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Research article

Monotonicity and inequalities related to complete elliptic integrals of the second kind

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Abstract: In the paper, the authors present some monotonicity properties of certain functions defined in terms of the complete elliptic integrals of the second kind and some elementary functions and, consequently, improve several known inequalities for the complete elliptic integrals of the second kind.

Keywords: monotonicity; inequality; complete elliptic integral of the second kind
Mathematics Subject Classification: Primary 33E05; Secondary 26A48, 26D15, 33C75

1. Introduction

For $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \dots$, the Gaussian hypergeometric function can be defined [4, pp. 32–47] by

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad |x| < 1,$$

where $(z)_0 = 1$ for $z \neq 0$,

$$(z)_n = \prod_{j=0}^{n-1} (z + j) = \frac{\Gamma(z + n)}{\Gamma(z)}, \quad n = 1, 2, \dots, \tag{1.1}$$

and

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{\prod_{k=0}^{n} (z + k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

is the classical Euler gamma function [17, 22].

For $r \in (0,1)$, the well-known complete elliptic integrals of the first and the second kinds are defined respectively by

$$\begin{cases} \mathcal{K}(r) = \dfrac{\pi}{2} f\Big(\dfrac{1}{2}, \dfrac{1}{2}; 1; r^2\Big) = \displaystyle\int_0^{\pi/2} \frac{1}{\sqrt{1 - r^2 \sin^2\theta}} \mathrm{d}\theta, \\ \mathcal{K}(0) = \dfrac{\pi}{2}, \quad \mathcal{K}(1) = \infty \end{cases} \tag{1.2}$$

and

$$\begin{cases} \mathcal{E}(r) = \dfrac{\pi}{2} F\Big(-\dfrac{1}{2}, \dfrac{1}{2}; 1; r^2\Big) = \displaystyle\int_0^{\pi/2} \sqrt{1 - r^2 \sin^2\theta}\, \mathrm{d}\theta, \\ \mathcal{E}(0) = \dfrac{\pi}{2}, \quad \mathcal{E}(1) = 1. \end{cases} \tag{1.3}$$

The complete elliptic integrals have many important applications in physics [3], engineering [7], geometric function theory [6, 30, 34], quasi-conformal analysis [16, 28, 32], theory of mean values [9, 12–14, 20, 24–26, 33], number theory [27, 35], and other related fields. Recently, the complete elliptic integrals have attracted the attention of numerous mathematicians. In particular, many remarkable properties and inequalities for the complete elliptic integrals can be found in the literature [2, 29, 31]. For more information on applications, please refer to [5,19,21,23,36,38] and related references therein.

In [11], Guo and Qi found some inequalities for $\mathcal{E}(r)$, one of these inequalities is

$$\frac{\pi}{2} - \frac{1}{2}\ln\frac{(1+r)^{r-1}}{(1-r)^{r+1}} < \mathcal{E}(r) < \frac{\pi - 1}{2} + \frac{1-r^2}{4r}\ln\frac{1+r}{1-r} \tag{1.4}$$

which holds true for all $0 < r < 1$.

In [37], Yin and Qi established some double inequalities for $\mathcal{E}(r)$ by virtue of the Lupaş integral inequality. For instance, the double inequality

$$\frac{\pi\sqrt{6 + 2\sqrt{1-r^2} - 3r^2}}{4\sqrt{2}} \le \mathcal{E}(r) \le \frac{\pi\sqrt{10 - 2\sqrt{1-r^2} - 5r^2}}{4\sqrt{2}} \tag{1.5}$$

holds for all $r \in (0,1)$.

The purpose of this paper is to present some monotonicity properties of certain functions defined in terms of the complete elliptic integrals of the second kind $\mathcal{E}(r)$ and some elementary functions and, consequently, derive some inequalities which improve the double inequalities (1.4) and (1.5).

Our main results are the following three theorems.

Theorem 1. *For $0 < r < 1$, the function*

$$r \mapsto \frac{\pi^2(2 - r^2) - 8\mathcal{E}^2(r)}{2 - r^2 - 2\sqrt{1-r^2}}$$

is strictly decreasing from $(0,1)$ onto $(\pi^2 - 8, \frac{\pi^2}{4})$. Consequently, for all $r \in (0,1)$, we have

$$\frac{\pi\sqrt{2(4-\alpha_1) + 2\alpha_1\sqrt{1-r^2} + (\alpha_1 - 4)r^2}}{4\sqrt{2}} \le \mathcal{E}(r) \le \frac{\pi\sqrt{2(4-\beta_1) + 2\beta_1\sqrt{1-r^2} + (\beta_1 - 4)r^2}}{4\sqrt{2}}, \tag{1.6}$$

where the constants $\alpha_1 = 1$ *and* $\beta_1 = 4(1-\frac{8}{\pi^2}) = 0.757\ldots$ *are the best possible in the sense that they can not be replaced by any larger and smaller constants respectively. The equality in the left-hand side of* (1.6) *is attained only when* $r \to 0^+$*, while the equality in the right-hand side of* (1.6) *is attained only when* $r \to 1^-$.

Theorem 2. *For* $0 < r < 1$*, the function*

$$r \mapsto \frac{r[\pi/2 - \mathcal{E}(r)]}{2r - (1-r^2)\ln[(1+r)/(1-r)]}$$

is strictly decreasing from $(0,1)$ *onto* $(\frac{\pi-2}{4}, \frac{3\pi}{32})$*. Consequently, for all* $r \in (0,1)$*, we have*

$$\frac{\pi - 4\alpha_2}{2} + \frac{\alpha_2(1-r^2)}{r}\ln\frac{1+r}{1-r} < \mathcal{E}(r) < \frac{\pi - 4\beta_2}{2} + \frac{\beta_2(1-r^2)}{r}\ln\frac{1+r}{1-r}, \tag{1.7}$$

where the constants $\alpha_2 = \frac{3\pi}{32} = 0.294\ldots$ *and* $\beta_2 = \frac{\pi-2}{4} = 0.285\ldots$ *are the best possible.*

Theorem 3. *For* $0 < r < 1$*, the function*

$$r \mapsto \frac{\mathcal{E}(r) - \pi/2 - \ln(1-r^2)}{r\ln[(1+r)/(1-r)] + \ln(1-r^2)}$$

is strictly increasing from $(0,1)$ *onto* $(1-\frac{\pi}{8}, \infty)$*. Consequently, for all* $r \in (0,1)$*, we have*

$$\mathcal{E}(r) > \frac{\pi}{2} + \alpha_3 r\ln\frac{1+r}{1-r} + (1+\alpha_3)\ln(1-r^2), \tag{1.8}$$

where the constant $\alpha_3 = 1 - \frac{\pi}{8} = 0.607\ldots$ *is the best possible.*

2. Lemmas

In order to prove our main results stated in the above three theorems, we need the following lemmas. In [4, pp. 474–475, Appendix E and Theorem 3.21 (7)], one can find that

$$\begin{aligned} &\frac{\mathrm{d}\mathcal{K}(r)}{\mathrm{d}r} = \frac{\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)}{r(1-r^2)}, \quad \frac{\mathrm{d}\mathcal{E}(r)}{\mathrm{d}r} = \frac{\mathcal{E}(r) - \mathcal{K}(r)}{r}, \\ &\frac{\mathrm{d}\,[\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]}{\mathrm{d}r} = r\mathcal{K}(r), \quad \frac{\mathrm{d}\,[\mathcal{K}(r) - \mathcal{E}(r)]}{\mathrm{d}r} = \frac{r\mathcal{E}(r)}{(1-r^2)}, \end{aligned} \tag{2.1}$$

and

$$\lim_{r\to 1}\Big[(1-r^2)^{\alpha/2}\mathcal{K}(r)\Big] = 0, \quad \alpha \geq \frac{1}{2}. \tag{2.2}$$

Lemma 1 ([4, Theorem 1.25])**.** *For* $-\infty < a < b < \infty$*, let* $f, g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$*, differentiable on* (a,b)*, and* $g'(x) \neq 0$ *on* (a,b)*. If* $\frac{f'(x)}{g'(x)}$ *is increasing (decreasing) on* (a,b)*, then so are the ratios*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $\frac{f'(x)}{g'(x)}$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2 ([1, Theorem 15] and [4, Theorem 3.21 and Exercises 3.43])**.** *Let $r \in (0,1)$. Then*

1. *the function $r \to \frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r^2}$ is strictly increasing from $(0,1)$ onto $(\frac{\pi}{4},1)$;*
2. *the function $r \to \frac{\mathcal{K}(r)-\mathcal{E}(r)}{r^2}$ is strictly increasing from $(0,1)$ onto $(\frac{\pi}{4},\infty)$;*
3. *the function $r \to \frac{\mathcal{E}^2(r)-(1-r^2)\mathcal{K}^2(r)}{r^4}$ is strictly increasing from $(0,1)$ onto $(\frac{\pi^2}{32},1)$;*
4. *the function $r \to \frac{(1-r^2)^{3/8}[\mathcal{K}(r)-\mathcal{E}(r)]}{r^2}$ is strictly decreasing from $(0,1)$ onto $(0,\frac{\pi}{4})$.*

Lemma 3 ([8, Theorem 1] and [10, 18])**.** *For $n > 0$, the double inequality*

$$\frac{1}{\sqrt{n+\mu_1}} \le \frac{\Gamma(n+1/2)}{\Gamma(n+1)} < \frac{1}{\sqrt{n+\mu_2}}$$

is valid, where the constants $\mu_1 = \frac{4}{\pi} - 1$ and $\mu_2 = \frac{1}{4}$ are the best possible.

Lemma 4. *For $r \in (0,1)$, the function*

$$r \mapsto \frac{(1-r^2)^{1/2}[\mathcal{E}^2(r) - (1-r^2)\mathcal{K}^2(r)]}{r^4}$$

is strictly decreasing from $(0,1)$ onto $(0,\frac{\pi^2}{32})$.

Proof. Let $I(r) = \frac{I_1(r)}{I_2(r)}$, where $I_1(r) = \mathcal{E}^2(r) - (1-r^2)\mathcal{K}^2(r)$ and $I_2(r) = \frac{r^4}{(1-r^2)^{1/2}}$ with $I_1(0) = I_2(0) = 0$. By virtue of those formulas in (2.1) and (2.2), an elementary computation shows that

$$\begin{aligned}
\frac{I_1'(r)}{I_2'(r)} &= \left(\frac{2\mathcal{E}(r)[\mathcal{E}(r)-\mathcal{K}(r)]}{r} + 2r\mathcal{K}^2(r) - \frac{2\mathcal{K}[\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]}{r}\right)\frac{(1-r^2)^{3/2}}{r^3[3(1-r^2)+1]} \\
&= 2\left(\frac{(1-r^2)^{3/8}[\mathcal{K}(r)-\mathcal{E}(r)]}{r^2}\right)^2 \frac{(1-r^2)^{3/4}}{3(1-r^2)+1} \\
&= 2\left(\frac{(1-r^2)^{3/8}[\mathcal{K}(r)-\mathcal{E}(r)]}{r^2}\right)^2 I_3\Big((1-r^2)^{1/2}\Big),
\end{aligned}$$

where $I_3(r) = \frac{r^{3/2}}{(3r^2+1)}$. It is easy to verify that the function $I_3(r)$ is strictly increasing from $(0,1)$ onto $(0,\frac{1}{4})$. From Lemma 1 and the fourth item in Lemma 2, it follows directly that $I(r)$ is strictly decreasing in $(0,1)$.

By L'Hôpital's rule and the fourth item in Lemma 2, we obtain

$$\lim_{r\to 0^+} I(r) = \lim_{r\to 0^+} \frac{I_1'(r)}{I_2'(r)} = 2\left(\frac{\pi}{4}\right)^2 \frac{1}{4} = \frac{\pi^2}{32},$$

while $I(1^-) = 0$ is clear. □

Lemma 5. *For $r \in (0,1)$, the function*

$$r \mapsto \frac{1}{r^2}\left[\mathcal{E}(r) - \frac{\pi}{2}(1-r^2)\right]$$

is strictly decreasing from $(0,1)$ onto $(1,\frac{3\pi}{8})$.

Proof. Let $J(r) = \frac{J_1(r)}{J_2(r)}$, where $J_1(r) = \mathcal{E}(r) - \frac{\pi}{2}(1 - r^2)$ and $J_2(r) = r^2$ with $J_1(0) = J_2(0) = 0$. Making use of those formulas in (2.1) and (2.2) and computing give

$$\frac{J_1'(r)}{J_2'(r)} = \frac{1}{2r}\left[\pi r - \frac{\mathcal{K}(r) - \mathcal{E}(r)}{r}\right] = \frac{\pi}{2} - \frac{\mathcal{K}(r) - \mathcal{E}(r)}{2r^2}.$$

Utilizing Lemma 1 and the second item in Lemma 2 yields that the function $J(r)$ is strictly decreasing on $(0, 1)$.

Making use of L'Hôpital's rule and the second item in Lemma 2 shows

$$\lim_{r\to 0^+} J(r) = \lim_{r\to 0^+} \frac{J_1'(r)}{J_2'(r)} = \frac{\pi}{2} - \frac{1}{2} \times \frac{\pi}{4} = \frac{3\pi}{8}.$$

It is clear that $J(1) = 1$. □

Lemma 6. *For $n \geq 0$, we have*

$$c_n = 2 - \frac{\pi}{8} \frac{[(\frac{1}{2})_n]^2}{(n+1)(n!)^2} > 0.$$

Proof. For $n = 0$, it is trivial.

For $n \geq 1$, since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, applying Lemma 3 and (1.1) arrives at

$$c_n = 2 - \frac{\pi}{8(n+1)}\left[\frac{\Gamma(n + 1/2)}{\sqrt{\pi}\,\Gamma(n+1)}\right]^2 > 2 - \frac{1}{8(n+1)(n+1/4)} > 0.$$

The proof of Lemma 6 is complete. □

3. Proofs of main results

Now we are in a position to prove our main results.

Proof of Theorem 1. Let $F(r) = \frac{f_1(r)}{f_2(r)}$, where

$$f_1(r) = \pi^2(2 - r^2) - 8\mathcal{E}^2(r) \quad \text{and} \quad f_2(r) = 2 - 2(1 - r^2)^{1/2} - r^2$$

with $f_1(0) = f_2(0) = 0$. From those formulas in (2.1) and (2.2), it follows that

$$\frac{f_1'(r)}{f_2'(r)} = \frac{8\mathcal{E}(r)[\mathcal{K}(r) - \mathcal{E}(r)]/r - \pi^2 r}{r[1/(1 - r^2)^{1/2} - 1]} = \frac{8\mathcal{E}(r)[\mathcal{K}(r) - \mathcal{E}(r)]/r^2 - \pi^2}{1/(1 - r^2)^{1/2} - 1} \triangleq \frac{f_3(r)}{f_4(r)},$$

where $f_3(r) = \frac{8\mathcal{E}(r)[\mathcal{K}(r)-\mathcal{E}(r)]}{r^2} - \pi^2$ and $f_4(r) = \frac{1}{(1-r^2)^{1/2}} - 1$.

By the second item in Lemma 2 and (1.3), it is easy to see that $f_3(0) = f_4(0) = 0$. Differentiating and using those formulas in (2.1) and (2.2) give

$$\frac{f_3'(r)}{f_4'(r)} = 8\frac{\left(\frac{r\mathcal{E}^2(r)}{(1-r^2)} - \frac{[\mathcal{K}(r)-\mathcal{E}(r)]^2}{r}\right)r^2 - 2r\mathcal{E}(r)[\mathcal{K}(r) - \mathcal{E}(r)]}{r^4}\frac{(1 - r^2)^{3/2}}{r}$$

$$= 8\frac{(1-r^2)^{1/2}[\mathcal{E}^2(r)-(1-r^2)\mathcal{K}^2(r)]}{r^4}.$$

Employing (2.1) and (2.2) and considering Lemmas 1 and 4 reveal that $F(r)$ is strictly decreasing in $(0,1)$.

Making use of L'Hôpital's rule and considering Lemma 4 demonstrates

$$\lim_{r\to 0^+} F(r) = \lim_{r\to 0^+}\frac{f_1'(r)}{f_2'(r)} = \lim_{r\to 0^+}\frac{f_3'(r)}{f_4'(r)} = 8\times\frac{\pi^2}{32} = \frac{\pi^2}{4}.$$

From (1.3), it is clear that $\lim_{r\to 1^-} F(r) = \pi^2 - 8$.

The double inequality (1.6) and its equality cases follow from the monotonicity of $F(r)$ on $(0,1]$. □

Proof of Theorem 2. Let $G(r) = \frac{g_1(r)}{g_2(r)}$, where

$$g_1(r) = r\left[\frac{\pi}{2} - \mathcal{E}(r)\right] \quad \text{and} \quad g_2(r) = 2r - (1-r^2)\ln\frac{1+r}{1-r}$$

with $g_1(0) = g_2(0) = 0$. From (2.1), a simple computation leads to

$$\frac{g_1'(r)}{g_2'(r)} = \frac{1}{2}\frac{[\pi/2 - \mathcal{E}(r)] + [\mathcal{K}(r) - \mathcal{E}(r)]}{r\ln[(1+r)/(1-r)]} \triangleq \frac{g_3(r)}{g_4(r)},$$

where $g_3(r) = \frac{[\frac{\pi}{2}-\mathcal{E}(r)]+[\mathcal{K}(r)-\mathcal{E}(r)]}{r}$ and $g_4(r) = 2\ln\frac{1+r}{1-r}$.

By the formulas in (1.2) and (1.3), the fourth item in Lemma 2, and $g_3(0) = g_4(0) = 0$, we obtain

$$\begin{aligned}\frac{g_3{}'(r)}{g_4{}'(r)} &= \frac{\mathcal{K}(r) - \mathcal{E}(r) + \frac{r^2\mathcal{E}(r)}{1-r^2} - (\frac{\pi}{2} - \mathcal{E}(r) + [\mathcal{K}(r) - \mathcal{E}(r)])}{r^2}\frac{1-r^2}{4}\\ &= \frac{r^2\mathcal{E}(r) - [\frac{\pi}{2} - \mathcal{E}(r)](1-r^2)}{4r^2} = \frac{\mathcal{E}(r) - \frac{\pi}{2}(1-r^2)}{4r^2} = \frac{J(r)}{4}.\end{aligned}$$

From the formulas in (2.1) and (2.2) and by Lemmas 1 and 5, it follows that $G(r)$ is strictly decreasing in $(0,1)$.

Making use of L'Hôpital's rule and Lemma 5 leads to

$$\lim_{r\to 0^+} G(r) = \lim_{r\to 0^+}\frac{g_1'(r)}{g_2'(r)} = \lim_{r\to 0^+}\frac{g_3'(r)}{g_4'(r)} = \frac{3\pi}{32}.$$

It is straightforward to obtain $\lim_{r\to 1^-} G(r) = \frac{\pi-2}{4}$.

The double inequality (1.7) in Theorem 2 follows immediately from the monotonicity properties of $G(r)$ on $(0,1]$. □

Proof of Theorem 3. It is general knowledge that

$$\frac{1}{1-r^2} = \sum_{n=0}^{\infty} r^{2n}, \quad |r| < 1. \tag{3.1}$$

Let $H(r)=\frac{h_1(r)}{h_2(r)}$, where $h_1(r)=\mathcal{E}(r)-\frac{\pi}{2}-\ln(1-r^2)$ and $h_2(r)=r\ln\frac{1+r}{1-r}+\ln(1-r^2)$ with $h_1(0)=h_2(0)=0$. A direct differentiation results in

$$\frac{h_1'(r)}{h_2'(r)}=\frac{\frac{2r}{(1-r^2)}-\frac{\mathcal{K}(r)-\mathcal{E}(r)}{r}}{\ln[(1+r)/(1-r)]}\triangleq\frac{h_3(r)}{h_4(r)},$$

where

$$h_3(r)=\frac{2r}{(1-r^2)}-\frac{\mathcal{K}(r)-\mathcal{E}(r)}{r}\quad\text{and}\quad h_4(r)=\ln\frac{1+r}{1-r}$$

with $h_3(0)=h_4(0)=0$. By those formulas in (2.1) and (2.2), utilizing (1.2) and (3.1) gives

$$\begin{aligned}\frac{h_3'(r)}{h_4'(r)}&=\left[\frac{2}{(1-r^2)}+\frac{4r^2}{(1-r^2)^2}-\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{(1-r^2)r^2}\right]\frac{1-r^2}{2}\\ &=\frac{1}{2}\left[\frac{4}{1-r^2}-\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r^2}-2\right]=\frac{1}{2}\left[4\sum_{n=0}^{\infty}r^{2n}-\frac{\pi}{4}\sum_{n=0}^{\infty}\frac{[(\frac{1}{2})_n]^2}{(n+1)(n!)^2}r^{2n}-2\right]\\ &=\sum_{n=0}^{\infty}\left[2-\frac{\pi}{8}\frac{[(\frac{1}{2})_n]^2}{(n+1)(n!)^2}\right]r^{2n}-1=\sum_{n=0}^{\infty}c_nr^{2n}-1=1-\frac{\pi}{8}+\sum_{n=1}^{\infty}c_nr^{2n}.\end{aligned}$$

Therefore, by Lemmas 1 and 6, we see that $H(r)$ is strictly increasing in $(0,1)$.

Making use of L'Hôpital's rule and Lemma 6 acquires

$$\lim_{r\to0^+}H(r)=\lim_{r\to0^+}\frac{h_1'(r)}{h_2'(r)}=\lim_{r\to0^+}\frac{h_3'(r)}{h_4'(r)}=1-\frac{\pi}{8},$$

while $\lim_{r\to1^-}H(r)=\infty$.

The inequality (1.8) in Theorem 3 follows from the monotonicity of $H(r)$. □

4. Comparisons

In this section, we compare our newly-established bounds for $\mathcal{E}(r)$ with the bounds in (1.4) and (1.5) in terms of remarks.

Remark 1. Let $\beta_1=4(1-\frac{8}{\pi^2})$ and $r\in(0,1)$. Let

$$S_1(r)=10-2\sqrt{1-r^2}-5r^2,\quad Q_1(r)=2(4-\beta_1)+2\beta_1\sqrt{1-r^2}+(\beta_1-4)r^2,$$

and

$$L_1(r)=S_1(r)-Q_1(r)=(1+\beta_1)\left(2-2\sqrt{1-r^2}-r^2\right).$$

It is easy to verify that the function $L_1(r)$ is strictly increasing on $(0,1)$. Hence, we have $L_1(r)>L_1(0)=0$. Consequently, the upper bound in (1.6) in Theorem 1 is better than the upper bound in (1.4).

Remark 2. For $\beta_2=\frac{\pi-2}{4}$ and $r\in(0,1)$, let

$$S_2(r)=\frac{\pi-1}{2}+\frac{1-r^2}{4r}\ln\frac{1+r}{1-r},\quad Q_2(r)=\frac{\pi-4\beta_2}{2}+\frac{\beta_2(1-r^2)}{r}\ln\frac{1+r}{1-r},$$

and

$$L_2(r) = S_2(r) - Q_2(r) = \frac{4\beta_2 - 1}{2} + \frac{1 - 4\beta_2}{4}\frac{1 - r^2}{r}\ln\frac{1 + r}{1 - r} = \frac{4\beta_2 - 1}{4}[2 - L_3(r)],$$

where

$$L_3(r) = \frac{1 - r^2}{r}\ln\frac{1 + r}{1 - r}.$$

Employing L'Hôpital's rule and Lemma 1 reveals that the function $L_3(r)$ is decreasing from $(0, 1)$ onto $(0, 2)$. Therefore, we have $L_2(r) > L_2(0) = 0$. Consequently, the right hand side of inequality in (1.7) in Theorem 2 is better than the right hand side of inequality in (1.5).

Remark 3. For $\alpha_3 = 1 - \frac{\pi}{8}$, let

$$S_3(r) = \frac{\pi}{2} + \alpha_3 r\ln\frac{1 + r}{1 - r} + (1 + \alpha_3)\ln(1 - r^2) \quad \text{and} \quad Q_3(r) = \frac{\pi}{2} - \frac{1}{2}\ln\frac{(1 + r)^{r-1}}{(1 - r)^{r+1}}.$$

A elementary computation gives

$$\begin{aligned} L_4(r) = S_3(r) - Q_3(r) &= \alpha_3 r\ln\frac{1 + r}{1 - r} + (1 + \alpha_3)\ln[(1 + r)(1 - r)] + \frac{1}{2}\ln\frac{(1 + r)^{r-1}}{(1 - r)^{r+1}} \\ &= \left(\frac{1}{2} + \alpha_3\right)\left(r\ln\frac{1 + r}{1 - r} + \ln[(1 + r)(1 - r)]\right). \end{aligned}$$

It is not difficult to verify that the function $L_4(r)$ is increasing on $(0, 1)$. Then $L_4(r) > L_4(0) = 0$. Therefore, the lower bound in (1.8) in Theorem 3 is better than the lower bound in (1.5).

Remark 4. On 16 March 2020, Vito Lampret (retired, University of Ljubljana, Slovenia) commented on the ResearchGate, wrote an e-mail to the second author, and stated that he established in his paper [15] a double inequality

$$g(n, r) < \mathcal{E}(r) < g(n, r) + \Delta_n(r)$$

for $n \in \mathbb{N}$ and $0 < r < 1$, where

$$g(n, r) = \frac{\pi}{2} - \frac{1}{4}\left[2 + \left(r - \frac{1}{r}\right)\ln\frac{1 + r}{1 - r}\right] + \sum_{i=1}^{n}\left(\frac{\pi}{2}w_i^2 - \frac{1}{2i + 1}\right)\frac{r^{2i}}{2i - 1},$$

$$w_i = \prod_{j=1}^{i}\frac{2j - 1}{2j}, \quad 0 < \Delta_n(r) < \frac{r^{2n+2}}{(2n + 1)^2}$$

for $n \in \mathbb{N}$ and $0 < r < 1$.

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Conflict of interest

The authors declare that they have no conflict of interest.

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