big(\frac{n+\tau}{n+1}\rho_1+\frac{1-\tau}{n+1}\rho_2\Big)\right| = \max\left\{|e^{\theta\rho_1}Y'(\rho_1)|, \left|e^{\theta\left(\frac{n\rho_1+\rho_2}{n+1}\right)}Y'\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\right|\right\} \tag{2.13}$$

and

$$\left|Y'\Big(\frac{1-\tau}{n+1}\rho_1+\frac{n+\tau}{n+1}\rho_2\Big)\right| = \max\left\{|e^{\theta\rho_2}Y'(\rho_2)|, \left|e^{\theta\left(\frac{\rho_1+n\rho_2}{n+1}\right)}Y'\Big(\frac{\rho_1+n\rho_2}{n+1}\Big)\right|\right\}. \tag{2.14}$$

Combining (2.6), (2.13) and (2.14), we get the desired inequality (2.12). □

Next, we discuss some special cases of Theorem 2.4 as follows.

Corollary 2.7. Let $n = 1$. Then Theorem 2.4 reduces to

$$\begin{aligned}
&\left|\int_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\Big(\frac{\rho_1+\rho_2}{2}\Big)\int_{\rho_1}^{\rho_2}\mathcal{V}(\varrho)d\varrho\right| \\
&\leq \frac{(\rho_2-\rho_1)}{2}\Bigg[\max\left\{|e^{\theta\rho_1}Y'(\rho_1)|, \left|e^{\theta\left(\frac{\rho_1+\rho_2}{2}\right)}Y'\Big(\frac{\rho_1+\rho_2}{2}\Big)\right|\right\} \\
&+\max\left\{|e^{\theta\rho_2}Y'(\rho_2)|, \left|e^{\theta\left(\frac{\rho_1+\rho_2}{2}\right)}Y'\Big(\frac{\rho_1+\rho_2}{2}\Big)\right|\right\}\Bigg]\int_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau.
\end{aligned}$$

Corollary 2.8. Let $\theta = 0$. Then Theorem 2.4 leads to

$$\begin{aligned}
&\left|\int_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\int_{\rho_1}^{\rho_2}\mathcal{V}(\varrho)d\varrho\right| \\
&\leq \frac{(\rho_2-\rho_1)}{n+1}\Bigg[\max\left\{|Y'(\rho_1)|, \left|Y'\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\right|\right\} \\
&+\max\left\{|Y'(\rho_2)|, \left|Y'\Big(\frac{\rho_1+n\rho_2}{n+1}\Big)\right|\right\}\Bigg]\int_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau.
\end{aligned}$$

Corollary 2.9. Let $\mathcal{V}(x) = 1$. Then Theorem 2.4 becomes

$$\begin{aligned}
&\left|Y\Big(\frac{n\rho_1+\rho_2}{n+1}\Big) - \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} Y(x)dx\right| \\
&\leq \frac{(\rho_2-\rho_1)}{2(n+1)}\Bigg[\max\left\{|Y'(\rho_1)|, \left|Y'\Big(\frac{n\rho_1+\rho_2}{(n+1)}\Big)\right|\right\} \\
&+\max\left\{|Y'(\rho_2)|, \left|Y'\Big(\frac{\rho_1+n\rho_2}{n+1}\Big)\right|\right\}\Bigg].
\end{aligned}$$

Remark 2.3. If $|Y'|$ is increasing in Theorem 2.4, then

$$\Big|\int\limits_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\int\limits_{\rho_1}^{\rho_2}\mathcal{V}(\varrho)d\varrho\Big| \tag{2.15}$$

$$\leq \frac{(\rho_2-\rho_1)}{n+1}\Big[|e^{\theta\rho_2}Y'(\rho_2)| + \Big|e^{\theta\left(\frac{\rho_1+n\rho_2}{n+1}\right)}Y'\Big(\frac{\rho_1+n\rho_2}{n+1}\Big)\Big|\Big]\int\limits_0^1\int_{\rho_1}^{\mathcal{L}.(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau$$

If $|Y'|$ is decreasing in Theorem 2.4, then

$$\Big|\int\limits_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\int\limits_{\rho_1}^{\rho_2}\mathcal{V}(\varrho)d\varrho\Big| \tag{2.16}$$

$$\leq \frac{(\rho_2-\rho_1)}{n+1}\Big[|e^{\theta\rho_1}Y'(\rho_1)| + \Big|e^{\theta\left(\frac{n\rho_1+\rho_2}{n+1}\right)}Y'\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\Big|\Big]\int\limits_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau.$$

Remark 2.4. From Theorem 2.4 we clearly see that
(1) Let $n = 1$ and $\theta = 0$. Then Theorem 2.4 and Remark 2.3 lead to Theorem 2.8 and Remark 2.9 of [72], respectively.
(2). Let $n = \mathcal{V}(\varrho) = 1$ and $\theta = 0$. Then we get Corollary 2.10 and Remark 2.11 of [72].

Theorem 2.5. Suppose that all the hypothesis of Theorem 2.2 are satisfied, $\theta \in \mathbb{R}$ and $\lambda \geq 1$. If $|Y'|^\lambda$ is θ-exponentially quasi-convex on $\mathcal{I}$, then we have

$$\Big|\int\limits_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\int\limits_{\rho_1}^{\rho_2}\mathcal{V}(\varrho)d\varrho\Big| \tag{2.17}$$

$$\leq \frac{(\rho_2-\rho_1)}{n+1}\Big[\Big(\max\Big\{|e^{\theta\rho_1}Y'(\rho_1)|^\lambda, \Big|e^{\theta\left(\frac{n\rho_1+\rho_2}{n+1}\right)}Y'\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\Big|^\lambda\Big\}\Big)^{\frac{1}{\lambda}}$$

$$+\Big(\max\Big\{|e^{\theta\rho_2}Y'(\rho_2)|^\lambda, \Big|e^{\theta\left(\frac{\rho_1+n\rho_2}{n+1}\right)}Y'\Big(\frac{\rho_1+n\rho_2}{n+1}\Big)\Big|^\lambda\Big\}\Big)^{\frac{1}{\lambda}}\Big]\int\limits_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau$$

for all $n \in \mathbb{N}$.

Proof. It follows from the exponentially quasi-convexity of $|Y'|^\lambda$ and (2.6) that

$$\Big|Y'\Big(\frac{n+\tau}{n+1}\rho_1 + \frac{1-\tau}{n+1}\rho_2\Big)\Big|^\lambda \leq \max\Big\{|e^{\theta\rho_1}Y'(\rho_1)|^\lambda, \Big|e^{\theta\left(\frac{n\rho_1+\rho_2}{n+1}\right)}Y'\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\Big|^\lambda\Big\} \tag{2.18}$$

and

$$\Big|Y'\Big(\frac{1-\tau}{n+1}\rho_1 + \frac{n+\tau}{n+1}\rho_2\Big)\Big|^\lambda \leq \max\Big\{|e^{\theta\rho_2}Y'(\rho_2)|^\lambda, \Big|e^{\theta\left(\frac{\rho_1+n\rho_2}{n+1}\right)}Y'\Big(\frac{\rho_1+n\rho_2}{n+1}\Big)\Big|^\lambda\Big\}. \tag{2.19}$$

A combination of (2.6), (2.18) and (2.19) lead to the required inequality (2.17). □

Corollary 2.10. Let $n = 1$. Then Theorem 2.5 reduces to

$$\Big|\int_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\Big(\frac{\rho_1+\rho_2}{2}\Big)\int_{\rho_1}^{\rho_2}\mathcal{V}(\varrho)d\varrho\Big|$$

$$\leq \frac{(\rho_2-\rho_1)}{2}\Big[\Big(\max\Big\{|e^{\theta\rho_1}Y'(\rho_1)|^{\lambda}, \Big|e^{\theta\big(\frac{\rho_1+\rho_2}{2}\big)}Y'\Big(\frac{\rho_1+\rho_2}{2}\Big)\Big|^{\lambda}\Big\}\Big)^{\frac{1}{\lambda}}$$

$$+\Big(\max\Big\{|e^{\theta\rho_2}Y'(\rho_2)|^{\lambda}, \Big|e^{\theta\big(\frac{\rho_1+1\rho_2}{2}\big)}Y'\Big(\frac{\rho_1+\rho_2}{2}\Big)\Big|^{\lambda}\Big\}\Big)^{\frac{1}{\lambda}}\Big]\int_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau.$$

Corollary 2.11. If $\theta = 0$, then Theorem 2.5 leads to the conclusion that

$$\Big|\int_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\int_{\rho_1}^{\rho_2}\mathcal{V}(\varrho)d\varrho\Big|$$

$$\leq \frac{(\rho_2-\rho_1)}{n+1}\Big[\max\Big\{|Y'(\rho_1)|, \Big|Y'\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\Big|\Big\}$$

$$+\max\Big\{|Y'(\rho_2)|, \Big|Y'\Big(\frac{\rho_1+n\rho_2}{n+1}\Big)\Big|\Big\}\Big]\int_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau.$$

3. Examples

In this section, we support our main results by presenting two examples.

Example 3.1. Let $\rho_1 = 0$, $\rho_2 = \pi$, $\theta = 2$, $n = 1$, $Y(\varrho) = \sin\varrho$ and $\mathcal{V}(\varrho) = \cos\varrho$. Then all the assumptions in Theorem 2.2 are satisfied. Note that

$$\Big|\int_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)\int_{\rho_1}^{\rho_2}\mathcal{V}(\varrho)d\varrho\Big|$$

$$= \Big|\int_0^{\pi}\sin\varrho\cos\varrho d\varrho - \sin\frac{\pi}{2}\int_0^{\pi}\cos\varrho d\varrho\Big| = 1 \tag{3.1}$$

and

$$\frac{\rho_2-\rho_1}{n+1}[|e^{\theta\rho_1}Y'(\rho_1)| + |e^{\theta\rho_2}Y'(\rho_2)|]\int_0^1\int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho d\tau$$

$$= \frac{\pi}{2}[|e^0\cos 0)| + |e^{2\pi}\cos\pi|]\int_0^1\int_0^{\mathcal{L}(0,\pi,\tau)}\cos\varrho d\varrho d\tau$$

$$= \frac{536.50\pi}{2} \int_0^1 \int_0^{\frac{(1-\tau)\pi}{2}} \cos \varrho d\varrho d\tau \approx 536.5. \tag{3.2}$$

From (3.1) and (3.2) we clearly Example 3.1 supports the conclusion of Theorem 2.2.

Example 3.2. Let $\rho_1 = 0$, $\rho_2 = 2$, $\theta = 0.5$, $n = 2$, $Y(\varrho) = \sqrt{\varrho + 2}$ and $\mathcal{V}(\varrho) = \varrho$. Then all the assumptions in Theorem 2.2 are satisfied. Note that

$$\Big| \int_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho - Y\Big(\frac{n\rho_1 + \rho_2}{n+1}\Big) \int_{\rho_1}^{\rho_2} \mathcal{V}(\varrho)d\varrho \Big|$$

$$= \Big| \int_0^2 \varrho \sqrt{\varrho + 2} d\varrho - \sqrt{\frac{8}{3}} \int_0^2 \varrho d\varrho \Big| \approx 0.3758 \tag{3.3}$$

and

$$\frac{\rho_2 - \rho_1}{n+1} [|e^{\theta\rho_1} Y'(\rho_1)| + |e^{\theta\rho_2} Y'(\rho_2)|] \int_0^1 \int_{\rho_1}^{\mathcal{L}(\rho_1,\rho_2,\tau)} \mathcal{V}(\varrho)d\varrho d\tau$$

$$= \frac{2}{3} [|e^{0.5(0)} \frac{1}{2\sqrt{2}})| + |e^{0.5(2)} \frac{1}{4}|] \int_0^1 \int_0^{\mathcal{L}(0,2,\tau)} \varrho d\varrho d\tau$$

$$= 0.6887 \int_0^1 \int_0^{\frac{2(1-\tau)}{3}} \varrho d\varrho d\tau \approx 1.0332. \tag{3.4}$$

From (3.3) and (3.4) we clearly see that Example 3.2 supports the conclusion of Theorem 2.2.

4. Applications

4.1. Application to weighted mean formula

Let Δ be a partition: $\rho_1 = \varrho_0 < \varrho_2 < \cdots < \varrho_{n-1} < \varrho_n = \rho_2$ of the interval $[\rho_1, \rho_2]$ and consider the quadrature formula

$$\int_{\rho_1}^{\rho_2} Y(\varrho)\mathcal{V}(\varrho)d\varrho = \mathcal{T}(Y, \mathcal{V}, p) + \mathcal{E}(Y, \mathcal{V}, p), \tag{4.1}$$

where

$$\mathcal{T}(Y, \mathcal{V}, p) = \sum_{j=0}^{\kappa-1} Y\Big(\frac{n\varrho_j + \varrho_{j+1}}{n+1}\Big) \int_{\varrho_j}^{\varrho_{j+1}} \mathcal{V}(\varrho)d\varrho$$

is weighted mean and $\mathcal{E}(Y, \mathcal{V}, p)$ is the related approximation error.

The aim of this subsection is to provide several new bounds for $\mathcal{E}(Y, \mathcal{V}, p)$.

Theorem 4.1. Let $\lambda \geq 1$, $\theta \in \mathbb{R}$, and $|Y'|^{\lambda}$ be θ-exponentially convex on $\mathcal{I}$. Then the inequality

$$|\mathcal{E}(Y, \mathcal{V}, p)| \leq \sum_{j=0}^{\kappa-1} (\varrho_{j+1} - \varrho_j) \Big(\frac{|e^{\theta \varrho_j} Y'(\varrho_j)|^{\lambda} + |e^{\theta \varrho_{j+1}} Y'(\varrho_{j+1})|^{\lambda}}{2} \Big)^{\frac{1}{\lambda}} \int_0^1 \int_{\varrho_j}^{\mathcal{L}(\varrho_j, \varrho_{j+1}, \tau)} \mathcal{V}(\varrho) d\varrho d\tau$$

holds for any $p \in \mathcal{I}$ if all the conditions of Theorem 2.2 are satisfied.

Proof. Applying Theorem 2.3 to the interval $[\varrho_j, \varrho_{j+1}]$ $(j = 0, 1, ..., \kappa - 1)$ of the partition Δ, we get

$$\Big| Y\Big(\frac{n\varrho_j + \varrho_{j+1}}{n+1} \Big) \int_{\varrho_j}^{\varrho_{j+1}} \mathcal{V}(\varrho) d\varrho - \int_{\varrho_j}^{\varrho_{j+1}} Y(\varrho) \mathcal{V}(\varrho) d\varrho \Big|$$

$$\leq (\varrho_{j+1} - \varrho_j) \Big(\frac{|e^{\theta \varrho_j} Y'(\varrho_j)|^{\lambda} + |e^{\theta \varrho_{j+1}} Y'(\varrho_{j+1})|^{\lambda}}{2} \Big)^{\frac{1}{\lambda}} \int_0^1 \int_{\varrho_j}^{\mathcal{L}(\varrho_j, \varrho_{j+1}, \tau)} \mathcal{V}(\varrho) d\varrho d\tau.$$

Summing the above inequality on j from 0 to $\kappa - 1$ and making use of the triangle inequality together with the exponential convexity of $|Y'|^{\lambda}$ lead to

$$\Big| \mathcal{T}(Y, \mathcal{V}, p) - \int_{\rho_1}^{\rho_2} Y(\varrho) \mathcal{V}(\varrho) d\varrho \Big|$$

$$\leq \sum_{j=0}^{\kappa-1} (\varrho_{j+1} - \varrho_j) \Big(\frac{|e^{\theta \varrho_j} Y'(\varrho_j)|^{\lambda} + |e^{\theta \varrho_{j+1}} Y'(\varrho_{j+1})|^{\lambda}}{2} \Big)^{\frac{1}{\lambda}} \int_0^1 \int_{\varrho_j}^{\mathcal{L}(\varrho_j, \varrho_{j+1}, \tau)} \mathcal{V}(\varrho) d\varrho d\tau,$$

this completes the proof of Theorem 4.1. □

Theorem 4.2. Let $\lambda \geq 1$, $\theta \in \mathbb{R}$, and $|Y'|^{\lambda}$ be θ-exponentially convex on $\mathcal{I}$. Then the inequality

$$|\mathcal{E}(Y, \mathcal{V}, p)|$$

$$\leq \frac{1}{n+1} \sum_{j=0}^{\kappa-1} (\varrho_{j+1} - \varrho_j) \Big[\Big[\max \Big\{ |e^{\theta \varrho_j} Y'(\varrho_j)|^{\lambda}, \Big| e^{\theta \left(\frac{n\varrho_j + \varrho_{j+1}}{n+1} \right)} Y'\Big(\frac{n\varrho_j + \varrho_{j+1}}{n+1} \Big) \Big|^{\lambda} \Big\} \Big]^{\frac{1}{\lambda}}$$

$$+ \Big[\max \Big\{ |e^{\theta \varrho_{j+1}} Y'(\varrho_{j+1})|^{\lambda}, \Big| e^{\theta \left(\frac{\varrho_j + n\varrho_{j+1}}{n+1} \right)} \tau Y'\Big(\frac{\varrho_j + n\varrho_{j+1}}{n+1} \Big) \Big|^{\lambda} \Big\} \Big]^{\frac{1}{\lambda}} \Big] \int_0^1 \int_{\varrho_j}^{\mathcal{L}(\varrho_j, \varrho_{j+1}, \tau)} \mathcal{V}(\varrho) d\varrho d\tau$$

holds for every partition Δ of $\mathcal{I}$ if all the hypothesis of Theorem 2.2 are satisfied.

Proof. Making use of Theorem 2.5 on the interval $[\varrho_j, \varrho_{j+1}]$ $(j = 0, 1, \cdots, \kappa - 1)$ of the partition $\triangle$, we get

$$\Big| Y\Big(\frac{n\varrho_j + \varrho_{j+1}}{n+1} \Big) \int_{\varrho_j}^{\varrho_{j+1}} \mathcal{V}(\varrho) d\varrho - \int_{\varrho_j}^{\varrho_{j+1}} Y(\varrho) \mathcal{V}(\varrho) d\varrho \Big|$$

$$\leq \frac{(\varrho_{j+1}-\varrho_j)}{n+1}\Bigg[\Bigg[\max\Big\{|e^{\theta\varrho_j}Y'(\varrho_j)|^\lambda, \Big|e^{\theta\left(\frac{n\varrho_j+\varrho_{j+1}}{n+1}\right)}Y'\Big(\frac{n\varrho_j+\varrho_{j+1}}{n+1}\Big)\Big|^\lambda\Big\}\Bigg]^{\frac{1}{\lambda}}$$

$$+\Bigg[\max\Big\{|e^{\theta\varrho_{j+1}}Y'(\varrho_{j+1})|^\lambda, \Big|e^{\theta\left(\frac{\varrho_j+n\varrho_{j+1}}{n+1}\right)}Y'\Big(\frac{\varrho_j+n\varrho_{j+1}}{n+1}\Big)\Big|^\lambda\Big\}\Bigg]^{\frac{1}{\lambda}}\Bigg]\int_0^1\int_{\varrho_j}^{\mathcal{L}(\varrho_j,\varrho_{j+1},\tau)}\mathcal{V}(\varrho)d\varrho d\tau.$$

Summing the above inequality on j from 0 to $\kappa-1$ and making use the triangle inequality together with the exponential convexity of $|Y'|^\lambda$ lead to the conclusion that

$$\Big|\mathcal{T}(Y,\mathcal{V},p)-\int_{\rho_1}^{\rho_2}Y(\varrho)\mathcal{V}(\varrho)d\varrho\Big|$$

$$\leq \frac{1}{n+1}\sum_{j=0}^{\kappa-1}(\varrho_{j+1}-\varrho_j)\Bigg[\Bigg[\max\Big\{|e^{\theta\varrho_j}Y'(\varrho_j)|^\lambda, \Big|e^{\theta\left(\frac{n\varrho_j+\varrho_{j+1}}{n+1}\right)}Y'\Big(\frac{n\varrho_j+\varrho_{j+1}}{n+1}\Big)\Big|^\lambda\Big\}\Bigg]^{\frac{1}{\lambda}}$$

$$+\Bigg[\max\Big\{|e^{\theta\varrho_{j+1}}Y'(\varrho_{j+1})|^\lambda, \Big|e^{\theta\left(\frac{\varrho_j+n\varrho_{j+1}}{n+1}\right)}Y'\Big(\frac{\varrho_j+n\varrho_{j+1}}{n+1}\Big)\Big|^\lambda\Big\}\Bigg]^{\frac{1}{\lambda}}\Bigg]\int_0^1\int_{\varrho_j}^{\mathcal{L}(\varrho_j,\varrho_{j+1},\tau)}\mathcal{V}(\varrho)d\varrho d\tau,$$

this completes the proof of Theorem 4.2. □

4.2. Applications to random variable

Let $0<\rho_1<\rho_2$, $r\in\mathbb{R}$, $\mathcal{V}:[\rho_1,\rho_2]\to[0,\infty]$ be continuous on $[\rho_1,\rho_2]$ and symmetric with respect to $\frac{n\rho_1+\rho_2}{n+1}$ and X be a continuous random variable having probability density function $\mathcal{V}$. Then the rth-moment $E_r(X)$ of X is given by

$$E_r(X)=\int_{\rho_1}^{\rho_2}\tau^r\mathcal{V}(\tau)d\tau$$

if it is finite.

Theorem 4.3. The inequality

$$\Big|E_r(X)-\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)^r\Big|\leq\frac{r(\rho_2-\rho_1)}{(n+1)^2}\Big[|e^{\theta\rho_1}\rho_1^{r-1}|+|e^{\theta\rho_2}\rho_2^{r-1}|\Big]$$

holds for $0<\rho_1<\rho_2$ and $r\geq 2$.

Proof. Let $Y(\tau)=\tau^r$. Then $|Y'(\tau)|=r\tau^{r-1}$ is exponentially convex function. Note that

$$\int_{\rho_1}^{\rho_2}Y(\varrho)\mathcal{V}(\varrho)d\varrho=E_r(X),\qquad\int_{\rho_1}^{L(\rho_1,\rho_2,\tau)}\mathcal{V}(\varrho)d\varrho\leq\int_{\rho_1}^{\frac{n\rho_1+\rho_2}{n+1}}\mathcal{V}(\varrho)d\varrho=\frac{1}{n+1}\quad(\tau\in[0,1]),$$

$$Y\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)=\Big(\frac{n\rho_1+\rho_2}{n+1}\Big)^r,\quad |e^{\theta\rho_1}Y'(\rho_1)|+|e^{\theta\rho_2}Y'(\rho_2)|=r\big(e^{\theta\rho_1}\rho_1^{r-1}+e^{\theta\rho_2}\rho_2^{r-1}\big).$$

Therefore, the desired result follows from inequality (2.2) immediately. □

Theorem 4.4. The inequality

$$\left|E_r(X) - \left(\frac{n\rho_1 + \rho_2}{n+1}\right)^r\right| \le \frac{r(\rho_2 - \rho_1)}{(n+1)^2}\left[|e^{\theta\rho_2}\rho_2^{r-1}| + \left|e^{\theta\left(\frac{n\rho_1+\rho_2}{n+1}\right)}\left(\frac{n\rho_1 + \rho_2}{n+1}\right)^{r-1}\right|\right]$$

holds for $0 < \rho_1 < \rho_2$ and $r \ge 1$.

Proof. Let $Y(\tau) = \tau^r$. Then $|Y(\tau)| = r\tau^{r-1}$ is increasing and exponentially quasi-convex, and the desired result can be obtained by use of inequality (2.15) and the similar arguments of Theorem 4.3. □

4.3. Applications to special means

A real-valued function $\Omega : (0,\infty) \times (0,\infty) \to (0,\infty)$ is said to be a bivariate mean if $\min\{\rho_1,\rho_2\} \le \Omega(\rho_1,\rho_2) \le \max\{\rho_1,\rho_2\}$ for all $\rho_1,\rho_2 \in (0,\infty)$. Recently, the properties and applications for the bivariate means and their related special functions have attracted the attention of many researchers [73–86]. In particular, many remarkable inequalities for the bivariate means can be found in the literature [87–96].

In this subsection, we use the results obtained in Section 2 to give some applications to the special bivariate means.

Let $\rho_1,\rho_2 > 0$ with $\rho_1 \ne \rho_2$. Then the arithmetic mean $A(\rho_1,\rho_2)$, weighted arithmetic mean $A(\rho_1,\rho_2;w_1,w_2)$ and n-th generalized logarithmic mean $L_n(\rho_1,\rho_2)$ are defined by

$$A(\rho_1,\rho_2) = \frac{\rho_1 + \rho_1}{2}, \quad A(\rho_1,\rho_2;w_1,w_2) = \frac{w_1\rho_1 + w_2\rho_2}{w_1 + w_2}$$

and

$$L_n(\rho_1,\rho_2) = \left[\frac{\rho_2^{n+1} - \rho_1^{n+1}}{(n+1)(\rho_2 - \rho_1)}\right]^{1/n}.$$

Let $\varrho > 0$, $r \in \mathbb{N}$, $Y(\varrho) = \varrho^r$ and $\mathcal{V} : [\rho_1,\rho_2] \to \mathbb{R}^+$ be a differentiable mapping such that it is symmetric with respect to $\frac{n\rho_1+\rho_2}{n+1}$. Then Theorem 2.2 implies that

$$\left|\left(\frac{n\rho_1 + \rho_2}{n+1}\right)^r \int_{\rho_1}^{\rho_2} \mathcal{V}(\varrho)d\varrho - \int_{\rho_1}^{\rho_2} \varrho^r\mathcal{V}(\varrho)d\varrho\right| \le \frac{r(\rho_2 - \rho_1)}{n+1}\left[|e^{\theta\rho_1}\rho_1^{n-1}| + |e^{\theta\rho_2}\rho_2^{n-1}|\right]\int_0^1 \int_{\rho_1}^{L(\rho_1,\rho_2,\tau)} \mathcal{V}(\varrho)d\varrho d\tau,$$

which can be rewritten as

$$\left|\left(A(\rho_1,\rho_2;n,1)\right)^r \int_{\rho_1}^{\rho_2} \mathcal{V}(\varrho)d\varrho - \int_{\rho_1}^{\rho_2} \varrho^r\mathcal{V}(\varrho)d\varrho\right|$$

$$\le \frac{2r(\rho_2 - \rho_1)}{n+1}\left[A(|e^{\theta\rho_1}\rho_1^{n-1}|, |e^{\theta\rho_2}\rho_2^{n-1}|)\right]\int_0^1 \int_{\rho_1}^{L(\rho_1,\rho_2,\tau)} \mathcal{V}(\varrho)d\varrho d\tau. \tag{4.2}$$

Let $\mathcal{V} = 1$. Then inequality (4.2) leads to Corollary 4.1 immediately.

Corollary 4.1. Let $\rho_2 > \rho_1 > 0$, $r \in \mathbb{N}$ and $r \ge 2$. Then one has

$$\left|\left(A(\rho_1,\rho_2;n,1)\right)^r - L_r^r(\rho_1,\rho_2)\right| \le \frac{r(\rho_2 - \rho_1)^2}{(n+1)^2}\left[A(|e^{\theta\rho_1}\rho_1^{n-1}|, |e^{\theta\rho_2}\rho_2^{n-1}|)\right].$$

5. Conclusion

We conducted a preliminary attempt to develop a novel formulation presumably for new Hermite-Hadamard type for proposing two new classes of exponentially convex and exponentially quasi-convex functions and presented their analogues. An auxiliary result was chosen because of its success in leading to the well-known Hermite-Hadamard type inequalities. An intriguing feature of an auxiliary is that this simple formulation has significant importance while studying the error bounds of different numerical quadrature rules. Such a potential the connection needs further investigation. We conclude that the results derived in this paper are general in character and give some contributions to inequality theory and fractional calculus as an application for establishing the uniqueness of solutions in boundary value problems, fractional differential equations, and special relativity theory. This interesting aspect of time is worth further investigation. Finally, the innovative concept of exponentially convex functions has potential application in rth-moments and special bivariate mean to show the reported result. Our findings are the refinements and generalizations of the existing results that stimulate futuristic research.

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Conflict of interest

The authors declare that they have no competing interests.

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