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Research article

A sum analogous to Kloosterman sum and its fourth power mean

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Abstract: The main purpose of this paper is using the analytic methods and the properties of the Legendre's symbol and quadratic residue mod p to study the computational problem of the fourth power mean of a sum analogous to Kloosterman sum, and give a sharp asymptotic formula for it.

Keywords: a sum analogous to Kloosterman sum; the fourth power mean; asymptotic formula; analytic method
Mathematics Subject Classification: 11L03, 11L05

1. Introduction

Let $q \geq 3$ be an integer, χ denotes any Dirichlet character mod q. Then for any integers m and n, the generalized Kloosterman sum $K(m,n,\chi;q)$ is defined as

$$K(m,n,\chi;q) = \sum_{a=1}^{q}{}' \chi(a) e\left(\frac{ma + n\bar{a}}{q}\right),$$

where, $e(y) = e^{2\pi i y}$, and $a \cdot \bar{a} \equiv 1 \bmod q$.

As is known to all, the research on the properties of $K(m,n,\chi;q)$ is a very important topic in analytic number theory, and many famous number theory problems are closely related to it. Because of this reason, many scholars have studied the properties of $K(m,n,\chi;q)$, and obtained a series of meaningful results, interested readers may refer to references [3–21]. For example, Zhang Wenpeng [3] proved the following conclusion:

Let $q > 2$ be an odd square-full number, n be any integer with $(n,q) = 1$. Then for any primitive character χ mod q, we have the identity

$$\sum_{m=1}^{q}{}' \left| \sum_{a=1}^{q} \chi(a)\, e\left(\frac{ma+n\overline{a}}{q}\right) \right|^4 = q^2\phi(q) \prod_{p^\alpha \| q} \left(\alpha + 1 - \frac{5}{p-1}\right),$$

where $p^\alpha \| q$ denotes that $p^\alpha | q$ and $p^{\alpha+1} \nmid q$.

If $q = p$ is a prime, then Zhang Wenpeng [4], Li Jianghua and Liu Yanni [5] also proved the identity

$$\sum_{m=1}^{p-1} |K(m,n,\chi;p)|^4 = \sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma+n\overline{a}}{p}\right) \right|^4$$
$$= \begin{cases} 2p^3 - 3p^2 - 3p - 1 & \text{if } \chi \text{ is the principal character mod } p, \\ 3p^3 - 8p^2 & \text{if } \chi \text{ is the Legendre symbol mod } p, \\ p^2(2p-7) & \text{if } \chi \text{ is a non-real character mod } p, \end{cases}$$

where $(n, p) = 1$.

In addition, Kloosterman sum has been widely used in the study of analytic number theory problem. In fact, Zhang Yitang's very important works [6] on the gaps between primes is obtained based on the sieve methods and some special mean value estimate for Kloosterman sums.

On the other hand, Lv Xingxing and Zhang Wenpeng [7] introduced a new sum, which can be regarded as a companion of the Kloosterman sum. It is defined by

$$H(m,n,k,\chi;p) = \sum_{a=1}^{p-1} \chi\left(ma + n\overline{a}\right) e\left(\frac{ka}{q}\right). \tag{1.1}$$

Then they used the analytic methods and the properties of character sums to study the hybrid power mean problem involving $H(m,n,k,\chi;p)$ and the quadratic Gauss sums, and obtained an exact computational formula for

$$\sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^2}{p}\right) \right|^2 \cdot \left| \sum_{b=1}^{p-1} \chi\left(b + \overline{b}\right) e\left(\frac{nb}{p}\right) \right|^2.$$

Shane Chern [8] also studied the mean value properties of $H(m,n,r,\chi;q)$, and obtained the identity

$$\sum_{\chi \bmod p} \left| \sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \chi\left(ma + n\overline{a}\right) e\left(\frac{ka}{p}\right) \right|^2 \right|^2 = (p-1)\left(p^4 - 7p^3 + 17p^2 - 5p - 25\right),$$

where p is an odd prime, n and k are integers with $(nk, p) = 1$.

But Shane Chern said in [8], his ultimate goal is to find the fourth power mean of $H(m,n,k,\chi;p)$. That is,

$$\sum_{\chi \bmod p} \sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \chi\left(ma + n\overline{a}\right) e\left(\frac{ka}{p}\right) \right|^4. \tag{1.2}$$

Unfortunately, he did not succeed, and ultimately he only proved a different form of the following:

$$\sum_{\chi \bmod p}\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi(ma+n\overline{a})\,e\left(\frac{ka}{p}\right)\right|^4=(p-1)\left(p^2-10p+37+pT(p)-T_L(p)\right),$$

where $T(p)$ and $T_L(p)$ are defined by

$$T(p)=\sum_{\substack{u=1\\ (a-1)^2(u\overline{a}-1)(ub-1)\equiv(b-1)^2(u\overline{b}-1)(ua-1) \bmod p}}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_0\left((a-1)^2(u\overline{a}-1)(ub-1)\right)$$

and

$$T_L(p)=\sum_{u=1}^{p-1}\left(\sum_{a=1}^{p-1}\left(\frac{(ua-1)(u\overline{a}-1)}{p}\right)\right)^2,$$

χ_0 denotes the principal character, and $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol mod p.

According to our experience, the main term in this formula is focused on $T(p)$, which is the reason why one can not see any asymptotic properties of (1.2).

Inspired by references [7, 8], we are interested in the computing problem of the following $2k$-th power mean

$$\sum_{m=0}^{p-1}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a+\overline{a})\,e\left(\frac{ma}{p}\right)\right|^{2k},\quad k\geq 2. \tag{1.3}$$

Although this problem looks very similar to the mean value (1.2), it is substantially different.

Of course, it seems no one to study this $2k$-th power mean and we have not seen any related results so far.

In this paper, we will use the number of the solutions of some congruence equations mod p and the properties of the Legendre's symbol to study the calculating problem of (1.3) with $k=2$, and give a sharp asymptotic formula for it. That is, we will prove the following result:

Theorem. Let p be an odd prime with $p\equiv 3 \bmod 4$, then we have the asymptotic formula

$$\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1}\chi(a+\overline{a})\,e\left(\frac{ma}{p}\right)\right|^4=3p^4+O\left(p^{\frac{7}{2}}\cdot\ln p\right).$$

Some notes: For prime p with $p\equiv 1 \bmod 4$, because the congruence equation $a+\overline{a}\equiv 0 \bmod p$ has two solutions, so it is a little bit complicated, and we do not have a corresponding result. For $p\equiv 3 \bmod 4$ and $k=2$, if one can get an exact value for the summation

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a^2b^2(a+b)^2-4ab(a+b)^2-4ab}{p}\right), \tag{1.4}$$

then we can obtain an exact calculating formula for (1.3).

Whether there exists an exact calculating formula for (1.4) is an open problem.

2. Some basic lemmas

In this section, we first give three simple lemmas, which are necessary in the proof of our main result. Of course, in order to prove these lemmas, we need some basic knowledge of elementary and analytic number theory, these content can be found in reference [1], here we do not need to repeat them. First we prove the following:

Lemma 1. For any odd prime p, we have the identity

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ ab\equiv cd \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1} 1 = 2p^2 - 5p + 3.$$

Proof. Note that if $a,\ b,\ c$ passed through a reduced residue system mod p, then da, db and cd also pass through a reduced residue system mod p, providing $(d, p) = 1$. So from these properties we have

$$\begin{aligned}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ ab\equiv cd \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1} 1 &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ abd^2\equiv cd^2 \bmod p\\ ad+bd\equiv cd+d \bmod p}}^{p-1} 1 = (p-1)\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ a+b\equiv ab+1 \bmod p}}^{p-1} 1\\
&= (p-1)\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ (a-1)(b-1)\equiv 0 \bmod p}}^{p-1} 1 = (p-1)\left(2\sum_{a=1}^{p-1} 1 - 1\right)\\
&= (p-1)(2p-3) = 2p^2 - 5p + 3.\end{aligned}$$

This proves Lemma 1.

Lemma 2. Let p be an odd prime with $p \equiv 3 \bmod 4$, then we have the asymptotic formula

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ (a+b)^2+1\equiv abcd \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1} 1 = p^2 + O\left(p^{\frac{3}{2}} \cdot \ln p\right).$$

Proof. Since $p \equiv 3 \bmod 4$, so for any integers a and b, we have $p \nmid \left((a+b)^2 + 1\right)$. From the properties of the reduced residue system, quadratic residue and Legendre's symbol mod p we have

$$\begin{aligned}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ (a+b)^2+1\equiv abcd \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1} 1 &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=0\\ (a+b)^2+1\equiv abcd \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1} 1 - \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{\substack{c=1\\ (a+b)^2+1\equiv 0 \bmod p\\ a+b\equiv c \bmod p}}^{p-1} 1\\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{\substack{c=1\\ (a+b)^2+1\equiv abc(a+b-c) \bmod p}}^{p-1} 1 = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{\substack{c=1\\ abc^2-ab(a+b)c+(a+b)^2+1\equiv 0 \bmod p}}^{p-1} 1\\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{\substack{c=1\\ (ab)^2c^2-(ab)^2(a+b)c+ab\left((a+b)^2+1\right)\equiv 0 \bmod p}}^{p-1} 1 = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{\substack{c=1\\ c^2-ab(a+b)c+ab\left((a+b)^2+1\right)\equiv 0 \bmod p}}^{p-1} 1\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{a=1 \\ (2c-ab(a+b))^2+4ab\left((a+b)^2+1\right)\equiv a^2b^2(a+b)^2 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=0}^{p-1} 1 = \sum_{\substack{a=1 \\ c^2+4ab\left((a+b)^2+1\right)\equiv a^2b^2(a+b)^2 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=0}^{p-1} 1 \\
&= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(1 + \left(\frac{a^2b^2(a+b)^2 - 4ab(a+b)^2 - 4ab}{p}\right)\right) \\
&= (p-1)^2 + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{a^2b^6(a+1)^2 - 4ab^4(a+1)^2 - 4ab^2}{p}\right) \\
&= (p-1)^2 + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{a^2(a+1)^2 - 4a\overline{b}^2(a+1)^2 - 4a\overline{b}^4}{p}\right) \\
&= (p-1)^2 + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{a^2(a+1)^2 - 4ab^2(a+1)^2 - 4ab^4}{p}\right) \\
&= (p-1)^2 + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{-a}{p}\right) \left(\frac{\left(2b^2 + (a+1)^2\right)^2 - a(a+1)^2 - (a+1)^4}{p}\right). \qquad (2.1)
\end{aligned}
$$

It is clear that there exist at most three integers $1 \le a \le p-1$ such that $p \mid (a+1)^2\left((a+1)^2 + a\right)$. If $p \nmid (a+1)^2\left((a+1)^2 + a\right)$, that is, $a \neq -1$ and $p \nmid (a+1)^2+a$, then $\left(2x^2 + (a+1)^2\right)^2 - a(a+1)^2 - (a+1)^4$ is not a complete square of an integral coefficient polynomial $f(x)$. So from the Weil's important work [2] we have the estimate

$$
\sum_{b=1}^{p-1} \left(\frac{\left(2b^2 + (a+1)^2\right)^2 - a(a+1)^2 - (a+1)^4}{p}\right) \ll \sqrt{p} \cdot \ln p. \qquad (2.2)
$$

Then combining (2.1) and (2.2) we have the estimate

$$
\sum_{\substack{a=1 \\ (a+b)^2+1\equiv abcd \bmod p \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} 1 = p^2 + O(3p) + O\left(p^{\frac{3}{2}} \cdot \ln p\right) = p^2 + O\left(p^{\frac{3}{2}} \cdot \ln p\right).
$$

This proves Lemma 2.

Lemma 3. Let p be a prime, then we have the identity

$$
\sum_{\substack{a=1 \\ (a+b)^2+1\equiv abcd \bmod p \\ ab\equiv cd \bmod p \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} 1 \le 4(p-1).
$$

Proof. It is clear that the conditions $ab \equiv cd \bmod p$ and $(a+b)^2 + 1 \equiv abcd \bmod p$ are equivalent to $ab \equiv cd \bmod p$ and $(a+b)^2 + 1 \equiv (ab)^2 \bmod p$, and the condition $(a+b)^2 + 1 \equiv (ab)^2 \bmod p$ is equivalent to $(a+b+ab)(a+b-ab) \equiv -1 \bmod p$.

Then we have

$$\sum_{\substack{a=1\\(a+b)^2+1\equiv abcd \bmod p\\ ab\equiv cd \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} 1 = \sum_{\substack{a=1\\(a+b+ab)(a+b-ab)\equiv -1 \bmod p\\ ab\equiv cd \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} 1$$

$$= \sum_{\substack{a=1\\(a+b+ab)\equiv -t \bmod p\\ a+b-ab\equiv \frac{1}{t} \bmod p\\ ab\equiv cd \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{t=1}^{p-1} 1 = \sum_{\substack{a=1\\(c+d)\equiv(a+b)\equiv(\frac{1}{t}-t)/2 \bmod p\\ cd\equiv ab\equiv(-t-\frac{1}{t})/2 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{t=1}^{p-1} 1$$

Notice that for any given $s, t \in \mathbb{F}_q^*$, the equation

$$\begin{cases} x + y = s \\ xy = t \end{cases}$$

has at most 2 zeros in $\mathbb{F}_q$, hence we have

$$\sum_{\substack{a=1\\(a+b)^2+1\equiv abcd \bmod p\\ ab\equiv cd \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} 1 \le 4(p-1).$$

This proves Lemma 3.

3. Proof of the theorem

In this section, we use the three simple lemmas in section two to prove our main result. In fact, from Lemma 1, Lemma 2, Lemma 3, the trigonometric identity

$$\sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) = \begin{cases} p, & \text{if } p \mid m; \\ 0, & \text{if } p \nmid m \end{cases}$$

and the orthogonality of characters mod p

$$\sum_{\chi \bmod p} \chi(a) = \begin{cases} p-1, & \text{if } a \equiv 1 \bmod p; \\ 0, & \text{otherwise} \end{cases}$$

. we have

$$\frac{1}{p(p-1)} \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi\left(a + \overline{a}\right) e\left(\frac{ma}{p}\right) \right|^4$$

$$
\begin{aligned}
&= \frac{1}{p} \sum_{\substack{a=1 \\ (a+\overline{a})(b+\overline{b})\equiv(c+\overline{c})(d+\overline{d}) \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{m=0}^{p-1} e\left(\frac{m(a+b-c-d)}{p}\right) \\
&= \sum_{\substack{a=1 \\ \overline{ab}(a^2+1)(b^2+1)\equiv\overline{cd}(c^2+1)(d^2+1) \bmod p \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} 1 = \sum_{\substack{a=1 \\ \overline{ab}((ab-1)^2+(a+b)^2)\equiv\overline{cd}((cd-1)^2+(c+d)^2) \bmod p \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} 1 \\
&= \sum_{\substack{a=1 \\ (ab-cd)((a+b)^2+1-abcd)\equiv 0 \bmod p \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} 1 = \sum_{\substack{a=1 \\ ab\equiv cd \bmod p \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} 1 \\
&\quad + \sum_{\substack{a=1 \\ (a+b)^2+1\equiv abcd \bmod p \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} 1 - \sum_{\substack{a=1 \\ (a+b)^2+1\equiv abcd \bmod p \\ ab\equiv cd \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} 1 \\
&= 2p^2 - 5p + 3 + p^2 + O\left(p^{\frac{3}{2}} \cdot \ln p\right) + O(p) \\
&= 3p^2 + O\left(p^{\frac{3}{2}} \cdot \ln p\right).
\end{aligned}
$$

This completes the proof of our Theorem.

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Conflict of interest

The authors declare no conflict of interest.

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