Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(3): 2526–2538.
DOI: 10.3934/math.2020166
Received: 17 September 2019
Accepted: 02 February 2020
Published: 10 March 2020

Research article

Coefficient inequalities for pseudo subclasses of analytical functions related to Petal type domains defined by error function

S. O. Olatunji[1] **and Hemen Dutta**[2,*]

[1] Department of Mathematical Sciences, Federal University of Technology, Akure, P.M.B. 704, Ondo State, Nigeria
[2] Department of Mathematics, Gauhati University, Guwahati-781014, India

* **Correspondence:** E-mail: hemen_dutta08@rediffmail.com.

Abstract: Pseudo is a force applied on a function to show the positive or negative effect of it on a class of function defined. The paper aims to investigate the coefficient inequalities for pseudo subclasses of analytical functions related to Petal type domains defined by error function. The early few coefficient bounds were obtained and relevant connection to Fekete-Szegö inequalities have also been derived. The results are new and several other results can be deduced easily from the main findings.

Keywords: analytic function; univalent function; starlike function; convex function; Fekete-Szegö inequalities; error function; Pseudo; Petal type domain
Mathematics Subject Classification: 30C45, 33C10, 30C20, 30C50, 30C75

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots , \tag{1.1}$$

which are analytic in the open unit disk $D = (z :| z |< 1)$ and normalized by $f(0) = 0$ and $f'(0) = 1$. Recall that, $S \subset A$ is the univalent function in $D = (z :| z |< 1)$ and has the star-like and convex functions as its sub-classes which their geometric condition satisfies $Re\left(\frac{zf'(z)}{f(z)}\right) > 0$ and $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. The two well-known sub-classes have been used to define different subclass of analytical functions in different direction with different perspective and their results are too voluminous in literature.

Two functions f and g are said to be subordinate to each other, written as $f \prec g$, if there exists a Schwartz function $w(z)$ such that

$$f(z) = g(w(z)), \quad z \epsilon D \tag{1.2}$$

where $w(0)$ and $| w(z) |< 1$ for $z \epsilon D$. Let P denote the class of analytic functions such that $p(0) = 1$ and $p(z) \prec \frac{1+z}{1-z}$, $z \epsilon D$. See [1] for details.

Goodman [2] proposed the concept of conic domain to generalize convex function which generated the first parabolic region as an image domain of analytic function. The same author studied and introduced the class of uniformly convex functions which satisfy

$$UCV = Re\left\{1 + (z - \psi)\frac{f''(z)}{f'(z)}\right\} > 0, (z, \psi \in A).$$

In recent time, Ma and Minda [3] studied the underneath characterization

$$UCV = Re\left\{1 + \frac{zf''(z)}{f'(z)} > \left|\frac{zf''(z)}{f'(z)}\right|\right\}, \quad z \epsilon D. \tag{1.3}$$

The characterization studied by [3] gave birth to first parabolic region of the form

$$\Omega = \{w; Re(w) >| w - 1 |\}, \tag{1.4}$$

which was later generalized by Kanas and Wisniowska ([5, 6]) to

$$\Omega_k = \{w; Re(w) > k \mid w - 1 \mid, k \geq 0\}. \tag{1.5}$$

The Ω_k represents the right half plane for $k = 0$, hyperbolic region for $0 < k < 1$, parabolic region for $k = 1$ and elliptic region for $k > 1$ [30].

The generalized conic region (1.5) has been studied by many researchers and their interesting results litter everywhere. Just to mention but a few Malik [7] and Malik et al. [8].

More so, the conic domain Ω was generalized to domain $\Omega[A, B]$, $-1 \leq B < A \leq 1$ by Noor and Malik [9] to

$$\Omega[A, B] = \{u + iv : [(B^2 - 1)(U^2 + V^2) - 2(AB - 1)u + (A^2 - 1)]^2$$
$$> [-2(B + 1)(u^2 v^2) + 2(A + B + C)u - 2(A + 1)]^2 + 4(A - B)^2 v^2\}$$

and it is called petal type region.

A function $p(z)$ is said to be in the class $UP[A, B]$, if and only if

$$p(z) \prec \frac{(A + 1)\tilde{p}(z) - (A - 1)}{(B + 1)\tilde{p}(z) - (B - 1)}, \tag{1.6}$$

where $\tilde{p}(z) = 1 + \frac{2}{\pi^2}\left(log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$.

Taking $A = 1$ and $B = -1$ in (1.8), the usual classes of functions studied by Goodman [1] and Kanas ([5, 6]) will be obtained.

Furthermore, the classes $UCV[A, B]$ and $ST[A, B]$ are uniformly Janoski convex and Starlike functions satisfies

$$Re\left(\frac{(B-1)\frac{(zf'(z))'}{f'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)} - (A+1)}\right) > \left|\frac{(B-1)\frac{(zf'(z))'}{f'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)} - (A+1)} - 1\right| \tag{1.7}$$

and

$$Re\left(\frac{(B-1)\frac{zf'(z)}{f'(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f'(z)} - (A+1)}\right) > \left|\frac{(B-1)\frac{zf'(z)}{f'(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f'(z)} - (A+1)} - 1\right|, \tag{1.8}$$

or equivalently

$$\frac{(zf'(z))'}{f'(z)} \in UP[A, B]$$

and

$$\frac{zf'(z)}{f'(z)} \in UP[A, B].$$

Setting $A = 1$ and $B = -1$ in (1.7) and (1.8), we obtained the classes of functions investigated by Goodman [2] and Ronning [10].

The relevant connection to Fekete-Szegö problem is a way of maximizing the non-linear functional $\left|a_3 - \lambda a_2^2\right|$ for various subclasses of univalent function theory. To know much of history, we refer the reader to [11–14] and so on.

The error function was defined because of the normal curve, and shows up anywhere the normal curve appears. Error function occurs in diffusion which is a part of transport phenomena. It is also useful in biology, mass flow, chemistry, physics and thermomechanics. According to the information at hand, Abramowitz [15] expanded the error function into Maclaurin series of the form

$$Erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)n!} \tag{1.9}$$

The properties and inequalities of error function were studied by [16] and [4] while the zeros of complementary error function of the form

$$erfc(z) = 1 - erf(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2}dt, \tag{1.10}$$

was investigated by [17], see for more details in [18, 19] and so on. In recent time, [20–22] and [23] applied error functions in numerical analysis and their results are flying in the air.

For f given by [15] and g with the form $g(z) = z + b_2z^2 + b_3z^3 + \cdots$ their Hadamard product (convolution) by $f * g$ and at is defined as:

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \tag{1.11}$$

Let E_rf be a normalized analytical function which is obtained from (1.9) and given by

$$E_rf = \frac{\sqrt{\pi}z}{2} e_rf(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}z^n}{(2n-1)(n-1)!} \tag{1.12}$$

Therefore, applying a notation (1.11) to (1.1) and (1.12) we obtain

$$\epsilon = A * E_rf = \left\{ \mathcal{F} : \mathcal{F}(z) = (f * E_rf)(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}a_n z^n}{(2n-1)(n-1)!}, f \in A \right\}, \tag{1.13}$$

where E_rf is the class that consists of a single function or E_rf. See concept in Kanas et al. [18] and Ramachandran et al. [19].

Babalola [24] introduced and studied the class of $\lambda-$pseudo starlike function of order $\beta(0 \leq \beta \leq 1)$ which satisfy the condition

$$Re\left(\frac{z(f'(z))^{\lambda}}{f(z)}\right) > \beta, \tag{1.14}$$

where $\lambda \geq 1 \in \Re(z \in D)$ and denoted by $\angle_{\lambda}(\beta)$. We observed from (1.14) that putting $\lambda = 2$, the geometric condition gives the product combination of bounded turning point and starlike function which satisfy

$$Ref'(z)\left(\frac{z(f'(z))}{f(z)}\right) > \beta$$

Olatunji [25] extended the class $\angle_{\lambda}(\beta)$ to $\angle_{\lambda}^{\beta}(s, t, \Phi)$ which the geometric condition satisfy

$$Re\left(\frac{(s-t)z(f'(z))^{\lambda}}{f(sz) - f(tz)}\right) > \beta,$$

where $s, t \in C, s \neq t, \lambda \geq 1 \in \Re, 0 \leq \beta < 1, z \in D$ and $\Phi(z)$ is the modified sigmoid function. The initial coefficient bounds were obtained and the relevant connection to Fekete-Szegö inequalities were generated. The contributions of authors like Altinkaya and Özkan [26] and Murugusundaramoorthy and Janani [27] and Murugusundaramoorthy et al. [28] can not be ignored when we are talking on λ-pseudo starlike functions.

Inspired by earlier work by [18, 19, 29]. In this work, the authors employed the approach of [13] to study the coefficient inequalities for pseudo certain subclasses of analytical functions related to petal type region defined by error function. The first few coefficient bounds and the relevant connection to Fekete-Szegö inequalities were obtained for the classes of functions defined. Also note that, the results obtained here has not been in literature and varying of parameters involved will give birth to corollaries.

For the purpose of the main results, the following lemmas and definitions are very necessary.

Lemma 1.1. *If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is a function with positive real part in D, then, for any complex μ,*

$$|p_2 - \mu p_1^2| \leq 2 \max\{1, |2\mu - 1|\}$$

and the result is sharp for the functions

$$p_0(z) = \frac{1+z}{1-z} \quad or \quad p(z) = \frac{1+z^2}{1-z^2} \quad (z \in D).$$

Lemma 1.2. *[29] Let $p \in UP[A, B], -1 \leq B < A \leq 1$ and of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Then, for a complex number μ, we have*

$$\left|p_2 - \mu p_1^2\right| \leq \frac{4}{\pi^2}(A - B) \max\left(1, \left|\frac{4}{\pi^2}(B+1) - \frac{2}{3} + 4\mu\left(\frac{A-B}{\pi^2}\right)\right|\right). \tag{1.15}$$

The result is sharp and the equality in (1.15) holds for the functions

$$p_1(z) = \frac{\frac{2(A+1)}{\pi^2}\left(log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 + 2}{\frac{2(B+1)}{\pi^2}\left(log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 + 2}$$

or

$$p_2(z) = \frac{\frac{2(A+1)}{\pi^2}\left(log\frac{1+z}{1-z}\right)^2 + 2}{\frac{2(B+1)}{\pi^2}\left(log\frac{1+z}{1-z}\right)^2 + 2}.$$

Proof. For $h \in P$ and of the form $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, we consider

$$h(z) = \frac{1 + w(z)}{1 - w(z)}$$

where $w(z)$ is such that $w(0) = 0$ and $|w(z)| < 1$. It follows easily that

$$w(z) = \frac{h(z)-1}{h(z)+1} = \frac{1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{c_3}{2} - \frac{c_2c_1}{2} + \frac{c_1^3}{8}\right)z^3 + \cdots \tag{1.16}$$

Now, if $\tilde{p}(z) = 1 + R_1 z + R_2 z^2 + \cdots$, then from (1.16), one may have,

$$\tilde{p}(w(z)) = 1 + R_1 w(z) + R_2(w(z))^2 + R_3(w(z))^3 \cdots \tag{1.17}$$

where $R_1 = \frac{8}{\pi^2}$, $R_2 = \frac{16}{3\pi^2}$, and $R_3 = \frac{184}{45\pi^2}$, see [30]. Substitute R_1, R_2 and R_3 into (1.17) to obtain

$$\tilde{p}(w(z)) = 1 + \frac{4c_1}{\pi^2}z + \frac{4}{\pi^2}(c_2 - \frac{c_1^2}{6})z^2 + \frac{4}{\pi^2}(c_3 - \frac{c_1c_2}{3} + \frac{2c_1^3}{45})z^3 + \cdots \tag{1.18}$$

Since $p \in UP[A, B]$, so from relations (1.16), (1.17) and (1.18), one may have,

$$p(z) = \frac{(A+1)\tilde{p}(w(z)) - (A-1)}{(B+1)\tilde{p}(w(z)) - (B-1)} = \frac{2 + (A+1)\frac{4}{\pi^2}c_1 z + (A+1)\frac{4}{\pi^2}(c_2 - \frac{c_1^2}{6})z^2 + \cdots}{2 + (B+1)\frac{4}{\pi^2}c_1 z + (B+1)\frac{4}{\pi^2}(c_2 - \frac{c_1^2}{6})z^2 + \cdots}$$

This implies that,

$$\begin{aligned} p(z) =& 1 + \frac{2(A-B)c_1}{\pi^2}z + \frac{2(A-B)}{\pi^2}\left(c_2 - \frac{c_1^2}{6} - \frac{2(B-1)c_1^2}{\pi^2}\right)z^2 \\ &+ \frac{8(A-B)}{\pi^2}\left[\left(\frac{(B+1)^2}{\pi^4} + \frac{B+1}{6\pi^2}\frac{1}{90}\right)c_1^2 - \left(\frac{B+1}{\pi^2} + \frac{1}{12}\right)c_1c_2 + \frac{c_3}{4}\right]z^3 + \cdots \end{aligned} \tag{1.19}$$

If $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, then equating coefficients of z and z^2, one may have,

$$p_1 = \frac{2}{\pi^2}(A-B)c_1$$

and

$$p_2 = \frac{2}{\pi^2}(A-B)\left(c_2 - \frac{c_1^2}{6} - \frac{2(B-1)c_1^2}{\pi^2}\right).$$

Now for a complex number μ, consider

$$p_2 - \mu p_1^2 = \frac{2(A-B)}{\pi^2}\left[c_2 - c_1^2\left(\frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{2\mu(A-B)}{\pi^2}\right)\right]$$

This implies that

$$\left|p_2 - \mu p_1^2\right| = \frac{2(A-B)}{\pi^2}\left|c_2 - c_1^2\left(\frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{2\mu(A-B)}{\pi^2}\right)\right|.$$

Using Lemma 1.1, one may have

$$\left|p_2 - \mu p_1^2\right| = \frac{4(A-B)}{\pi^2} max\{1, |2v-1|\},$$

where $v = \frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{2\mu(A-B)}{\pi^2}$, which completes the proof of the Lemma. □

Definition 1.3. A function $\mathcal{F}\epsilon A$ is said to be in the class $UCV[\lambda, A, B]$, $-1 \leq B < A \leq 1$, if and only if,

$$Re\left(\frac{(B-1)\frac{(z(\mathcal{F}'(z)^{\lambda}))'}{\mathcal{F}'(z)} - (A-1)}{(B+1)\frac{(z(\mathcal{F}'(z)^{\lambda}))'}{\mathcal{F}'(z)} - (A+1)}\right) > \left|\frac{(B-1)\frac{(z(\mathcal{F}'(z)^{\lambda}))'}{\mathcal{F}'(z)} - (A-1)}{(B+1)\frac{(z(\mathcal{F}'(z)^{\lambda}))'}{\mathcal{F}'(z)} - (A+1)} - 1\right|, \tag{1.20}$$

where $\lambda \geq 1\epsilon R$ or equivalently $\frac{(z(\mathcal{F}'(z)^{\lambda}))'}{\mathcal{F}'(z)}\epsilon UP[A, B]$.

Definition 1.4. A function $\mathcal{F}\epsilon A$ is said to be in the class $US[\lambda, A, B]$, $-1 \leq B < A \leq 1$, if and only if,

$$Re\left(\frac{(B-1)\frac{z(\mathcal{F}'(z)^{\lambda})}{\mathcal{F}(z)} - (A-1)}{(B+1)\frac{z(\mathcal{F}'(z)^{\lambda})}{\mathcal{F}(z)} - (A+1)}\right) > \left|\frac{(B-1)\frac{z(\mathcal{F}'(z)^{\lambda})}{\mathcal{F}(z)} - (A-1)}{(B+1)\frac{z(\mathcal{F}'(z)^{\lambda})}{\mathcal{F}(z)} - (A+1)} - 1\right|, \tag{1.21}$$

where $\lambda \geq 1\epsilon R$ or equivalently $\frac{z(\mathcal{F}'(z)^{\lambda})}{\mathcal{F}(z)}\epsilon UP[A, B]$.

Definition 1.5. A function $\mathcal{F}\epsilon A$ is said to be in the class $UM_{\alpha}[\lambda, A, B]$, $-1 \leq B < A \leq 1$, if and only if,

$$Re\left(\frac{(B-1)\left[(1-\alpha)\frac{z(\mathcal{F}'(z)^{\lambda})}{\mathcal{F}(z)} + \alpha\frac{(z(\mathcal{F}'(z)^{\lambda}))'}{\mathcal{F}'(z)}\right] - (A-1)}{(B+1)\left[(1-\alpha)\frac{z(\mathcal{F}'(z)^{\lambda})}{\mathcal{F}(z)} + \alpha\frac{(z(\mathcal{F}'(z)^{\lambda}))'}{\mathcal{F}'(z)}\right] - (A+1)}\right)$$
$$> \left|\frac{(B-1)\left[(1-\alpha)\frac{z(\mathcal{F}'(z)^{\lambda})}{\mathcal{F}(z)} + \alpha\frac{(z(\mathcal{F}'(z)^{\lambda}))'}{\mathcal{F}'(z)}\right] - (A-1)}{(B+1)\left[(1-\alpha)\frac{z(\mathcal{F}'(z)^{\lambda})}{\mathcal{F}(z)} + \alpha\frac{(z(\mathcal{F}'(z)^{\lambda}))'}{\mathcal{F}'(z)}\right] - (A+1)} - 1\right|,$$

where $\alpha \geq 0$ and $\lambda \geq 1\epsilon R$ or equivalently $(1-\alpha)\frac{z(\mathcal{F}'(z)^{\lambda})}{f(z)} + \alpha\frac{(z(f'(z)^{\lambda}))'}{f'(z)} \in UP[A, B]$.

2. Main results

In this section, we shall state and prove the main results, and several corollaries can easily be deduced under various conditions.

Theorem 2.1. *Let $\mathcal{F} \in US[\lambda, A, B]$, $-1 \leq B < A \leq 1$, and of the form (1.13). Then, for a real number μ, we have*

$$|a_3 - \mu a_2^2| \leq \frac{40(A-B)}{|1-3\lambda|\pi^2} \max\left\{1, \left|\frac{4(B+1)}{\pi^2} - \frac{1}{3} - \frac{2(A-B)}{(1-2\lambda)^2\pi^2}\left(2(2\lambda^2 - 4\lambda + 1) - \frac{9\mu(1-3\lambda)}{5}\right)\right|\right\}.$$

Proof. If $\mathcal{F} \in US[\lambda, A, B]$, $-1 \le B < A \le 1$, the it follows from relations (1.18),(1.19), and (1.20),

$$\frac{z(\mathcal{F}'(z)^{\lambda})}{\mathcal{F}(z)} = \frac{(A+1)\tilde{p}(w(z)) - (A-1)}{(B+1)\tilde{p}(w(z)) - (B-1)},$$

where $w(z)$ is such that $w(0) = 0$ and $| \; w(z) \; |< 1$. The right hand side of the above expression get its series form from (1.13) and reduces to

$$\frac{z(\mathcal{F}'(z)^{\lambda})}{\mathcal{F}(z)} = 1 + \frac{2(A-B)c_1}{\pi^2}z + \frac{2(A-B)}{\pi^2}\left(c_2 - \frac{c_1^2}{6} - \frac{2(B-1)c_1^2}{\pi^2}\right)z^2$$

$$+\frac{8(A-B)}{\pi^2}\left[\left(\frac{(B+1)^2}{\pi^4} + \frac{B+1}{6\pi^2}\frac{1}{90}\right)c_1^2 - \left(\frac{B+1}{\pi^2} + \frac{1}{12}\right)c_1c_2 + \frac{c_3}{4}\right]z^3 + \cdots . \quad (2.1)$$

If $\mathcal{F}(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}a_n z^n}{(2n-1)(n-1)!}$, then one may have

$$\frac{z(\mathcal{F}'(z)^{\lambda})}{\mathcal{F}(z)} = 1 + \frac{1-2\lambda}{3}a_2 z + \left(\frac{2\lambda^2 - 4\lambda + 1}{9}a_2^2 - \frac{1-3\lambda}{10}a_3\right)z^2 + \cdots \quad (2.2)$$

From (2.1) and (2.2), comparison of coefficient of z and z^2 gives,

$$a_2 = \frac{6(A-B)}{(1-2\lambda)\pi^2}c_1 \quad (2.3)$$

and

$$\frac{2\lambda^2 - 4\lambda + 1}{9}a_2^2 - \frac{1-3\lambda}{10}a_3 = \frac{2(A-B)}{\pi^2}\left(c_2 - \frac{1}{6}c_1^2 - \frac{2(B+1)}{\pi^2}c_1^2\right).$$

This implies, by using (2.3), that

$$a_3 = \frac{-20(A-B)}{(1-3\lambda)\pi^2}\left[c_2 - \frac{1}{6}c_1^2 - \frac{2(B+1)}{\pi^2}c_1^2 - \frac{2(2\lambda^2 - 4\lambda + 1)(A-B)}{(1-2\lambda)^2\pi^2}c_1^2\right].$$

Now, for a real number μ consider

$$\left|a_3 - \mu a_2^2\right| =$$

$$\left|-\frac{20(A-B)}{(1-3\lambda)\pi^2}\left(c_2 - \frac{1}{6}c_1^2 - \frac{2(B+1)}{\pi^2}c_1^2\right) + \frac{40(A-B)^2(2\lambda^2 - 4\lambda + 1)}{(1-2\lambda)^2(1-3\lambda)\pi^4} - \frac{36\mu(A-B)^2c_1^2}{(1-2\lambda)^2\pi^4}\right|$$

$$= \frac{20(A-B)}{(1-3\lambda)\pi^2}\left|c_2 - c_1^2\left(\frac{1}{6} + \frac{2(B+1)}{\pi^2} - \frac{2(A-B)(2\lambda^{2-4\lambda+1})}{(1-2\lambda)^2\pi^2} + \frac{9\mu(A-B)(1-3\lambda)}{5(1-2\lambda)^2\pi^2}\right)\right|$$

$$= \frac{20(A-B)}{(1-3\lambda)\pi^2}\left|c_2 - v c_1^2\right|$$

where $v = \frac{1}{6} + \frac{2(B+1)}{\pi^2} - \frac{(A-B)}{(1-2\lambda)^2\pi^2}\left(2(2\lambda^2 - 4\lambda + 1) - \frac{9\mu(1-3\lambda)}{5}\right)$. □

Theorem 2.2. *Let $\mathcal{F} \in UCV[\lambda, A, B]$, $-1 \leq B < A \leq 1$, and of the form (1.13). Then, for a real number μ, we have*

$$|a_3 - \mu a_2^2| \leq \frac{40(A-B)}{3|1+3\lambda|\pi^2} \max\left\{1, \left|\frac{4(B+1)}{\pi^2} - \frac{1}{3} - \frac{2(1+3\lambda)(A-B)}{(1+2\lambda)^2\pi^2}(\lambda - \frac{27\mu}{20})\right|\right\}$$

Proof. If $\mathcal{F} \in UCV[\lambda, A, B]$, $-1 \leq B < A \leq 1$, then it follows from relations (1.18), (1.19), and (1.21),

$$\frac{(z\mathcal{F}'(z)^{\lambda})'}{\mathcal{F}'(z)} = \frac{(A+1)\tilde{p}(w(z)) - (A-1)}{(B+1)\tilde{p}(w(z)) - (B+1)},$$

where $w(z)$ is such that $w(0) = 0$ and $|\, w(z) \,| < 1$. The right hand side of the above expression get its series form from (1.13) and reduces to,

$$\frac{(z\mathcal{F}'(z)^{\lambda})'}{\mathcal{F}'(z)} = 1 + \frac{2(A-B)c_1}{\pi^2}z + \frac{2(A-B)}{\pi^2}\left(c_2 - \frac{c_1^2}{6} - \frac{2(B+1)}{\pi^2}c_1^2\right)z^2$$
$$+ \frac{8(A-B)}{\pi^2}\left[\left(\frac{B+1}{\pi^4} + \frac{B+1}{6\pi^2} + \frac{1}{90}\right)c_1^3 - \left(\frac{B+1}{\pi^2} + \frac{1}{12}\right)c_1c_2 + \frac{c_3}{4}\right]z^3 + \cdots \tag{2.4}$$

If $\mathcal{F}(z) = z + \sum \dfrac{(-1)^{n-1}a_n z^n}{(2n-1)(n-1)!}$, then we have,

$$\frac{(z\mathcal{F}'(z)^{\lambda})'}{\mathcal{F}'(z)} = 1 - \frac{2(1+2\lambda)}{3}a_2 z + (1+3\lambda)\left(\frac{3a_3}{10} + \frac{2\lambda}{9}a_2^2\right)z^2 + \cdots \tag{2.5}$$

From (2.4) and (2.5), comparison of coefficients of z and z^2 gives,

$$a_2 = -\frac{3(A-B)c_1}{(1+2\lambda)\pi^2} \tag{2.6}$$

and

$$(1+3\lambda)\left(\frac{3a_3}{10} + \frac{2\lambda}{9}a_2^2\right) = \frac{2(A-B)}{\pi^2}\left(c_2 - \frac{c_1^2}{6} - \frac{2(B+1)c_1^2}{\pi^2}\right)$$

This implies, by using (2.6), that

$$a_3 = \frac{10}{3}\left[\frac{2(A-B)}{(1+3\lambda)\pi^2}\left(c_2 - \frac{c_1^2}{6} - \frac{2(B+1)c_1^2}{\pi^2}\right) + \frac{2\lambda(A-B)^2c_1^2}{(1+2\lambda)^2\pi^4}\right].$$

Now, for a real number μ, consider

$$\left|a_3 - \mu a_2^2\right| = \left|-\frac{20(A-B)}{3(1+3\lambda)\pi^2}\left(c_2 - \frac{1}{6}c_1 - \frac{2(B+1)}{\pi^2}c_1^2\right) + \frac{20(A-B)^2c_1^2}{3(1+2\lambda)\pi^4} - \frac{9\mu(A-B)^2c_1^2}{(1+2\lambda)^2\pi^4}\right|$$

$$= \frac{20(A-B)}{3(1+3\lambda)\pi^2}\left|c_2 - c_1^2\left(\frac{1}{6} + \frac{2(B+1)}{\pi^2} - \frac{\lambda(1+3\lambda)(A-B)}{(1+2\lambda)^2\pi^2} + \frac{27\mu(A-B)(1+3\lambda)}{20(1+2\lambda)^2\pi^2}\right)\right|$$

$$= \frac{20(A-B)}{3(1+3\lambda)\pi^2}\left|c_2 - vc_1^2\right|,$$

where

$$v = \frac{1}{6} + \frac{2(B+1)}{\pi^2} - \frac{(1+3\lambda)(A-B)}{(1+2\lambda)^2\pi^2}\left(\lambda - \frac{27\mu}{20}\right).$$

□

Theorem 2.3. *$\mathcal{F} \in M_\alpha[\lambda, A, B]$, $-1 \le B < A \le 1$, $\alpha \ge 0$ and of the form (1.13). Then, for a real number μ, we have*

$$|a_3 - \mu a_2^2| \le \frac{40(A-B)}{\pi^2|3(\lambda+\alpha+2\alpha\lambda)+\alpha-1|}\max\left\{1, \left|\frac{4(B+1)}{\pi^2} - \frac{1}{3}\right.\right.$$
$$\left.\left. - \frac{4(A-B)}{[1-2\lambda-\alpha(3+2\lambda)]^2\pi^2}\left(2\lambda^2(1+2\alpha) + 2\lambda(3\alpha-2) + 1 - \alpha - \frac{9\mu(3(\lambda+\alpha+2\alpha\lambda)+\alpha-1)}{10}\right)\right|\right\}.$$

Proof. Let $\mathcal{F} \in M_\alpha[\lambda, A, B]$, $-1 \le B < A \le 1$, $\alpha \ge 0$ and of the form (1.13). Then, for a real number μ, we have

$$(1-\alpha)\frac{z(\mathcal{F}'(z))^\lambda}{\mathcal{F}(z)} + \alpha\frac{(z(\mathcal{F}'(z))^\lambda)'}{\mathcal{F}'(z)} = \frac{(A+1)\tilde{p}(w(z)) - (A-1)}{(B+1)\tilde{p}(w(z)) - (B-1)}, \tag{2.7}$$

where $w(z)$ is such that $w(z_0) = 0$ and $|w(z)| < 1$. The right hand side of the above expression get its series form from (2.7) and reduces to

$$(1-\alpha)\frac{z(\mathcal{F}'(z))^\lambda}{\mathcal{F}(z)} + \alpha\frac{(z(\mathcal{F}'(z))^\lambda)'}{\mathcal{F}'(z)} = 1 + \frac{2(A-B)G}{\pi^2}z + \frac{2(A-B)}{\pi^2}(c_2 - \frac{c_1^2}{6} - \frac{2(B+1)}{\pi^2}c_1^2)z^2 + ... \tag{2.8}$$

If $\mathcal{F}(z) = z + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}a_nz^n}{(2n-1)(n-1)!}$, then one may have

$$(1-\alpha)\frac{z(\mathcal{F}'(z))^\lambda}{\mathcal{F}(z)} + \alpha\frac{(z(\mathcal{F}'(z))^\lambda)'}{\mathcal{F}'(z)} = (1-\alpha)\Big[1 + \frac{1-2\lambda}{3}a_2z + (\frac{2\lambda^2-4\lambda+1}{9}a_2^2 - \frac{1-3\lambda}{10}a_3)z^2 + ...\Big]$$
$$+ \alpha\Big[1 - \frac{2(1+2\lambda)}{3}a_2z + (1+3\lambda)(\frac{3a_3}{10} + \frac{2\lambda}{9}a^{2_2})z^2 + ...\Big] \tag{2.9}$$

from (2.8) and (2.9), comparison of coefficients of z and z^2 gives

$$a_2 = \frac{6(A-B)c_1}{[1-2\lambda-\alpha(3+2\lambda)]\pi^2} \tag{2.10}$$

and

$$\frac{3(\lambda+\alpha+2\alpha\lambda)+\alpha-1}{10}a_3 - \frac{2\lambda^2(1+2\lambda)+\alpha-1}{9}a_2^2 = \frac{2(A-B)}{\pi^2}\Big(c_2 - \frac{c_1^2}{6} - \frac{2(B+1)}{\pi^2}c_1^2\Big)$$

This implies, by using (2.10), that

$$a_3 = \frac{10}{3(\lambda+\alpha+2\alpha\lambda)+\alpha-1}\Big[\frac{2(A-B)}{\pi^2}\Big(c_2-\frac{c_1^2}{6}-\frac{2(B+1)}{\pi^2}c_1^2\Big) + \frac{4(A-B)^2[2\lambda^2(1+2\lambda)+2\lambda(3\alpha-2)+1-\alpha]}{[1-2\lambda-\alpha(3+2\lambda)]^2\pi^4}c_1^2\Big]$$

Now, for a real number μ, consider

$$\left|a_3-\mu a_2^2\right| = \left|\frac{10}{3(\lambda+\alpha+2\alpha\lambda)+\alpha-1}\Big[\frac{2(A-B)}{\pi^2}\Big(c_2-\frac{c_1^2}{6}-\frac{2(B+1)}{\pi^2}c_1^2\Big) + \frac{4(A-B)^2[2\lambda^2(1+2\lambda)+2\lambda(3\alpha-2)+1-\alpha]}{[1-2\lambda-\alpha(3+2\lambda)]^2\pi^4}c_1^2\Big] - \frac{36(A-B)^2\mu G^2}{[1-2\lambda-\alpha(3+2\lambda)]^2\pi^4}\right|$$

$$= \left|\frac{20(A-B)}{\pi(3(\lambda+\alpha+2\alpha\lambda)+\alpha-1)}\right|c_2-c_1^2\Big[\frac{1}{6}+\frac{2(B+1)}{\pi^2}-\frac{2(A-B)[2\lambda^2(1+2\alpha)+2\lambda(3\alpha-2)+1-\alpha]}{(1-2\lambda-\alpha(3+2\lambda))^2\pi^2} + \frac{18\mu(A-B)[3(\lambda+\alpha+2\alpha\lambda)+\alpha-1]}{10[1-2\lambda-\alpha(3+2\lambda)]^2\pi^2}$$

$$= \frac{20(A-B)}{\pi(3(\lambda+\alpha+2\alpha\lambda)+\alpha-1)}\left|c_2-vc_1^2\right|,$$

where

$$v = \frac{1}{6}+\frac{2(B+1)}{\pi^2}-\frac{2(A-B)[2\lambda^2(1+2\alpha)+2\lambda(3\alpha-2)+1-\alpha]}{(1-2\lambda-\alpha(3+2\lambda))^2\pi^2}+\frac{18\mu(A-B)[3(\lambda+\alpha+2\alpha\lambda)+\alpha-1]}{10[1-2\lambda-\alpha(3+2\lambda)]^2\pi^2}.$$

□

3. Conclusions

The force applied on certain subclasses of analytical functions associated with petal type domain defined by error function has played a vital role in this work. The results obtained are new and varying the parameters involved in the classes of function defined, these will bring new more results that has not been in existence.

Acknowledgments

The authors would like to thank the referees for their valuable comments and suggestions.

Conflict of interest

The authors declare that they have no conflict of interests.

References

1. A. W. Goodman, *Univalent Functions*, vol. 1, Tempa: Mariner Publishing Company, 1983.
2. A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math., **56** (1991), 87–92.
3. W. Ma, D. Minda, *A unified treatment of some special classes of univalent factors,* In: Proceedings of the Conference on Complex Analysis, Tianjin, China, 19–23 June, 1992, Conference on Proceedings Lecture Notes for Analysis, Cambridge: International Press, 1994, 157–169.
4. D. Coman, *The radius of starlikeness for the error function*, Stud. Univ. Babes-Bolyai. Math., **36** (1991), 13–16.
5. S. Kanas, A. Wisniowska, *Conic regions and k-uniform convexity*, J. Compt. Appl. Math., **105** (1999), 327–336.
6. S. Kanas, A. Wisniowska, *Conic domains and starlike functions*, Rev. Roum. Math. Pure Appl., **45** (2000), 647–657.
7. S. N. Malik, *Some topics in Geometric functions theory*, Saarbrücken: LAP LAMBERT Academic Publishing, 2017.
8. S. N. Malik, M. Raza, M. Arif, et al., *Coeffifficient estiates of some subclasses of analytic functions related with conic domains*, Anal. Univ. Ovidius Const. Ser. Mat., **21** (2013), 181–188.
9. K. I. Noor, S. N. Malik, *On Coefficient inequalities of functions associated with conic domains*, Compt. Math. Appl., **62** (2011), 2209–2217.
10. A. Ronning, *Uniformly convex functions and corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118** (1993), 189–196.
11. S. Kanas, *An unified approach to Fekete-Szegö problem*, Appl. Math. Compt., **218** (2012), 8453–8461.
12. B. Kowalczyk, A. Lecko, *The Fekete-Szegö inequality for close-to-convex functions with respect to a certain starlike function dependent on a real parameter*, J. Inequal. Appl., **2014** (2014), 1–16.
13. N. Magesh, V. K. Balaji, C. Abirami, *Certain classes of Bi-univalent functions related to shell-like curves connected with Fibonancci numbers*, arXiv: 1810: 06216v1 [math CV] 15 Oct, 2018.
14. F. Yousef, R. A. Frasin, T. A. Hawary, *Fekete-Szegö inequality for analytic and bi-univalent functions subordinate to Chebyshev polynomials*, arXiv: 1801.095331v2 [math CV] 22 Oct, 2018.
15. M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*, New York: Dover Publications Inc., 1965.
16. H. Alzer, *Error function inequalities*, Adv. Comput. Math., **33** (2010), 349–379.
17. A. Elbert, A. Laforgia, *The zeros of the complimentary error function*, Numer. Algorithms, **49** (2008), 153–157.
18. S. Kanas, L. Vanitha, C. Ramanchandran, *Certain results on q-starlike and q-convex error functions*, Math. Slovaca., **68** (2018), 361–368.
19. C. Ramanchandran, K. Dhanalakshmi, L. Vanitha, *Hankel Determinant for a subclass of analytic functions associated nth error functions bounded by conical regions*, Int. J. Math. Anal., **11** (2017), 571–581.

20. L. Wang, Y. Zhao, *Unscented transformation with scaled symmetric sampling strategy for precision estimation of total least squares*, Stud. Geophys. Geod., **61** (2017), 385–411.
21. L. Wang, Y. Zhao, *Scaled unscented transformation for nonlinear error propagation: Accuracy, sensitivity and applications*, J. Surv. Eng., **144** (2018), 04017022.
22. L. Wang, Y. Zhao, *Second order approximating function method for precision estimation of total least squares*, J. Surv. Eng., **145** (2019), 04018011.
23. L. Wang, C. Zou, *Accuracy analysis and applications of the Sterling interpolation method for nonlinear function error propagation*, Measurement, **146** (2019), 55–64.
24. K. O. Babalola, *On λ-pseudo starlike function*, J. Class. Anal., **3** (2013), 137–147.
25. S. O. Olatunji, *Sigmoid function in the space of univalent λ − pseudo starlike functions with Sakaguchi type functions*, J. Progressive Res. Math., **7** (2016), 1164–1172.
26. S. Altinkaya, Y. S. Özkan, *On Salagean type pseudo-starlike functions*, Acta Comment. Univ. Tartu. Math., **21** (2017), 275–285.
27. G. Murugusundaramoorthy, T. Janani, *Sigmoid function in the space of univalent $\lambda-$pseudo starlike functions*, Int. J Pure Appl. Math., **101** (2015), 33–41.
28. G. Murugusundaramoorthy, T. Janani, K. Vijaya, *New subclass of pseudo-meromorphic bi-univalent functions of complex order*, arXiv: 1709.00337v1 [math CV]31 Aug, 2017.
29. S. N. Malik, S. Mahmood, M. Raza, et al., *Coefficient inequalities of functions associated with petal type domain*, Mathematics, **6** (2018), 1–11.
30. S. Kanas, *Coefficient estimates in subclasses of the Caratheodory class related to conical domains*, Acta Math. Univ. Comenian. (N.S.), **74** (2005), 149–161.