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Research article

On redundancy, separation and connectedness in multiset topological spaces

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Abstract: This paper makes an attempt to study M-topology as a novel structure and emphasizes its importance by projecting how it differs from general topology. Primarily, the issue of redundancy in M-topology is addressed by pointing out the importance of complementation with appropriate examples. Unlike general topology, M-topology induces two subspace M-topologies on a submset. In general topology, these two definitions of the subspace topologies coincide. The situations in which these two subspace M-topologies coincide are also analyzed for the purpose. Furthermore, two types of M-connectedness and M-separations in M-topology are introduced and it is proved that neither of which implies the other.

Keywords: multiset; M-topology; subspace M-topology; M-connectedness
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1. Introduction

After the formulation of modern set theory by George Cantor in late nineteenth century, many generalizations of set theory were introduced to overcome some of the naturally occurring difficulties that arise during the modelling of real word problems. The fuzzy set theory introduced by Zadeh [4] was the first among these to provide a general framework. Rough sets by Pawlak [3], Multisets by Yager [5], Soft sets by Moldostov [10], Genuine sets by Demicri [11] are some of the other alternatives.

One among these is the concept of multiset. A multiset is a set equipped with a multiplicity or count function. The multiplicity function maps each member of the set to a non-negative integer that gives how many times it occurs in the multiset. i.e., multiset is a set with repeated elements. Nature demands it because there are identical things like repeated hydrogen atoms in a water molecule, repeated observations in a statistical data, repeated roots of polynomials, etc.

Many authors like Yager [5], Miyamoto [7], Hickman [8], Blizard [9, 29, 30] have studied the properties of multisets. So many mathematical structures were already developed on multisets. The concepts of multiset topology were introduced by Girish K. P. and Sunil Jacob John [1, 2]. Many of the M-topological concepts like M-compactness [13], semicompactness [17], generalized closed sets [16], rb-closed sets and rb-convergence [15] of multisets also were introduced. The concept of quasi coincidence has a major role in the neighborhood structure of a multipoint in the multiset topology and it was introduced by Karishma Shravan and Binod Chandra Tripathy in 2019 [14].

Previously, many Mathematical structures and concepts were already developed in different generalized set theories. A few of them are separation axiom on multiset theory [20], the topological structure of generalized rough multisets [21], combinatorics of multisets [24], multi weak structures [31], the theory of bags and lists [19], etc. The concept of soft multisets is a fast-growing generalization of sets and concepts of connectedness [22, 23], compactness [25], soft multi continuous functions [26], generalized closed soft multiset [27], semi compactness [28] that were already developed in this structure. Recently, some applications of M-topology on DNA mutations [32] are also developed.

In a paper published in 2017, A. Ghareeb [12] claimed that M-topology is redundant and unnecessary complicated in the theoretical sense. He proves that $(P^*(U), \sqsubseteq)$ and $(Ds(\phi(U), \subseteq)$ are isomorphic as lattices. That is, the mapping preserves union and intersection operations which we use for the definition of the topology. Unfortunately, that map does not preserve complementation. So one cannot say that they are homeomorphic as in a topological sense, even though the definition of topology is based on these two operations. We know that complementation has a major role in various concepts in topology. In order to prove the claim, the author [12] considered the down set topology in which all open sets are down sets. Therefore, a closed subset which is the complement of an open subset is not an open set. So these types of topologies do not have a subset which is both open and closed. But in M-topology introduced in [1, 2], it is possible to find submsets that are open as well as closed. Without the existence of clopen sets one cannot discuss the concepts of separation axiom and connectedness. Lack of clopen sets in down set topology implies that all down set topological spaces are connected. That is not the case in the M-topology. We can define two types of separations in the M-topology of a submset and there are spaces that are not connected also.

In this paper primarily we prove that the mapping defined for the purpose is not preserving complementation. Due to this, unlike the general topology, we can define two subspace M-topologies on a submset. Similar concepts of subspaces in ordinary set theoretic topology coincide. Using it we can define two types of M-connectedness and M-separation for a submset. For a submset A, it is possible that $A \cap A^C \neq \phi$. But it is not possible in general set theory and it will make differences in M-topology. Another similar structure in literature is fuzzy topology, which is well developed and mostly in parallel with general topology.

This paper further introduces a new concept of subspace M-topology on a submset. We compared this subspace M-topology with already had concepts and analyzed the situations in which they coincide. We compared one of the important topological properties, M-connectedness in the light of these two concepts and we found that M-connectedness of a submset in a subspace M-topology does not imply the M-connectedness of that submset in other subspace M-topology. We identified the properties in terms of count that satisfied by the M-topology for an M-separation in both of the subspace M-topologies.

2. Preliminaries

In this section, we give necessary definition and results discussed in [1], [2], [6] and [12], which we require for the purpose.

Definition 2.1. *[1] An mset M drawn from the set X is represented by a function Count M or* C_M *defined as* $C_M : X \to N$ *where N represents the set of non negative integers.*

Here $C_M(x)$ is the number of occurrences of the element x in the mset M. We represent the mset M drawn from the set $X = \{x_1, ..., x_n\}$ as $M = \{m_1/x_1, m_2/x_2, ..., m_n/x_n\}$ where m_i is the number of occurrences of the element $x_i, i = 1, 2, ..., n$ in the mset M. Those elements which are not included in the mset have zero count.

Example 2.1. *Let* $X = \{a, b, c, d\}$ *Then* $M = \{8/a, 7/b, 6/c, 5/d\}$ *is an mset drawn from X.*

The mset operations are defined as follows:
If M and N are two msets drawn from the set X, then

- $M = N \Leftrightarrow C_M(x) = C_N(x) \forall x \in X$.
- $M \subseteq N \Leftrightarrow C_M(x) \leq C_N(x) \forall x \in X$.
- If $P = M \cup N \Leftrightarrow C_P(x) = max\{C_M(x), C_N(x)\}\ \forall x \in X$.
- If $P = M \cap N \Leftrightarrow C_P(x) = min\{C_M(x), C_N(x)\} \forall x \in X$.
- If $P = M \oplus N \Leftrightarrow C_P(x) = C_M(x) + C_N(x) \forall x \in X$.
- If $P = M \ominus N \Leftrightarrow C_P(x) = max\{C_M(x) - C_N(x), 0\}\ \forall x \in X$. where $\oplus$ and $\ominus$ represents M-set addition and M-set subtraction respectively.

Definition 2.2. *[1] The support set of an mset M, denoted by* M^* *is an ordinary subset of X and is defined as* $M^* = \{x \in X : C_M(x) > 0\}$. M^* *is also called root set.*

Definition 2.3. *[1] A domain X, is defined as a set of elements from which msets are constructed. The mset space* $[X]^w$ *is the set of all msets whose elements are in X such that no element in the mset occurs more than w times.*

The set $[X]^\infty$ *is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in an mset.*

Definition 2.4. *[1] An mset M is said to be an empty mset if for all* $x \in X$, $C_M(x) = 0$.

The concept of subset of a mset is defined in terms of count function. Therefore there are different types of submsets in multiset theory.

Definition 2.5. *[1] (Whole Submset) A submset N of M is a whole submset of M with each element in N having full multiplicity as in M. i.e.,* $C_N(x) = C_M(x)$ *for every x in N.*

Definition 2.6. *[1] (Partial Whole Submset) A submset N of M is a partial whole submset of M with at least one element in N having full multiplicity as in M. i.e.,* $C_N(x) = C_M(x)$ *for some x in N.*

Definition 2.7. *[1] (Full submset) A submset N of M is a full submset of M if each element in M is an element in N with the same or lesser multiplicity as in M. i.e.,* $M^* = N^*$ *with* $C_N(x) \leq C_M(x)$ *for every x in N.*

Definition 2.8. *[1] (Power Mset) Let M be an mset. The power mset $\mathcal{P}(M)$ of M is the set of all submsets of M. We have $N \in \mathcal{P}(M)$ if and only if $N \subseteq M$.*

Definition 2.9. *[1] The power set of an mset is the support set of the power mset and is denoted by $\mathcal{P}^*(M)$.*

2.1. Multiset topology

In this section, we recall the definition of multiset topology and some properties.

Definition 2.10. *[1] Let M be an mset drawn from a set X and $\tau \subseteq \mathcal{P}^*(M)$. Then τ is called a multiset topology of M if τ satisfies the following properties.*

1. *The mset M and the empty mset ϕ are in τ.*
2. *The mset union of the elements of any sub collection of τ is in τ.*
3. *The mset intersection of the elements of any finite sub collection of τ is in τ.*

If M is an M-topological space with M-topology τ, we say that a submset U of M is an open submset of M if U belongs to the collection τ.

Definition 2.11. *[2] Let (M, τ) be an M-topological space and N is a submset of M. The collection $\tau_N = \{N \cap U; U \in \tau\}$ is an M-topology on N, called the subspace M-topology. With this M-topology, N is called a subspace of M and its open msets consisting of all mset intersections of open msets of M with N.*

Definition 2.12. *[2] (Closed Submset)A submset N of an M-topological space M is said to be closed if the mset $M \ominus N$ is open.*

Definition 2.13. *[1] If M is an mset, then the M-basis for an M- topology on M in $[X]^w$ is a collection $\mathcal{B}$ of submsets of M (called M basis elements) such that*

1. *For each $x \in^m M$, for some $m > 0$, there is at least one M-basis element $B \in \mathcal{B}$ containing m/x. i.e., for each indistinguishable element in M, there is at least one M-basis element in $\mathcal{B}$ having that element with same multiplicity as in M.*
2. *If m/x belongs to the intersection of two M-basis elements B_1 and B_2, then there exists an M-basis element P containing m/x such that $P \subseteq B_1 \cap B_2$ with $C_P(x) = C_{B_1 \cap B_2}(x)$ and $C_P(y) \leq C_{B_1 \cap B_2}(y)$ for all $y \neq x$.*

If a collection $\mathcal{B}$ satisfies the conditions of M-basis, then the M-topology τ generated by B can be defined as follows.

Definition 2.14. *[1] A submset U of M is said to be an open mset in M (i.e., to be an element of τ) if for each $x \in^k U$, there is an M-Basis element $B \in \mathcal{B}$ such that $x \in^k B$ and $C_B(y) \leq C_U(y)$ for all $y \neq x$.*

Definition 2.15. *[18] The topology on the metric space (M, d) induced by (the metric) d is defined as the topology τ generated by the basis consisting of the set of all open ϵ-balls in M.*

Definition 2.16. *[6] (M-connectedness in M-topology) Let (M, τ) be an M-topological space. An M-separation of M is a pair M_1, M_2 of disjoint nonempty open submsets of M whose union is M. An M-space (M, τ) is said to be M-connected if there does not exist an M-separation of M. A submset N of an M-space M is M-connected if N is M-connected as a subspace of M.*

Remark 2.1. *If there exists an M-separation, the pair of sets M_1 and M_2 are proper whole submsets of M.*

In other words, an M-topological space M has an M-separation if and only if there exists a nonempty proper open whole submset N of M such that $M \ominus N$, the complement of N is also open.

Definition 2.17. *[12] Let U be an mset in $[X]^w$. Define $\varphi(U)$ as following:*

$$\varphi(U) = \cup_{x \in U^*} \{x\} \times \{n \in N : 0 < n \leq C_U(x)\}.$$

Then (y, n) in $\varphi(U)$ if and only if $0 < n \leq C_U(y)$

Definition 2.18. *[12] Let U be an mset in $[X]^w$. For arbitrary $(x, n), (y, m) \in \varphi(U)$, denote $(x, n) \leq (y, m)$ if $x = y$ and $n \leq m$. Thus " $\leq$ " is a partial order of $\varphi(U)$ and the family of all down sets is denoted by $Ds(\varphi(U))$.*

Result 2.1. *[12] Let U be an mset in $[X]^w$. Then*

$$\varphi : (P^*(U), \sqcap, \sqcup) \to (Ds(\varphi(U)), \cap, \cup)$$

is an isomorphism.

3. Closed subspace M-topology

In this section, we show that the complementation has a major role in topology. Many of the concepts in general topology are defined in terms of open sets. If we define the same concept in terms of closed sets, then it will coincide with the definition of the same by using open sets. But in M-topology it may differ due to a lack of preserving complementation by the mapping between the lattices. Even the subspace M-topologies of a submset are different if we define it in terms of open and closed sets.

Remark 3.1. *Let U be an mset in $[X]^w$. Then the map*

$$\varphi : (P^*(U), \sqcap, \sqcup) \to (Ds(\varphi(U)), \cap, \cup)$$

does not preserve the operation complementation. Consequently, for a topology $\tau \subset P^*(U)$ on U, the corresponding topological space $(\varphi(U), \varphi^{\to}(\tau))$ need not be in the same topological structure of (U, τ).

We will prove it by giving an example. If $U = \{4/x, 6/y\}$ and $\tau = \{U, \phi, \{2/x, 3/y\}\}$. Clearly τ is a topology on U. Then $V = \{2/x, 3/y\}$ is an open subset in this topology.

$$\varphi(V) = \{(x, 1), (x, 2), (y, 1), (y, 2)(y, 3)\}$$

which is already a down set and $\varphi(V) \in Ds(\varphi(U))$.

$$\varphi^{\to}(\tau) = \{\varphi(U), \phi, \varphi(V)\} \subset Ds(\varphi(U))$$

In U, $V^C = \{2/x, 3/y\}^c = \{2/x, 3/y\} = V$. $\therefore$ In τ, V is open as well as closed. i.e.,V is a clopen set. But in $(\varphi(U), \varphi^{\to}(\tau))$, the corresponding open set $\varphi(V)$ is not a clopen set, since

$$\varphi(V)^C = \{(x, 3), (x, 4), (y, 4), (y, 5), (y, 6)\}$$

is not a down set and does not belong to $\varphi^{\rightarrow}(\tau)$.

Therefore the topological structure of (U, τ) is different from topological structure of $(\varphi(U), \varphi^{\rightarrow}(\tau))$.

Clearly $\varphi(V)^C \neq \varphi(V^C)$.

$\therefore$ the mapping does not preserve complementation.

We introduce a new concept of subspace M-topology on a submset by using closed sets. Thus a submset N of an M-topological space M will have two subspace M-topologies. In general, these two subspace M-topologies need not be equal. But in certain circumstances, they will coincide. Firstly we found that whatever may be the topology on an mset whole submset has this property. Secondly, we found a condition that if given M-topology has a property with respect to that submset these subspace M-topologies will coincide.

Definition 3.1. *Let (M, τ) be an M-topological space and N be a submset of M. Then the closed subspace M-topology on N is defined by*

$$\tau_c = \{N \ominus (N \cap C) : \textit{where } C \textit{ is closed in } M\}$$

For convenience we denote subspace M-topology using open sets by τ_o and we call it open subspace M-topology. Therefore

$$\tau_o = \{N \cap U : \text{where } U \text{ is open in } M\}$$

In general, $\tau_c \neq \tau_o$.

Firstly we want to prove that τ_c is a M-topology on N.

Proposition 3.1. *The collection τ_c is an M-topology on N.*

Proof. Here $\tau_c = \{N \ominus (N \cap C) : \text{where } C \text{ is closed in } M\}$

In order to prove that the collection τ_c is an M-topology on N, firstly we need to prove that N and ϕ belong to the collection τ_c. The collection τ is a topology. $\therefore C = \phi$ is a closed submset of M. The submset corresponding to $C = \phi$ in τ_c is

$$N \ominus (N \cap C) = N \ominus (N \cap \phi) = N \ominus \phi = N$$

$$\therefore N \in \tau_c$$

If we take $C = M$ which is also a closed subset of M. Then we get

$$N \ominus (N \cap C) = N \ominus (N \cap M) = N \ominus N = \phi$$

$$\therefore \phi \in \tau_c$$

Thus $\phi, N \in \tau_c$.

Now we want to prove that τ_c is closed under arbitrary union. Let $\{U_\alpha\}$ be a collection of submsets of N in τ_c.

Therefore, corresponding to each U_α, $\exists$ a closed subset C_α such that

$$U_\alpha = N \ominus (N \cap C_\alpha)$$

Now consider $\bigcup_\alpha U_\alpha = \bigcup_\alpha (N \ominus (N \cap C_\alpha))$.
If $x \in \bigcup_\alpha U_\alpha$ then $x \in \bigcup_\alpha (N \ominus (N \cap C_\alpha))$.

$$\begin{aligned}
C_{\cup U_\alpha}(x) &= max_\alpha \{C_{U_\alpha}(x)\} \\
&= max_\alpha \{C_{N \ominus (N \cap C_\alpha)}(x)\} \\
&= max_\alpha \{max\{C_N(x) - C_{N \cap C_\alpha}(x), 0\}\} \\
&= max_\alpha \{C_N(x) - C_{N \cap C_\alpha}(x)\} \\
&= max_\alpha \{C_N(x) - min\{C_N(x), C_{C_\alpha}(x)\}\} \\
&= C_N(x) - min_\alpha \{min\{C_N(x), C_{C_\alpha}(x)\}\} \\
&= C_N(x) - min\{min_\alpha \{C_N(x), C_{C_\alpha}(x)\}\} \\
&= C_N(x) - min\{C_N(x), min_\alpha \{C_{C_\alpha}(x)\}\} \\
&= C_N(x) - min\{C_N(x), C_{\cap C_\alpha}(x)\} \\
&= C_N(x) - C_{N \cap (\cap C_\alpha)}(x) \\
&= C_{N \ominus N \cap (\cap C_\alpha)}(x)
\end{aligned}$$

$$\therefore \cup U_\alpha = N \ominus N \cap (\cap C_\alpha)$$

The intersection of an arbitrary collection of closed submsets is closed in τ. $\therefore C = \cap C_\alpha$ is a closed subset of M. $\therefore U = \cup U_\alpha$ is of the form $N \ominus (N \cap C)$ for some closed set $C = \cap C_\alpha$.

$$\therefore \cup U_\alpha \in \tau_c.$$

Now we want to prove that intersection of finitely many sets in τ_c is an element of τ_c. If it is true for the intersection of two sets, using induction we can extend this result for finitely many sets in τ_c.

Let U_1 and U_2 be two sets in τ_c. We want to prove that $U_1 \cap U_2 \in \tau_c$.
There exists closed sets C_1 and C_2 such that $U_i = N \ominus (N \cap C_i)$ for $i = 1, 2$.

$$\begin{aligned}
C_{U_1 \cap U_2}(x) &= min\{C_{U_1}(x), C_{U_2}(x)\} \\
&= min\{C_{N \ominus (N \cap C_1)}(x), C_{N \ominus (N \cap C_2)}(x)\} \\
&= min\{C_N(x) - C_{N \cap C_1}(x), C_N(x) - C_{N \cap C_2}(x)\} \\
&= C_N(x) - max\{C_{N \cap C_1}(x), C_{N \cap C_1}(x)\} \\
&= C_N(x) - max\{min\{C_N(x), C_{C_1}(x)\}, min\{C_N(x), C_{C_2}(x)\}\} \\
&= C_N(x) - min\{C_N(x), max\{C_{C_1}(x), C_{C_2}(x)\}\} \\
&= C_{N \ominus (N \cap (C_1 \cup C_2))}(x)
\end{aligned}$$

$U_1 \cap U_2 = N \ominus (N \cap (C_1 \cup C_2)$. The union of finite collection of closed submsets is closed in τ. Therefore $C_1 \cup C_2$ is a closed submset of M. Therefore $U_1 \cap U_2$ is of the required form. $U_1 \cap U_2 \in \tau_c$.

Hence τ_c is a topology on N.

□

Remark 3.2. *In the case of general topology, these two definitions coincide and give the same subspace topology. i.e,* $\tau_o = \tau_c$. *But in M-topology, they need not be equal.*

Example 3.1. *Let* $M = \{10/x, 10/y, 10/z\}$, $\tau = \{M, \phi, \{9/x, 8/y, 7/z\}\}$ *and* $N = \{5/x, 3/y, 6/z\}$.
Then $\tau_o = \{N \cap M, N \cap \phi, N \cap \{9/x, 8/y, 7/z\}\} = \{N, \phi\}$
The closed submsets of M are $M \ominus M = \phi, M \ominus \phi = M$, *and* $M \ominus \{9/x, 8/y, 7/z\} = \{1/x, 2/y, 3/z\}$.
Taking intersections with N, we get

$$\phi, N, \{1/x, 2/y, 3/z\}$$

Taking complements of these sets in N, i.e., $N \ominus (N \cap C)$, *we get*

$$\tau_c = \{N, \phi, \{4/x, 1/y, 3/z\}\} \neq \tau_o$$

Now we will analyze the situations for which these two subspace M-topologies are same.

Theorem 3.1. *Let* (M, τ) *be an M-topological space and N be a whole submset. Then these subspace M-topologies on N are same. i.e.,* $\tau_c = \tau_o$.

Proof. Given that N is a whole submset. $\therefore \forall x \in N, C_N(x) = C_M(x)$.

We want to prove that $\tau_c = \tau_o$. It is enough to prove that $N \cap U = N \ominus (N \cap C) = N \ominus (N \cap (M \ominus U))$ for every open set U.

Let U be an open submset of M and $C = M \ominus U$. Let $x \in N$. Let us check the count of x in these two sets.

$C_{N \cap U}(x) = min\{C_N(x), C_U(x)\} = C_U(x)$, since $C_N(x) = C_M(x)$ and $C_U(x) \leq C_M(x)$.

$$\begin{aligned} C_{N \cap C}(x) &= min\{C_N(x), C_C(x)\} \\ &= C_C(x) \\ &= max\{C_M(x) - C_U(x), 0\} \\ &= C_M(x) - C_U(x) \end{aligned}$$

$$\begin{aligned} C_{N \ominus (N \cap C)}(x) &= max\{C_N(x) - C_{N \cap C}(x), 0\} \\ &= C_N(x) - C_{N \cap C}(x) \\ &= C_N(x) - (C_M(x) - C_U(x)) \\ &= C_U(x) \end{aligned}$$

$C_{N \ominus (N \cap C)}(x) = C_U(x), \forall x \in N$.
For $x \notin N$, $C_{N \cap U}(x) = min\{C_N(x), C_U(x)\} = 0$.
Since $C_{N \cap C}(x) = 0, C_{N \ominus (N \cap C)}(x) = 0$. In this case also $C_{N \ominus (N \cap C)}(x) = C_{N \cap U}(x)$.
Therefore $N \cap U = N \ominus (N \cap C)$.
$\therefore \tau_c = \tau_o$.

□

Remark 3.3. *Whatever may be the M-topology on M, if N is a whole submset, then* $\tau_c = \tau_o$.

Definition 3.2. *Let M be an mset and N be a partial whole submset of M, then* $x \in N$ *is called whole element of N if* $C_N(x) = C_M(x)$ *and called a part element of N if* $C_N(x) < C_M(x)$.

Theorem 3.2. *Let N be a partial whole submset of M. If the M-topology* τ *on M has the property that the part elements of N appear as whole elements in all open sets which contains it, then* $\tau_c = \tau_o$.

Proof. If x is a whole element of N, proceed as in the proof of previous theorem. Suppose x is a part element of N.Then

$C_{N\cap U}(x) = min\{C_N(x), C_U(x)\} = C_N(x)$.
$C_C(x) = C_{M\ominus U}(x) = max\{C_M(x) - C_U(x), 0\} = 0$.
$C_{N\cap C}(x) = min\{C_N(x), C_C(x)\} = min\{C_N(x), 0\} = 0$.
$C_{N\ominus(N\cap C)}(x) = max\{C_N(x) - C_{N\cap C}(x), 0\} = C_N(x)$.
Therefore
$C_{N\cap U}(x) = C_{N\ominus(N\cap C)}(x).\forall x$.
$\therefore N \cap U = N \ominus (N \cap C)$.
Therefore $\tau_c = \tau_o$.

□

Example 3.2. *Let* $M = \{5/a, 6/b, 7/c\}$ *and* $N = \{3/a, 2/b, 7/c\}$ *be a submset of M. Clearly N is a partial whole submset. Consider the topology.*

$$\tau = \{M, \{4/a, 1/b, 5/c\}, \phi\}$$

Here $U_1 = \{4/a, 1/b, 5/c\}$ *is not in the category mentioned in the theorem, since part elements a and b of N are not whole elements in* U_1.

$$\tau_o = \{N, \{3/a, 1/b, 5/c\}, \phi\}$$
$$\tau_c = \{N, \{2/a, 5/c\}, \phi\}$$

Hence $\tau_c \neq \tau_o$.
Consider the topologies

$$\tau' = \{M, \{5/a, 6/b, 5/c\}, \phi\}$$
$$\tau'' = \{M, \{5/a, 3/c\}, \phi\}$$

In these topologies, part elements of N occur as whole elements in all open sets containing it.Therefore

$$\tau'_o = \{N, \{3/a, 2/b, 5/c\}, \phi\} = \tau'_c \text{ and}$$
$$\tau''_o = \{N, \{3/a, 3/c\}, \phi\} = \tau''_c.$$

Definition 3.3. *(Whole M-topology) If all open submsets of an M-topology* τ *on X are whole submsets, then we call it as whole M-topology.*

Example 3.3. *Let* $X = \{5/x, 4/y.3/z\}$

$$\tau = \{X, \phi, \{5/x\}, \{5/x, 4/y\}\}$$

Here τ *is a whole M-topology.*

Definition 3.4. *(Pseudo metric on an mset) A pseudo metric d on an mset M is a function $d : M \times M \to R$ which satisfies following conditions.*

(i) $d(x, y) \geq 0$.
(ii) $d(x, x) = 0$ $\forall x \in M$, *ie, the distance between two indistinguishable elements is zero.*
(iii) $d(x, y) = d(y, x)$ $\forall$ $x, y \in M$.
(iv) $d(x, y) \leq d(x, z) + d(z, y)$ $\forall$ $x, y, z \in M$.

For the definition of pseudo metric on M, we are not considering the multiplicity of an ordered pair $(x, y) \in M \times M$. Therefore every $(x, y) \in M \times M$ is also an element of $M^* \times M^*$ and we can consider d as a function from $M^* \times M^*$ to R. By defining $d^* : M^* \times M^* \to R$ as $d^*(x, y) = d(x, y)$ we get a pseudo metric on M^*.

Proposition 3.2. *A function $d : M \times M \to R$ is a pseudo metric on M if and only if corresponding d^* is a pseudo metric on M^*.*

Proposition 3.3. *The topology induced by a pseudo metric on an mset is a whole M-topology.*

Proof. It is enough to prove that every base element of the topology is a whole submset. Here a base element

$$B = B(x, r) = \{y \in M : d(x.y) < r\}$$

We will prove that B is a whole submset.

Here $x \in B$ and $d(x.x) = 0 < r \Rightarrow C_B(x) = C_M(x)$.

If $y \in B$ then $d(x, y) < r$ Therefore, for all other indistinguishable y's in M.

$$d(x, y) \leq d(x, y) + d(y, y) = d(x, y) < r$$

$$\therefore y \in B, \forall\ y \in M$$

Hence $C_B(y) = C_M(y)$. ie, if $y \in B$ then all indistinguishable $y \in M$ are in B. $\therefore$ B is whole submset of M. Hence τ is a whole M-topology. □

Definition 3.5. *Let M be an mset drawn from X and $\mathcal{F}$ be a collection of submsets of M. Then*

$$\mathcal{F}' = \{U^* \subset X : U \in \mathcal{F}\}$$

is called the corresponding supporting ordinary collection of subsets of M^*.

If $\mathcal{F}'$ is a collection of subsets of X, then

$$\mathcal{F} = \{U \subset M : U \text{ is a whole submset and } U^* \in \mathcal{F}'\}$$

is the corresponding whole submset collection of submsets of M.

Proposition 3.4. *A collection τ of whole submsets of M is a M-topology on M if and only if corresponding supporting ordinary collection τ' is a topology on M^*.*

Proof. Straight forward. □

Theorem 3.3. *An M-topological space τ on a multiset M is pseudo metrizable if it is a whole M-topology and the supporting topology τ' on M^* is pseudo metrizable in ordinary set topology.*

Proof. Straight forward. □

Example 3.4. *Let $X = \{a, b, c\}$. Define $d : X \times X \to R$ by $d(b, c) = d(c, b) = 1 = d(a, c) = d(c, a)$ and $d(x, y) = 0$ for all other $(x, y) \in X \times X$.*

Then

$$\tau' = \{X, \phi, \{a, b\}, \{c\}\}$$

is the topology induced by d.

If $M = \{5/a, 3/b, 8/c\}$ then corresponding M-topology is

$$\tau = \{M, \phi, \{5/a, 3/b\}, \{8/c\}\}$$

Theorem 3.4. *In a whole M-topology, open subspace M-topology and closed subspace M-topology on a submset N are same. i.e., $\tau_c = \tau_o$.*

Proof. Let τ be a whole M-topology on M and N be a submset of M.

For any open set U, it is enough to prove that $N \cap U = N \ominus (N \cap C)$ where $C = M \ominus U$.

$$C_{N\cap U}(x) = min\{C_N(x), C_U(x)\} = \begin{cases} C_N(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

$$C_C(x) = C_{M\ominus U}(x) = max\{C_M(x) - C_U(x), 0\} = \begin{cases} 0 & \text{if } x \in U \\ C_M(x) & \text{if } x \notin U \end{cases}$$

$$C_{N\cap C} = min\{C_N(x), C_C(x)\} = \begin{cases} 0 & \text{if } x \in U \\ C_N(x) & \text{if } x \notin U \end{cases}$$

$$C_{N\ominus(N\cap C)} = max\{C_N(x) - C_{N\cap C}(x), 0\} = \begin{cases} C_N(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

Therefore $N \cap U = N \ominus (N \cap C)$.

$\therefore \tau_c = \tau_o$.

□

4. M-connectedness and subspace M-topologies

The M-connectedness of a submset N is determined by the subspace M-topology on N and we have two types of subspace M-topologies on a submset. Therefore we can define two types of M-connectedness on a submset N.

Definition 4.1. *(MO-Connectedness) A submset N of a multiset M is said to be MO-connected if it is connected in open subspace M-topology.*

Definition 4.2. *(MC-Connectedness) A submset N of a multiset M is said to be MC-connected if it is connected in closed subspace M-topology.*

There exist MO-connected submsets which are not MC-connected and vice versa.

Example 4.1. *Let* $M = \{10/x, 10/y, 10/z\}$ *and* $N = \{8/x, 6/y, 3/z\}$

$\tau = \{M, \phi, \{9/x, 4/z\}, \{8/y\}, \{9/x, 8/y, 4/z\}\}$

$\tau_o = \{N, \phi, \{8/x, 3/z\}, \{6/y\}\}$ *and N has a separation in the open subspace M-topology. Therefore N is not connected in open subspace M-topology.*

$\tau_c = \{N, \phi, \{7/x\}, \{4/y\}.\{7/x, 4/y\}\}$

N has no separation in closed subspace M-topology, and hence it is MC-connected.

If $\tau' = \{M, \phi, \{10/x, 10/y, 5/z\}, \{2/x, 4/y, 10/z\}, \{2/x, 4/y, 5/z\}\}$

$\tau'_o = \{N, \phi, \{2/x, 4/y, 3/z\}\}$ *and* $\tau'_c = \{N, \phi, \{8/x, 6/y\}, \{3/z\}\}$

Hence N is MO-connected and not MC-connected.

There exists submset which is connected in both subspace M-topologies.

Example 4.2. *Consider M and N as in above example and we consider indiscrete topology on M. Then N is MO-connected and MC-connected.*

There exists submset which has separation in both subspace M-topologies.

Example 4.3. *Let* $M = \{10/x, 10/y, 10/z\}$ *and* $N = \{8/x, 6/y, 3/z\}$. $\tau = \{M, \phi, \{10/x, 10/z\}, \{10/y\}\}$ *Then open subspace M-topology and closed subspace M-topology are given by*

$\tau_o = \{N, \phi, \{8/x, 3/z\}, \{6/y\}\}$ *and* $\tau_c = \{N, \phi, \{8/x, 3/z\}, \{6/y\}\}$

Therefore N has separation in both subspace M-topologies.

In general topology, existence two open subsets is important for a separation. But in M-topology we need an additional condition in terms of count of elements also.

Theorem 4.1. *Let* (M, τ) *be an M-topological space and N be a submset of M. Then N has a separation in open subspace M-topology if and only if there exists two open submsets* V_1 *and* V_2 *such that each* $x \in N$ *is an element of exactly one of the* V_i*'s with* $C_{V_i}(x) \geq C_N(x)$ *and* $N \not\subset V_i$ *for i=1,2.*

Proof. Firstly suppose there exists two open submsets V_1 and V_2 such that each $x \in N$ is an element of exactly one of the V_i's with $C_{V_i}(x) \geq C_N(x)$ and $N \not\subset V_i$ for $i = 1, 2$.

Define $U_i = N \cap V_i$ for $i = 1, 2$. Then $C_{U_i}(x) = C_N(x)$ or 0.

Then $U_i \in \tau_o$ and $\forall x \in N$, $x \in V_1$ or V_2, therefore $x \in U_1$ or U_2 i.e., $N \subset U_1 \cup U_2$.

The sets U_1 and U_2 are submsets of N, $\therefore N = U_1 \cup U_2$

$N \not\subset V_i \implies N \neq U_i \implies U_1$ and U_2 are nonempty, since $N = U_1 \cup U_2$.

$\therefore N$ has a separation.

Conversely, suppose N has a separation, then $N = U_1 \cup U_2$ where $U_i \in \tau_o$. Then there exists open submsets V_1 and V_2 such that $U_i = N \cap V_i$. Then clearly V_i has the properties that each $x \in N$ is an element of exactly one of the V_i's with $C_{V_i}(x) \geq C_N(x)$ and $N \not\subset V_i$ for $i = 1, 2$.

□

Theorem 4.2. *Let* (M, τ) *be an M-topological space and N be a submset of M. Then N has a separation in closed subspace M-topology if and only if there exists two open partial whole submsets* V_1 *and* V_2 *such that each* $x \in N$ *is a whole element of exactly one of the* V_i*'s and whole element of one set is a part element of other set with a count less than or equal to* $C_M(x) - C_N(x)$ *and there exists* $x, y \in N$ *such that x is not a whole element of* V_1 *and y is not a whole element of* V_2.

Proof. Suppose there exists two open partial whole submsets V_1 and V_2 such that each $x \in N$ is an element of exactly one of the V_i's and whole element of one set is a part element of other set with count less than or equal to $C_M(x) - C_N(x)$ and there exists $x, y \in N$ such that x is not a whole element of V_1 and y is not a whole element of V_2.

Define $C_i = M \ominus V_i$. Then C_i is a closed submset and contains all elements of M except whole elements of V_i.

If $x \in N$ is a part element of V_i. Then $C_{V_i}(x) \leq C_M(x) - C_N(x)$, $\therefore C_{C_i}(x) \geq C_N(x)$.

$\therefore N \cap C_i$ contains only the part elements of V_i with count $C_N(x)$.

$\therefore N \ominus (N \cap C_i)$ contains whole elements of V_i which are in N with count $C_N(x)$.

$\therefore U_i = N \ominus (N \cap C_i)$ is an open set in closed subspace topology, which contains whole elements of V_i with count $C_N(x)$.

If $x \in N$, it is whole element of exactly one of V_i's.

$\therefore x$ is an element of exactly one of U_i's with count $C_N(x)$.

$\therefore N \subset U_1 \cup U_2$.

Since each U_i is a submset of N, $\therefore N = U_1 \cup U_2$.

By assumption $\exists x \in N$ which is not a whole element of V_1. $\therefore x$ is a whole element of V_2, and $x \notin C_2$.

$\therefore x \notin N \cap C_2 \implies x \in N \ominus (N \cap C_2) = U_2$ with count $C_N(x)$.

$\therefore U_2$ is not empty.

Similarly, we can prove that U_1 is nonempty.

Conversely, suppose N has a separation in closed subspace M-topology.

There exists nonempty submsets U_1 and U_2 which are open in closed subspace M-topology such that $U_i \subset N$ and $N = U_1 \cup U_2$.

Then there exists closed submsets C_i such that $U_i = N \ominus (N \cap C_i)$.

$\therefore \exists$ open submsets V_i such that $C_i = M \ominus V_i$

If $x \in N$ then $x \in U_1$ or $x \in U_2$. Suppose $x \in U_1$.Then

$$\max\{C_N(x) - C_{N\cap C_1}(x), 0\} = C_{U_1}(x) = C_N(x).$$
$$\therefore C_N(x) - C_{N\cap C_1}(x) = C_N(x).$$
$$\therefore C_{N\cap C_1}(x) = 0$$

$\implies x \notin C_1 \implies x \in V_1$ with $C_{V_1}(x) = C_M(x)$.

$\therefore x$ belongs to one of the V_i's as whole element.

$x \in U_1 \implies x \notin U_2, \therefore C_{U_2}(x) = 0$.

$$\max\{C_N(x) - C_{N\cap C_2}(x), 0\} = 0$$

since $N \cap C_2 \subset N$, $C_N(x) = C_{N\cap C_2}(x)$. Therefore

$$C_N(x) \leq C_{C_2}(x)$$
$$\text{i.e., } C_N(x) \leq C_M(x) - C_{V_2}(x)$$
$$\therefore C_{V_2}(x) \leq C_M(x) - C_N(x)$$

That x is a part element of V_2 with count $\leq C_M(x) - C_N(x)$

Here $N = U_1 \cup U_2$, U_1 and U_2 are nonempty $\implies$ there exists an $x \in U_1$ which is not an element of U_2.

i.e, $x \in N \ominus (N \cap C_1)$ with count $C_N(x)$. Then $N_{C_1}(x) = 0$ and x is whole element of V_1. Similarly there exists a $y \in N$ which is a whole element of V_2.

□

Theorem 4.3. *Let (M, τ) be an M-topological space and N be a whole submset. Then N is MO-connected if and only if N is MC-connected.*

Proof. By theorem 3.1, both subspace M-topologies on N coincide and hence the result. □

Theorem 4.4. *In whole topology a submset N is MO-connected if and only if it is MC-connected.*

Proof. By theorem 3.1, both subspace M-topologies on N coincide and hence the result. □

5. Conclusions and future work

In this paper, we emphasized the importance of complementation in topology and how M-topology differs from the general topology. We introduced the concept of a new subspace M-topology on a submset which is distinct from the subspace M-topology already defined by using open sets, in many situations. In ordinary set theoretic topology, we discuss all topological properties of a subset by considering the subspace topology. In M-topologies we have two subspace M-topologies and we could discuss all M-topological properties of a submset in the light of these two concepts. We compared the property of M-connectedness in these two subspace M-topologies. There is a scope of a study to compare other M-topological properties in these two subspace M-topologies also.

One can define two subspace M-topologies on a submset of an mset. We plan to analyze many topological properties like semi open sets, generalized closed sets, semi compactness, interior, boundary, exterior, separation axioms and $\gamma-$operations in the light of these two subspace M-topologies. Similar way we can define one more subspace topology in some situations of fuzzy topology also. So we are planning to extend our research work to the fuzzy topological spaces. This approach is possible in rough multiset theory and soft multiset theory also.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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