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Research article

Forming localized waves of the nonlinearity of the DNA dynamics arising in oscillator-chain of Peyrard-Bishop model

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Abstract: In this article, the mathematical modeling of DNA vibration dynamics has been considered that describes the nonlinear interaction between adjacent displacements along with the Hydrogen bonds with utilizing five techniques, namely, the improved $\tan(\phi/2)$-expansion method (ITEM), the $\exp(-\Omega(\eta))$-expansion method (EEM), the improved $\exp(-\Omega(\eta))$-expansion method (IEEM), the generalized (G'/G)-expansion method (GGM), and the exp-function method (EFM) to get the new exact solutions. This model of the equation is analyzed using the aforementioned schemes. The different kinds of traveling wave solutions: solitary, topological, periodic and rational, are fall out as a by-product of these schemes. Finally, the existence of the solutions for the constraint conditions is also shown.

Keywords: improved $\tan(\phi/2)$-expansion method; $\exp(-\Omega(\eta))$-expansion method; improved $\exp(-\Omega(\eta))$-expansion method; generalized (G'/G)-expansion method; the exp-function method; solitary, topological, periodic and rational solutions
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1. Introduction

Traveling wave and soliton solutions are one of the most interesting and fascinating areas of research in different fields of engineering and physical sciences. These models are basic ingredients of sciences in which play important roles in numerous areas such as biology, physics, chemistry, fluid mechanics and many engineering and science applications among others [1–5]. Furthermore, the approaches to solving these types of equations alongside nonlinear PDEs ranging from analytical to numerical methods are very important in many engineering and sciences applications. Some of these methods include finding the exact solutions by using the special techniques in which can be manifested to new works with vigorous references. Consequently, it is imperative to address the dynamics of these soliton pulses from a mathematical aspect. This will lead to a deeper understanding of the engineering perspective of these solutions [6–13].

In this paper, we will study the different kinds of traveling wave solutions in mathematical model in DNA dynamics from a purely mathematical viewpoint. Therefore, the importance of this paper will be to extract the exact traveling wave solution for the nonlinear model. This model is described for first by Peyrard-Bishop, that takes into consideration the inclusion of nonlinear interaction between adjacent displacements along with the Hydrogen bonds [14]. There are several integration tools available to solve the model. Many such nonlinear equations as DNA dynamics have been examined with regards to soliton theory, where complete integrability was emphasized by various analytical techniques.

For investigating the appearance of solitonic structures of the oscillator-chain of Peyrard-Bishop model has been analyzed by [14, 15]. The balance between weak nonlinearity and dispersion in DNA dynamic model with linear dispersion and nonlinear dispersion arise in works of Dusuel et al. [16] and Alvarez et al. [17]. Treatment of mathematical and physical modeling of equations of DNA dynamics show that those can be reduced to a significant nonlinear formations. The nonlinearity of the DNA dynamic model arises in localized waves in which have a few considerable features, as example in transporting energy without dissipation. A few methods in which physical properties of DNA dynamics have been investigated by the numerous authors [18–23]. There are techniques usually used in biological systems such as the discrete derivative operator (DDO) technique applying to long-range interactions systems [24, 25], the semi-discrete approximation [26–28], the solitary perturbation technique [29, 30], the modified extended tanh function method [31–33].

Author of [34] made use of the Hirota bilinear method of the bidirectional Sawada-Kotera equation to obtain new lump-type solutions and interaction phenomenon. In [35], author found the lump soliton and novel solitary wave solutions for the (3+1)-dimensional extended Jimbo-Miwa equations. Manafian and co-author found the interaction phenomenon to the (2+1)-dimensional Breaking Soliton equation [36]. Moreover, Ilhan et al. determined lump wave solutions and the interactions between lump solutions for a variable-coefficient Kadomtsev-Petviashvili equation [37]. Authors of [38] obtained the stationary solutions of various nonlinear Schrödinger equations. Younas and co-authors [38] studied the nonlinear chirp solitons for the model of Schröodinger-Hirota equation with concluding the bright, dark and singular solitons. Ali and co-workers [39] utilized the extended trial equation method and retrieved Jacobi elliptic, periodic, bright and singular solitons for paraxial nonlinear Schröodinger equation. In [40], the first and second-order rogue wave solutions were gained for the coupled Schrödinger equations. Arif et al. investigated the solitons and lump wave solutions to the graphene thermophoretic motion system [41]. Structures of this paper as follows, the nonlinear DNA dynamics model has been summarized in section 2. In sections 3–7, an overview of the integration schemes are given along with the analysis of the model including the improved $\tan(\phi/2)$-expansion method, $\exp(-\Omega(\eta))$-expansion method, improved $\exp(-\Omega(\eta))$-expansion method, generalized (G'/G)-expansion method, and exp-function method, respectively. The next section gives the discussions about the model. In the last section, the conclusions have been given.

2. Peyrard-Bishop DNA dynamic model equation

It is popular that DNA molecule is a double helix. This means that it consists of two complementary polymeric chains twisted around each other [42]. The B-form DNA in theWatsonCrick model is a double helix, which contains of two strands. The masses of nucleotides do not vary too much which means that one can assume a homogeneous crystal structure. The strands are coupled to each other through the hydrogen bonds, so that these bonds are weak while the harmonic longitudinal are strong the PB model neglects all the displacements beside the transversal [43]. The Hamiltonian model of Peyrard and Bishop [43], and the equations found in the literature, is modeled by the Morse potential as

$$V_M(u_n - v_n) = D\left[e^{-a(u_n - v_n)} - 1\right]^2, \tag{2.1}$$

in which u_n and v_n are the displacements of the nucleotides. Also, the Hamiltonian for the DNA chain was described by Zdravković [43]. Moreover, the improved version of the PB model, introduced by Dauxois [44]. The Hamiltonian for describing the strand aperture the hydrogen bonds can be stated as [45]

$$H(u) = \frac{1}{2m}q_n^2 + \frac{k_1}{2}\Delta^2 u_n + \frac{k_2}{4}\Delta^4 u_n + \delta\left(e^{-\sqrt{2}au_n} - 1\right)^2, \quad \Delta u_n = u_{n+1} - u_n, \tag{2.2}$$

in which k_1 and k_2 denote the strength for the linear and nonlinear couplings respectively and $q_n = m\dot{u}_n$ is the momentum for the displacement u_n. Searching Starting with the hamiltonian (2.2) the equation of motion in the continuum limit can be stated by the following form

$$\frac{\partial^2 u}{\partial t^2} - \left(l_1 + 3l_2\frac{\partial^2 u}{\partial x^2}\right) - 2\sqrt{2}aD\, e^{-au}\left(e^{-au} - 1\right), \tag{2.3}$$

with $l_1 = \frac{k_1}{m}d^2$, $l_2 = \frac{k_2}{m}d^4$, $D = \frac{\delta}{m}$, $\alpha \equiv \sqrt{2}a$ and being d the inter-site nucleotide distance in the DNA ladder ([46–48]). In this paper, consider the Peyrard-Bishop DNA dynamic model equation as follows

$$u_{tt} - \left(l_1 + 3l_2u_x^2\right)u_{xx} - 2\alpha\Omega e^{-\alpha u}\left(e^{-\alpha u} - 1\right) = 0, \tag{2.4}$$

where l_1, l_2, α and $\Omega = D$ are constants. By make the following transformations

$$u(x,t) = u(\xi), \quad \xi = x - \beta t, \tag{2.5}$$

then the Eq. (2.4), can be reduced to the ordinary differential equation as

$$\beta^2 u'' - \left(l_1 + 3l_2(u')^2\right)u'' - 2\alpha\Omega e^{-\alpha u}\left(e^{-\alpha u} - 1\right) = 0. \tag{2.6}$$

By multiplying the Eq. (2.6) by u' and integrating once with respect to ξ, we get

$$\frac{(\beta^2 - l_1)}{2}(u')^2 - \frac{3}{4}l_2(u')^4 + \Omega e^{-\alpha u}\left(e^{-\alpha u} - 2\right) + R = 0. \tag{2.7}$$

By starting hypothesis is taken to be

$$v(\xi) = e^{-\alpha u(\xi)}. \tag{2.8}$$

By appending (2.8) into Eq. (2.7), the nonlinear equation is achieved as follows

$$\frac{(\beta^2 - l_1)}{2\alpha^2}v^2(v')^2 - \frac{3}{4\alpha^4}l_2(v')^4 + \Omega v^5\left(v - 2\right) + Rv^4 = 0. \tag{2.9}$$

3. Description of the ITEM

In this section, the improved $\tan(\phi/2)$-expansion method [8, 9] has been summarized to obtain the solutions of nonlinear partial differential equations (NPDEs). Hence, consider the NPDEs in the following way:

$$\mathcal{N}(u, u_x, u_t, u_{xx}, u_{tt}, ...) = 0, \tag{3.1}$$

where $\mathcal{N}$ is a polynomial of u and its partial derivatives in which the relationship of higher order derivatives and nonlinear terms. To find the traveling wave solutions, we outline the following sequence of steps towards the extended tanh method:
Step 1. Firstly, by using traveling wave transformation

$$\xi = x - \beta t, \tag{3.2}$$

where β is non-zero arbitrary constant, permits to reduce Eq. (6.1) to an ODE of $u = u(\xi)$ in the following form

$$Q(u, u', -\beta u', u'', \beta^2 u'', ...) = 0. \tag{3.3}$$

Step 2. Assuming that the solution of Eq. (6.1) can be expressed by the following ansatz:

$$u(\xi) = S(\phi) = \sum_{k=0}^{m} A_k \left[\tan(\phi/2)\right]^k, \tag{3.4}$$

where $A_k (0 \le k \le m)$ are the parameters to be determined and $A_m \neq 0$ and $\phi = \phi(\xi)$ satisfies in the ordinary differential equation as follows:

$$\phi'(\xi) = a \sin(\phi(\xi)) + b \cos(\phi(\xi)) + c. \tag{3.5}$$

The particular solutions of Eq. (3.5) will be read as:
Family 1: **When** $\Delta = a^2 + b^2 - c^2 < 0$ **and** $b - c \neq 0$**,** then $\phi(\xi) = 2\tan^{-1}\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right]$.
Family 2: **When** $\Delta = a^2 + b^2 - c^2 > 0$ **and** $b - c \neq 0$**,** then $\phi(\xi) = 2\tan^{-1}\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\bar{\xi}\right)\right]$.
For see the rest seventeen families refer to Ref. [8, 9]. Also, $\bar{\xi} = \xi + C, p, A_k, B_k (k = 1, 2, ..., m), a, b$ and c are constants to be determined later.
Step 3. To determine the positive integer m, we usually balance linear terms of the highest order in the resulting equation with the highest order nonlinear terms appearing in equation (3.3).
Step 4. We collect all the terms with the same order of $\tan(\phi/2)^k, (k = 0, 1, 2, ...)$ together. Equate each coefficient of the polynomials to zero, yields the set of algebraic equations for $A_0, A_k (k = 1, 2, ..., m), a, b$ and c with the aid of the Maple.
Step 5. Solving the algebraic equations in Step 4, then substituting $A_0, A_1, ..., B_m, a, b, c$ in (3.4).

3.1. Application of the ITEM for Eq. (2.4)

Consider the homogeneous balance principle between the highest order derivatives $(v')^4$ and nonlinear terms v^6, and get

$$A_{4m+4}\tan^{4m+4}(\phi/2) \cong (v')^4 = v^6 \cong A_{6m}\tan^{6m}(\phi/2) \Longrightarrow 4m+4 = 6m \Longrightarrow m = 2.$$

Therefore, the equation (3.4) takes the form

$$v(\xi) = \sum_{k=0}^{2} A_k \tan^k(\phi/2). \tag{3.6}$$

Substitute equation (3.6) and its derivatives into equation (2.9). Algebraic equations set can be obtained after equating the coefficients of $\tan^p(\phi/2)$ for $p = 0, 1, ..., 12$, and setting equal to zero. After solving the nonlinear algebraic equations, the following values of $a, b, c, \beta, A_0, A_1, A_2$ can be obtained:
Set I.

$$\beta = \sqrt{2\,l_1 + 2\,\sqrt{3\,\Omega\,l_2 - 3\,Rl_2}},\ a = \frac{\Xi_1}{3A_2 l_2},\ b = \frac{\sqrt[4]{108\Omega\,{A_2}^2{l_2}^3}\alpha}{3l_2},\quad c = c,\quad \Delta = \frac{\alpha^2 A_2\left(\beta^2 - 2\,l_1\right) - 3\,l_2\,(b-c)^2}{3A_2 l_2}, \tag{3.7}$$

$$A_0 = -\frac{2\,\Omega\,\alpha^4{A_2}^2 + 3l_2(b-c)^3}{2\Omega\,\alpha^4 A_2},\quad A_1 = -\frac{\Xi_1(b-c)^3}{2\Omega\,\alpha^4{A_2}^2},\quad A_2 = A_2,$$

$$\Xi_1 = \sqrt{3A_2 l_2\,(\alpha^2\beta^2 A_2 - 2\,\alpha^2 A_2 l_1 - 3\,b^2 A_2 l_2 + 3\,c^2 A_2 l_2 - 3\,b^2 l_2 + 6\,bcl_2 - 3\,c^2 l_2)}.$$

By utilizing of Family 1, the trigonometric function solution becomes

$$u_1(x,t) = -\frac{1}{\alpha}\ln\left[-\frac{2\,\Omega\,\alpha^4{A_2}^2 + 3l_2(b-c)^3}{2\Omega\,\alpha^4 A_2} - \frac{\Xi_1(b-c)^3}{2\Omega\,\alpha^4{A_2}^2}\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right]\right. \tag{3.8}$$

$$\left.+A_2\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right]^2\right],\quad \bar{\xi} = x - \sqrt{2\,l_1 + 2\,\sqrt{3\,\Omega\,l_2 - 3\,Rl_2}}t + C.$$

The existence of the solution for the constraint condition is as $\frac{A_2(\beta^2-2l_1)}{3l_2} < \left(\frac{\frac{\sqrt[4]{108\Omega\,{A_2}^2{l_2}^3}\alpha}{3l_2} - c}{\alpha}\right)^2$.

By utilizing of Family 2, the hyperbolic function solution becomes

$$u_1(x,t) = -\frac{1}{\alpha}\ln\left[-\frac{2\,\Omega\,\alpha^4{A_2}^2 + 3l_2(b-c)^3}{2\Omega\,\alpha^4 A_2} - \frac{\Xi_1(b-c)^3}{2\Omega\,\alpha^4{A_2}^2}\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right]\right. \tag{3.9}$$

$$\left.+A_2\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right]^2\right],\quad \bar{\xi} = x - \sqrt{2\,l_1 + 2\,\sqrt{3\,\Omega\,l_2 - 3\,Rl_2}}t + C.$$

The existence of the solution for the constraint condition is as $\frac{A_2(\beta^2-2l_1)}{3l_2} > \left(\frac{\frac{\sqrt[4]{108\Omega\,{A_2}^2{l_2}^3}\alpha}{3l_2} - c}{\alpha}\right)^2$.

4. The $\exp(-\Omega(\eta))$-expansion method

This section elucidates a systematic explanation of the $\exp(-\Omega(\eta))$-expansion method to obtain the solutions of nonlinear partial differential equations (NPDEs). Hence, take the NPDEs in the following way:

$$\mathcal{N}(u, u_x, u_t, u_{xx}, u_{tt}, ...) = 0, \tag{4.1}$$

where $\mathcal{N}$ is a polynomial of u and its partial derivatives in which the relationship of higher order derivatives and nonlinear terms. To find the traveling wave solutions, we outline the following sequence of steps towards the extended tanh method:

Step 1. Firstly, by utilizing the traveling wave transformation

$$\xi = x - \beta t, \tag{4.2}$$

where β is non-zero arbitrary constant, permits to reduce equation (4.1) to an ODE of $u = u(\xi)$ in the following form

$$Q(u, u', -\beta u', u'', \beta^2 u'', ...) = 0, \tag{4.3}$$

Step 2. Assuming that the solution of equation (4.1) can be expressed by the following ansatz:

$$U(\xi) = \sum_{j=0}^{m} A_j F^j(\xi), \tag{4.4}$$

where $F(\eta) = \exp(-\Phi(\xi))$ and $A_j (0 \leq j \leq m)$, are the parameters to be determined $A_m \neq 0$, and, $\Phi = \Phi(\xi)$ satisfying the ODE given below

$$\Phi' = \mu F^{-1}(\xi) + F(\xi) + \lambda. \tag{4.5}$$

The particular solutions of equation (4.5) will be read as:

Solution-1: When $\mu \neq 0$ and $\lambda^2 - 4\mu > 0$, therefore we attain $\Phi(\eta) = \ln\left(-\frac{\sqrt{\lambda^2-4\mu}}{2\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + E)\right) - \frac{\lambda}{2\mu}\right)$.

Solution-2: When $\mu \neq 0$ and $\lambda^2 - 4\mu < 0$, therefore we attain $\Phi(\eta) = \ln\left(\frac{\sqrt{-\lambda^2+4\mu}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}(\xi + E)\right) - \frac{\lambda}{2\mu}\right)$.

Solution-3: When $\mu = 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$, therefore we attain $\Phi(\eta) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi+E))-1}\right)$.

Solution-4: When $\mu \neq 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu = 0$, therefore we attain $\Phi(\eta) = \ln\left(-\frac{2\lambda(\xi+E)+4}{\lambda^2(\xi+E)}\right)$.

Solution-5: When $\mu = 0$, $\lambda = 0$, and $\lambda^2 - 4\mu = 0$, therefore we attain $\Phi(\eta) = \ln(\xi + E)$, where $A_j (0 \leq j \leq m)$, E, λ and μ are also the constants to be explored later.

Step 3. To determine the positive integer m, we usually balance the linear terms of the highest order in the resulting equation with the highest order nonlinear terms appearing in equation (4.3).

Step 4. We collect all the terms with the same order of $F(\xi)^k, (k = 0, 1, 2, ...)$ together. Equate each coefficient of the polynomials of $F(\xi)^k$ to zero, yields the set of algebraic equations for $A_0, A_k (k = 1, 2, ..., m), \lambda$ and μ with the aid of the Maple.

Step 5. Solving the algebraic equations in Step 4, then substituting $A_0, A_1, ..., A_m, \lambda, \mu$ in (4.4).

4.1. Application of the EEM for Eq. (2.4)

Consider the homogeneous balance principle between the highest order derivatives $(v')^4$ and nonlinear terms v^6, we obtain $4m + 4 = 6m$, then $m = 2$. The exact solution can be expressed in the following form

$$v(\xi) = \sum_{k=0}^{2} A_k F^k(\eta), \tag{4.6}$$

Substitute equation (4.6) and its derivatives into equation (2.9). The algebraic equations set can be obtained after equating the coefficients of $F(\xi)$ for p = 0,1,...,20, and setting equal to zero. After solving the nonlinear algebraic equations, the following values of $\lambda, \mu, \beta, A_0, A_1, A_2$ can be obtained:

Set I.

$$\Sigma_1 = -125\,\Omega^5\alpha^3 de^3\Xi_3\Xi_1^3\left(\lambda^2-4\,\mu\right) - 250\,\Omega^4 R\alpha^4 de^2\Xi_1\Xi_2\Xi_3 - 1250\,\Omega^5 R\alpha^5 e^3\Xi_3\Xi_1^2 + 5000\,\Omega^6\alpha^6 d\,(45\,\Omega-4\,R)\,\Xi_1^3 - \tag{4.7}$$

$$6\,\alpha^2 l_1 l_2\Xi_2^4 - 9\,{l_2}^2\Xi_3\Xi_2^3\left(\lambda^2-4\,\mu\right) + 5000\,\Omega^5\alpha^6 l_1\Xi_1^4 - 3\,\Omega\,\alpha\, l_2\Xi_2 (50\,\Omega\, R\alpha^2 e\Xi_2\Xi_3 + 2\,\alpha\, d\left(2025\,\Omega^2+65\,\Omega\, R+R^2\right)\Xi_2^2 +$$

$$5\,\Omega\, e\Xi_1\Xi_3\left(225\,\Omega^2\alpha\, e+5\,\Omega\, R\alpha\, e+2025\,\Omega^2 d+115\,\Omega\, Rd+R^2 d\right)\left(\lambda^2-4\,\mu\right)),$$

$$\Sigma_2 = 25000\,\Omega^6 R\alpha^6 d\Xi_1^3 + 1250\,\Omega^5 R\alpha^5 e^3\Xi_3\Xi_1^2 + 15\,\Omega^2\alpha\, del_2\Xi_1\Xi_3\Xi_2^2\left(\lambda^2-4\,\mu\right)$$

$$-18\,{l_2}^2\Xi_2^3\left(225\,\Omega^2\lambda^2+5\,\Omega\, R\lambda^2+11250\,\Omega^2\mu+670\,\Omega\, R\mu+6\,R^2\mu\right) + 25\,\Omega^2\alpha^3 e\Xi_3\left(5\,\Omega^3 de^2\Xi_1^3\left(\lambda^2-4\,\mu\right)+6\, l_2 R\Xi_2^2\right)$$

$$75\,\Omega^2\alpha^2 l_2\Xi_2\left(\Omega\, e^2\Xi_3\Xi_1^2\right)\left(\lambda^2-4\,\mu\right) - 4\,Rd\Xi_2 + 250\,\Omega^4\alpha^4\Xi_1\left(30\,\Omega\, l_2\Xi_1^3\left(\lambda^2+8\,\mu\right)+Rde^2\Xi_2\Xi_3\right),$$

$$\Xi_1 = 45\,\Omega+R,\quad \Xi_2 = 2025\,\Omega^2+115\,\Omega\, R+R^2,\quad \Xi_3 = 1575\,\Omega^2+105\,\Omega\, R+R^2,\quad \beta = \frac{\sqrt{(2500\,\Omega^5\alpha^4\Xi_1^4-3\, l_2\Xi_2^4)\Sigma_1}}{\alpha(2500\,\Omega^5\alpha^4\Xi_1^4-3\, l_2\Xi_2^4)},$$

$$d = \frac{\sqrt{3\Omega l_2}}{\Omega},\quad e = \frac{\sqrt[4]{12\Omega^3 l_2}}{\Omega},\quad A_0 = \frac{\Sigma_2}{6\alpha^2\Omega d(2500\,\Omega^5\alpha^4\Xi_1^4-3\, l_2\Xi_2^4)},\quad A_1 = \frac{2\lambda\, d}{\alpha^2},\quad A_2 = \frac{2d}{\alpha^2}.$$

By utilizing of Family 1, the hyperbolic function solution becomes

$$u_1(x,t) = -\frac{1}{\alpha}\ln\left[A_0 + \frac{2\lambda\, d}{\alpha^2}\left(-\frac{\sqrt{\lambda^2-4\mu}}{2\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\eta+E)\right)-\frac{\lambda}{2\mu}\right)^{-1}\right. \tag{4.8}$$

$$\left.+\frac{2d}{\alpha^2}\left(-\frac{\sqrt{\lambda^2-4\mu}}{2\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\eta+E)\right)-\frac{\lambda}{2\mu}\right)^{-2}\right],\quad \eta = x-\beta t.$$

The existence of the solution for the constraint condition is as $l_2(\sqrt{\alpha^4(\Omega^2\alpha^2 e^2 - 12Rl_2)} - e\alpha^3\Omega) > 0$.
By utilizing of Family 2, the trigonometric function solution becomes

$$u_2(x,t) = -\frac{1}{\alpha}\ln\left[A_0 + \frac{2\lambda\, d}{\alpha^2}\left(\frac{\sqrt{-\lambda^2+4\mu}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}(\eta+E)\right)-\frac{\lambda}{2\mu}\right)^{-1}\right. \tag{4.9}$$

$$\left.+\frac{2d}{\alpha^2}\left(\frac{\sqrt{-\lambda^2+4\mu}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}(\eta+E)\right)-\frac{\lambda}{2\mu}\right)^{-2}\right],\quad \eta = x-\beta t.$$

The existence of the solution for the constraint condition is as $l_2(\sqrt{\alpha^4(\Omega^2\alpha^2e^2 - 12Rl_2)} - e\alpha^3\Omega) < 0$. By utilizing of Family 3, the hyperbolic function solution becomes

$$u_3(x,t) = -\frac{1}{\alpha}\ln\left[A_0 + \frac{2\lambda d}{\alpha^2}\left(\frac{\lambda}{\exp\left(\lambda(\eta+E)\right)-1}\right) + \frac{2d}{\alpha^2}\left(\frac{\lambda}{\exp\left(\lambda(\eta+E)\right)-1}\right)^2\right],\ \eta = x - \beta t. \quad (4.10)$$

The existence of the solution for the constraint condition is as $l_2(\sqrt{\alpha^4(\Omega^2\alpha^2e^2 - 12Rl_2)} - e\alpha^3\Omega) > 0$, and

$$\lambda = \frac{\sqrt{3l_2\left(-e\alpha^3\Omega + 12\,\mu\, l_2 + \sqrt{\Omega^2\alpha^6e^2 - 12\,R\alpha^4 l_2}\right)}}{3l_2},\quad \lambda^2 - 4\mu = \frac{-e\alpha^3\Omega + \sqrt{\alpha^4\left(\Omega^2\alpha^2e^2 - 12\,Rl_2\right)}}{3l_2}.$$

5. The improved $\exp(-\Omega(\eta))$-expansion method

In this section the improved $\exp(-\Omega(\eta))$-expansion method is utilized to obtain the solutions of nonlinear partial differential equations (NPDEs). Hence, consider the NPDEs in the following way:

$$\mathcal{N}(u, u_x, u_t, u_{xx}, u_{tt}, ...) = 0, \quad (5.1)$$

where $\mathcal{N}$ is a polynomial of u and its partial derivatives in which the relationship of higher order derivatives and nonlinear terms. To find the traveling wave solutions, we outline the following sequence of steps towards the extended tanh method:

Step 1. Firstly, by using the traveling wave transformation

$$\xi = x - \beta t, \quad (5.2)$$

where β is non-zero arbitrary constant, permits to reduce equation (4.1) to an ODE of $u = u(\xi)$ in the following form

$$Q(u, u', -\beta u', u'', \beta^2 u'', ...) = 0, \quad (5.3)$$

Step 2. Assuming that the solution of equation (5.1) can be expressed by the following ansatz:

$$U(\xi) = \sum_{j=0}^{m} A_j F^j(\xi) + \sum_{j=1}^{m} B_j F^j(\xi), \quad (5.4)$$

where $F(\eta) = \exp(-\Phi(\xi))$ and $A_j(0 \leq j \leq m), B_j(1 \leq j \leq m)$, are the parameters to be determined $A_m \neq 0$, and, $\Phi = \Phi(\xi)$ satisfying the ODE given below

$$\Phi' = \mu F^{-1}(\xi) + F(\xi) + \lambda. \quad (5.5)$$

The particular solutions of equation (5.5) will be read like before section.

Step 3. To determine the positive integer m, we usually balance the linear terms of the highest order in the resulting equation with the highest order nonlinear terms appearing in equation (4.3).

Step 4. We collect all the terms with the same order of $F(\xi)^k, (k = 0, 1, 2, ...)$ together. Equate each coefficient of the polynomials of $F(\xi)^k$ to zero, yields the set of algebraic equations for $A_0, A_k, B_k(k = 1, 2, ..., m), \lambda$ and μ with the aid of the Maple.

Step 5. Solving the algebraic equations in Step 4, then substituting $A_0, A_1, B_1, ..., A_m, B_m, \lambda, \mu$ in (5.4).

5.1. Application of the IEEM for Eq. (2.4)

The exact solution will be the same as the previous section as

$$v(\xi) = \sum_{k=0}^{2} A_k F^k(\eta) + \sum_{k=1}^{2} B_k F^{-k}(\eta), \tag{5.6}$$

Substitute equation (5.6) and its derivatives into equation (2.9). The algebraic equations set can be obtained after equating the coefficients of $F(\xi)$ for $p = 0, 1, ..., 20$, and setting equal to zero. After solving the nonlinear algebraic equations, the following values of $\lambda, \mu, \beta, A_0, A_1, B_1, A_2, B_2$ can be obtained:

Set I.

$$\begin{aligned}\Sigma_1 &= 5000\,\Omega^5\alpha^6\Xi_1^3\left(45\,\Omega^2 d - 4\,\Omega R d + 45\,\Omega\, l_1 + R l_1\right) - 1250\,\Omega^5 R\alpha^5 e^3\Xi_3\Xi_1^2 - 250\,\Omega^4 R\alpha^4 d e^2\Xi_1\Xi_2\Xi_3 \\ &-25\,\Omega^2\alpha^3 e\Xi_2\left(5\,\Omega^3 d e^2\Xi_1^3\left(\lambda^2 - 4\,\mu\right) + 6\,R l_2\Xi_2^2\right) - 3\,\alpha^2 l_2\Xi_2\left(25\,\Omega^3 e^2\Xi_3\Xi_1^2\left(\lambda^2 - 4\,\mu\right) + \right. \\ &\left. 2\,\Xi_2^2\left(2025\,\Omega^3 d + 65\,\Omega^2 R d + \Omega R^2 d + 2025\,\Omega^2 l_1 + 115\,\Omega R l_1 + R^2 l_1\right)\right) - 15\,\Omega^2\alpha\, d e l_2\Xi_1\Xi_3\Xi_2^2\left(\lambda^2 - 4\,\mu\right) \\ &-9\,{l_2}^2\Xi_3\Xi_2^3\left(\lambda^2 - 4\,\mu\right), \quad d = \frac{\sqrt{3\Omega l_2}}{\Omega}, \quad e = \frac{\sqrt[4]{12\Omega^3 l_2}}{\Omega}, \quad \beta = \frac{\sqrt{(2500\,\Omega^5\alpha^4\Xi_1^4 - 3\,l_2\Xi_2^4)\Sigma_1}}{\alpha(2500\,\Omega^5\alpha^4\Xi_1^4 - 3\,l_2\Xi_2^4)},\end{aligned} \tag{5.7}$$

$$A_0 = \frac{\Sigma_2}{6\alpha^2\Omega d(2500\,\Omega^5\alpha^4\Xi_1^4 - 3\,l_2\Xi_2^4)}, \quad A_1 = \frac{2\lambda\, d}{\alpha^2}, \quad A_2 = \frac{2d}{\alpha^2}, \quad B_1 = 0, \quad B_2 = \frac{\Sigma_2}{360 d l_2\Sigma_3},$$

$$\begin{aligned}\Sigma_2 &= 25000\,\Omega^6 R\alpha^6 d\Xi_1^3 + 1250\,\Omega^5 R\alpha^5 e^3\Xi_3\Xi_1^2 + 15\,\Omega^2\alpha\, d e l_2\Xi_1\Xi_3\Xi_2^2\left(\lambda^2 - 4\,\mu\right) + 25\,\Omega^2\alpha^3 e\Xi_3\left(5\,\Omega^3 d e^2\Xi_1^3\left(\lambda^2 - 4\,\mu\right) + 6\right. \\ &-18\,{l_2}^2\Xi_2^3\left(225\,\Omega^2\lambda^2 + 5\,\Omega R\lambda^2 + 11250\,\Omega^2\mu + 670\,\Omega R\mu + 6\,R^2\mu\right) + 75\,\Omega^2\alpha^2 l_2\Xi_2\left(\Omega\, e^2\Xi_3\Xi_1^2\left(\lambda^2 - 4\,\mu\right) - 4\,R d\Xi_2^2\right) \\ &+250\,\Omega^4\alpha^4\Xi_1\left(30\,\Omega\, l_2\Xi_1^3\left(\lambda^2 + 8\,\mu\right) + R d e^2\Xi_2\Xi_3\right),\end{aligned}$$

$$\Sigma_3 = 6250000\,\Omega^{10}\alpha^8\Xi_1^8 - 3\,l_2\left(\Xi_2^4\right)\left(5000\,\Omega^5\alpha^4\Xi_1^4 - 3\,l_2\Xi_2^4\right),$$

$$\Xi_1 = 45\,\Omega + R, \quad \Xi_2 = 2025\,\Omega^2 + 115\,\Omega R + R^2, \quad \Xi_3 = 1575\,\Omega^2 + 105\,\Omega R + R^2.$$

By utilizing of Family 1, the hyperbolic function solution becomes

$$\begin{aligned}u_1(x,t) = -\frac{1}{\alpha}\ln\Bigg[A_0 &+ \frac{2\lambda\, d}{\alpha^2}\left(-\frac{\sqrt{\lambda^2 - 4\mu}}{2\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\overline{\eta}\right) - \frac{\lambda}{2\mu}\right)^{-1} \\ &+ \frac{2d}{\alpha^2}\left(-\frac{\sqrt{\lambda^2 - 4\mu}}{2\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\overline{\eta}\right) - \frac{\lambda}{2\mu}\right)^{-2} + \frac{\Sigma_2}{360 d l_2\Sigma_3}\left(-\frac{\sqrt{\lambda^2 - 4\mu}}{2\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\overline{\eta}\right) - \frac{\lambda}{2\mu}\right)^2\Bigg].\end{aligned} \tag{5.8}$$

The existence of the solution for the constraint condition is as $l_2(\sqrt{\alpha^4(\Omega^2\alpha^2 e^2 - 12 R l_2)} - e\alpha^3\Omega) > 0$.
By utilizing of Family 2, the trigonometric function solution becomes

$$u_2(x,t) = -\frac{1}{\alpha}\ln\Bigg[A_0 + \frac{2\lambda\, d}{\alpha^2}\left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2}\overline{\eta}\right) - \frac{\lambda}{2\mu}\right)^{-1} \tag{5.9}$$

$$+\frac{2d}{\alpha^2}\left(\frac{\sqrt{-\lambda^2+4\mu}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\overline{\eta}\right)-\frac{\lambda}{2\mu}\right)^{-2}+\frac{\Sigma_2}{360dl_2\Sigma_3}\left(\frac{\sqrt{-\lambda^2+4\mu}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\overline{\eta}\right)-\frac{\lambda}{2\mu}\right)^{2}\Bigg].$$

The existence of the solution for the constraint condition is as $l_2(\sqrt{\alpha^4(\Omega^2\alpha^2e^2-12Rl_2)}-e\alpha^3\Omega)<0$. By utilizing of Family 3, the kink-soliton solution becomes

$$u_3(x,t)=-\frac{1}{\alpha}\ln\left[A_0+\frac{2\lambda d}{\alpha^2}\left(\frac{\lambda}{\exp(\lambda\overline{\eta})-1}\right)+\frac{2d}{\alpha^2}\left(\frac{\lambda}{\exp(\lambda\overline{\eta})-1}\right)^2+\frac{\Sigma_2}{360dl_2\Sigma_3}\left(\frac{\lambda}{\exp(\lambda\overline{\eta})-1}\right)^{-2}\right]. \tag{5.10}$$

The existence of the solution for the constraint condition is as $l_2(\sqrt{\alpha^4(\Omega^2\alpha^2e^2-12Rl_2)}-e\alpha^3\Omega)>0$ and

$$\overline{\eta}=x-\beta t+E,\ \ \lambda=\frac{\sqrt{3l_2\left(-e\alpha^3\Omega+12\,\mu\,l_2+\sqrt{\Omega^2\alpha^6e^2-12\,R\alpha^4l_2}\right)}}{3l_2},\ \ \lambda^2-4\mu=\frac{-e\alpha^3\Omega+\sqrt{\alpha^4\left(\Omega^2\alpha^2e^2-12\,Rl_2\right)}}{3l_2}.$$

Set II.

$$\Sigma_1=5000\,\Omega^5\alpha^6\Xi_1^3\left(45\,\Omega^2d-4\,\Omega\,Rd+45\,\Omega\,l_1+Rl_1\right)-1250\mu\,\Omega^5R\alpha^5e^3\Xi_3\Xi_1^2-250\mu^2\,\Omega^4R\alpha^4de^2\Xi_1\Xi_2\Xi_3 \tag{5.11}$$

$$-25\,\Omega^2\alpha^3e\mu\Xi_3\left(5\,\Omega^3de^2\Xi_1^3\left(\lambda^2-4\,\mu\right)+6\,Rl_2\mu^2\Xi_2^2\right)-3\,\alpha^2l_2\mu^2\Xi_2\left(25\,\Omega^3e^2\Xi_3\Xi_1^2\left(\lambda^2-4\,\mu\right)+\right.$$

$$\left.2\mu^2\,\Xi_2^2\left(2025\,\Omega^3d+65\,\Omega^2Rd+\Omega\,R^2d+2025\,\Omega^2l_1+115\,\Omega\,Rl_1+R^2l_1\right)\right)-15\mu^3\,\Omega^2\alpha\,del_2\Xi_1\Xi_3\Xi_2^2\left(\lambda^2-4\,\mu\right)$$

$$-9\mu^4\,l_2{}^2\Xi_3\Xi_2^3\left(\lambda^2-4\,\mu\right),\ \ d=\frac{\sqrt{3\Omega l_2}}{\Omega},\ \ e=\frac{\sqrt[4]{12\Omega^3l_2}}{\Omega},\ \ \beta=\frac{\sqrt{(2500\,\Omega^5\alpha^4\Xi_1^4-3\mu^4\,l_2\Xi_2^4)\Sigma_1}}{\alpha(2500\,\Omega^5\alpha^4\Xi_1^4-3\mu^4\,l_2\Xi_2^4)},$$

$$\Sigma_2=1250\mu\Omega^5R\alpha^5e^3\Xi_3\Xi_1^2+15\mu^3\Omega^2\alpha del_2\Xi_1\Xi_3\Xi_2^2\left(\lambda^2-4\mu\right)+250\Omega^4\alpha^4\Xi_1\left(30\Omega l_2\Xi_1^3\left(\lambda^2+8\mu\right)+Rde^2\mu^2\Xi_2\Xi_3\right)$$

$$-18\mu^4\,l_2{}^2\Xi_2^3\left(225\,\Omega^2\lambda^2+5\,\Omega\,R\lambda^2+11250\,\Omega^2\mu+670\,\Omega\,R\mu+6\,R^2\mu\right)+25\mu\,\Omega^2\alpha^3e\Xi_3\left(5\,\Omega^3de^2\Xi_1^3\left(\lambda^2-4\,\mu\right)+6\mu^2\,l_2R\right.$$

$$75\mu^2\,\Omega^2\alpha^2l_2\Xi_2(\Omega\,e^2\left(1575\,\Omega^2+105\,\Omega\,R+R^2\right)\Xi_1^2\left(\lambda^2-4\,\mu\right)-4\mu^2\,Rd\Xi_2)+25000\Omega^6R\alpha^6d\Xi_1^3,$$

$$A_0=\frac{\Sigma_2}{6\alpha^2\Omega d(2500\,\Omega^5\alpha^4\Xi_1^4-3\mu^4\,l_2\Xi_2^4)},\ \ A_1=0,\ \ A_2=0,\ \ B_1=\frac{2d\lambda\mu}{\alpha^2},\ \ B_2=\frac{2d\mu^2}{\alpha^2},$$

$$\Xi_1=45\,\Omega+R,\quad \Xi_2=2025\,\Omega^2+115\,\Omega\,R+R^2,\quad \Xi_3=1575\,\Omega^2+105\,\Omega\,R+R^2.$$

By utilizing of Family 1, the hyperbolic function solution becomes

$$u_1(x,t)=-\frac{1}{\alpha}\ln\left[A_0+\frac{2d\lambda\mu}{\alpha^2}\left(-\frac{\sqrt{\lambda^2-4\mu}}{2\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\overline{\eta}\right)-\frac{\lambda}{2\mu}\right)+\right. \tag{5.12}$$

$$\left.\frac{2\mu^2d}{\alpha^2}\left(-\frac{\sqrt{\lambda^2-4\mu}}{2\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\overline{\eta}\right)-\frac{\lambda}{2\mu}\right)^2\right],\quad \overline{\eta}=x-\beta t+E.$$

The existence of the solution for the constraint condition is as $l_2\mu(\sqrt{\alpha^4(\Omega^2\alpha^2e^2-12\mu^2Rl_2)}-e\alpha^3\Omega)>0$. By utilizing of Family 2, the trigonometric function solution becomes

$$u_2(x,t)=-\frac{1}{\alpha}\ln\left[A_0+\frac{2d\lambda\mu}{\alpha^2}\left(\frac{\sqrt{-\lambda^2+4\mu}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\overline{\eta}\right)-\frac{\lambda}{2\mu}\right)+\right. \tag{5.13}$$

$$\left.\frac{2\mu^2d}{\alpha^2}\left(\frac{\sqrt{-\lambda^2+4\mu}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\overline{\eta}\right)-\frac{\lambda}{2\mu}\right)^2\right],\quad \overline{\eta}=x-\beta t+E.$$

The existence of the solution for the constraint condition is as $l_2\mu(\sqrt{\alpha^4(\Omega^2\alpha^2e^2-12\mu^2Rl_2)}-e\alpha^3\Omega)<0$. By utilizing of Family 3, the exponential function solution becomes

$$u_3(x,t)=-\frac{1}{\alpha}\ln\left[A_0+\frac{2\mu\lambda\,d}{\alpha^2}\left(\frac{\exp(\lambda\overline{\eta})}{\lambda}\right)+\frac{2d\mu}{\alpha^2}\left(\frac{\exp(\lambda\overline{\eta})}{\lambda}\right)^2\right],\quad \overline{\eta}=x-\beta t+E. \tag{5.14}$$

$$\lambda=\frac{\sqrt{3l_2\left(-e\alpha^3\Omega+12\,\mu\,l_2+\sqrt{\Omega^2\alpha^6e^2-12\,R\alpha^4l_2}\right)}}{3l_2},\quad \lambda^2-4\mu=\frac{-e\alpha^3\Omega+\sqrt{\alpha^4\left(\Omega^2\alpha^2e^2-12\,Rl_2\right)}}{3l_2}.$$

The existence of the solution for the constraint condition is as $l_2\mu(\sqrt{\alpha^4(\Omega^2\alpha^2e^2-12\mu Rl_2)}-e\alpha^3\Omega)>0$.

6. Description of the GGM

As the fourth method, the generalized (G'/G)-expansion method has been summarized to obtain the solutions of NPDEs. Hence, consider the NPDEs of in the following way:

$$\mathcal{N}(u,u_x,u_t,u_{xx},u_{tt},...)=0, \tag{6.1}$$

where $\mathcal{N}$ is a polynomial of u and its partial derivatives in which the relationship of higher order derivatives and nonlinear terms. To find the traveling wave solutions, we outline the following sequence of steps towards the GGM:

Step 1. Firstly, by using traveling wave transformation

$$\xi=x-\beta t, \tag{6.2}$$

where β is non-zero arbitrary constant, permits to reduce equation (6.1) to an ODE of $u=u(\xi)$ in the following form

$$Q(u,u',-\beta u',u'',\beta^2u'',...)=0, \tag{6.3}$$

Step 2. Assuming that the solution of equation (6.1) can be expressed by the following ansatz:

$$u(\xi)=S(\Phi(\xi))=\sum_{k=0}^{m}A_k\Phi(\xi)^k, \tag{6.4}$$

where, $A_k(0\le k\le m)$ are constants to be determined, such that $A_m\neq 0$, and $\Phi(\xi)=G'(\xi)/G(\xi)$ satisfies the following ODE:

$$k_1GG''-k_2GG'-k_3(G')^2-k_4G^2=0. \tag{6.5}$$

The particular solutions of equation (6.5) will be read as:
Family 1: **When** $k_2 \neq 0$, $f = k_1 - k_3$ **and** $s = k_2^2 + 4k_4(k_1 - k_3) > 0$, then $\Phi(\xi) = \frac{k_2}{2f} + \frac{\sqrt{s}}{2f}\frac{C_1 \sinh\left(\frac{\sqrt{s}}{2k_1}\xi\right)+C_2\cosh\left(\frac{\sqrt{s}}{2k_1}\xi\right)}{C_1\cosh\left(\frac{\sqrt{s}}{2k_1}\xi\right)+C_2\sinh\left(\frac{\sqrt{s}}{2k_1}\xi\right)}$.
Family 2: **When** $k_2 \neq 0$, $f = k_1 - k_3$ **and** $s = k_2^2 + 4k_4(k_1 - k_3) < 0$, then $\Phi(\xi) = \frac{k_2}{2f} + \frac{\sqrt{-s}}{2f}\frac{-C_1 \sin\left(\frac{\sqrt{-s}}{2k_1}\xi\right)+C_2\cos\left(\frac{\sqrt{-s}}{2k_1}\xi\right)}{C_1\cos\left(\frac{\sqrt{-s}}{2k_1}\xi\right)+C_2\sin\left(\frac{\sqrt{-s}}{2k_1}\xi\right)}$.
Family 3: **When** $k_2 \neq 0$, $f = k_1 - k_3$ **and** $s = k_2^2 + 4k_4(k_1 - k_3) = 0$, then $\Phi(\xi) = \frac{k_2}{2f} + \frac{C_2}{C_1 + C_2\xi}$.
Family 4: **When** $k_2 = 0$, $f = k_1 - k_3$ **and** $g = fk_4 > 0$, then $\Phi(\xi) = \frac{\sqrt{g}}{f}\frac{C_1 \sinh\left(\frac{\sqrt{g}}{k_1}\xi\right)+C_2\cosh\left(\frac{\sqrt{g}}{k_1}\xi\right)}{C_1\cosh\left(\frac{\sqrt{g}}{k_1}\xi\right)+C_2\sinh\left(\frac{\sqrt{g}}{k_1}\xi\right)}$.
Family 5: **When** $k_2 = 0$, $f = k_1 - k_3$ **and** $g = fk_4 < 0$, then $\Phi(\xi) = \frac{\sqrt{-g}}{f}\frac{-C_1 \sin\left(\frac{\sqrt{-g}}{k_1}\xi\right)+C_2\cos\left(\frac{\sqrt{-g}}{k_1}\xi\right)}{C_1\cos\left(\frac{\sqrt{-g}}{k_1}\xi\right)+C_2\sin\left(\frac{\sqrt{-g}}{k_1}\xi\right)}$.
Family 6: **When** $k_4 = 0$ **and** $f = k_1 - k_3$, then $\Phi(\xi) = \frac{C_1 k_2^2 \exp\left(\frac{-k_2}{k_1}\xi\right)}{fk_1 + C_1 k_1 k_2 \exp\left(\frac{-k_2}{k_1}\xi\right)}$.
Family 7: **When** $k_2 \neq 0$ **and** $f = k_1 - k_3 = 0$, then $\Phi(\xi) = -\frac{k_4}{k_2} + C_1 \exp\left(\frac{k_2}{k_1}\xi\right)$,
Family 8: **When** $k_1 = k_3$, $k_2 = 0$ **and** $f = k_1 - k_3 = 0$, then $\Phi(\xi) = C_1 + \frac{k_4}{k_1}\xi$,
Family 9: **When** $k_3 = 2k_1$, $k_2 = 0$ **and** $k_4 = 0$, then $\Phi(\xi) = -\frac{1}{C_1 + \left(\frac{k_3}{k_1} - 1\right)\xi}$, where $d_0, d_j, e_j (j = 1, ..., m), k_1, k_2, k_3$ and k_4 are constants to be determined later.
Step 3. To determine the positive integer m, we usually balance linear terms of the highest order in the resulting equation with the highest order nonlinear terms appearing in equation (3.3).
Step 4. We collect all the terms with the same order of $\Phi(\xi)^k, (k = 0, 1, 2, ...)$ together. Equate each coefficient of the polynomials of i to zero, yields the set of algebraic equations for $A_0, A_k(k = 1, 2, ..., m), k_1, k_2, k_3$, and k_4 with the aid of the Maple.
Step 5. Solving the algebraic equations in Step 4, then substituting $A_0, A_1, ..., B_m, k_1, k_2, k_3, k_4$ in (6.4).

6.1. Application of GGM

By processing the generalized G'/G-expansion method and considering the homogeneous balance principle, we get the exact solution in the following form

$$u(\xi) = A_0 + A_1\Phi(\xi) + A_1\Phi(\xi)^2. \tag{6.6}$$

Solving the nonlinear algebraic equations, we have the following sets of coefficients for the solutions of (6.6) as given below:
Subset I.

$$\beta = \frac{\sqrt{A_2(2\,\alpha^2 A_2 {k_1}^2 l_1 - 12\,{k_1}^2 l_2 \epsilon_1\,(\epsilon_1 - 2)\,(A_0 - 1) - 3\,l_2\left(4\,A_0 {k_1}^2 - A_2 {\epsilon_3}^2 - 4\,{k_1}^2\right))}}{k_1 \alpha A_2}, \tag{6.7}$$

$$\varepsilon_1 = \frac{12^{3/4}\sqrt[4]{\Omega\,{A_2}^2 {l_2}^3}\alpha}{12 l_2}, \quad \varepsilon_2 = \frac{\sqrt{3}\sqrt{l_2 k_1\left(\sqrt[4]{\Omega\,{A_2}^2 {l_2}^3}\,12^{3/4}\alpha + 12\,l_2\right)}}{6 l_2},$$

$$\varepsilon_3=\frac{1}{3A_2l_2}\sqrt{6A_2l_2((A_0-1)\left(6\,\epsilon_2 l_2\,(\epsilon_2-2\,k_1)+6\,{k_1}^2l_2\right)+\sqrt{\varepsilon_4}},$$

$$\varepsilon_4=36\,\epsilon_2{l_2}^2\,(\epsilon_2-2\,k_1)\left({\epsilon_2}^2-2\,\epsilon_2k_1+2\,{k_1}^2\right)(A_0-1)^2-3\,\alpha^4{A_2}^2{k_1}^4l_2\left(\Omega\,{A_0}^2-2\,\Omega\,A_0+R\right)+36\,{k_1}^4{l_2}^2\,(A_0-1)^2,$$

$$A_1=\frac{12\varepsilon_3l_2(\varepsilon_1-1)^3}{k_1\Omega\alpha^4A_2},\ \ A_2=A_2,\ \ k_1=k_1,\ \ k_2=\varepsilon_3,\ \ k_3=\varepsilon_1,\ \ k_4=\frac{\Omega\,\alpha^4A_0A_2k_1}{12l_2\,({\epsilon_1}^3-3\,{\epsilon_1}^2+3\,\epsilon_1-1)},$$

$$s=k_2^2+4k_4(k_1-k_3)={\epsilon_3}^2-\frac{\Omega\,\alpha^4A_0A_2{k_1}^2}{3\,(\epsilon_1-1)^2\,l_2}.$$

Based on the Family 1, the exact soliton solution can be written as

$$u_1(x,t)=-\frac{1}{\alpha}\ln\left[A_0+\frac{12\varepsilon_3l_2(\varepsilon_1-1)^3}{k_1\Omega\alpha^4A_2}\left\{\frac{k_2}{2f}+\frac{\sqrt{s}}{2f}\frac{C_1\sinh\left(\frac{\sqrt{s}}{2k_1}\xi\right)+C_2\cosh\left(\frac{\sqrt{s}}{2k_1}\xi\right)}{C_1\cosh\left(\frac{\sqrt{s}}{2k_1}\xi\right)+C_2\sinh\left(\frac{\sqrt{s}}{2k_1}\xi\right)}\right\}+\right.\tag{6.8}$$

$$\left.A_2\left\{\frac{k_2}{2f}+\frac{\sqrt{s}}{2f}\frac{C_1\sinh\left(\frac{\sqrt{s}}{2k_1}\xi\right)+C_2\cosh\left(\frac{\sqrt{s}}{2k_1}\xi\right)}{C_1\cosh\left(\frac{\sqrt{s}}{2k_1}\xi\right)+C_2\sinh\left(\frac{\sqrt{s}}{2k_1}\xi\right)}\right\}^2\right],$$

in which $\xi=x-\frac{\sqrt{A_2(2\,\alpha^2A_2{k_1}^2l_1-12\,{k_1}^2l_2\epsilon_1(\epsilon_1-2)(A_0-1)-3\,l_2\left(4\,A_0{k_1}^2-A_2{\epsilon_3}^2-4\,{k_1}^2\right))}}{k_1\alpha A_2}t$. The existence of the solution for the constraint condition is as $|\varepsilon_3|>\frac{\alpha^2|k_1|}{|\varepsilon_1-1|}\sqrt{\frac{A_0A_2\Omega}{3l_2}}$.

Based on the Family 2, the exact periodic solution can be written as

$$u_2(x,t)=-\frac{1}{\alpha}\ln\left[A_0+\frac{12\varepsilon_3l_2(\varepsilon_1-1)^3}{k_1\Omega\alpha^4A_2}\left\{\frac{k_2}{2f}+\frac{\sqrt{-s}}{2f}\frac{-C_1\sin\left(\frac{\sqrt{-s}}{2k_1}\xi\right)+C_2\cos\left(\frac{\sqrt{-s}}{2k_1}\xi\right)}{C_1\cos\left(\frac{\sqrt{-s}}{2k_1}\xi\right)+C_2\sin\left(\frac{\sqrt{-s}}{2k_1}\xi\right)}\right\}+\right.\tag{6.9}$$

$$\left.A_2\left\{\frac{k_2}{2f}+\frac{\sqrt{-s}}{2f}\frac{-C_1\sin\left(\frac{\sqrt{-s}}{2k_1}\xi\right)+C_2\cos\left(\frac{\sqrt{-s}}{2k_1}\xi\right)}{C_1\cos\left(\frac{\sqrt{-s}}{2k_1}\xi\right)+C_2\sin\left(\frac{\sqrt{-s}}{2k_1}\xi\right)}\right\}^2\right],$$

in which $\xi=x-\frac{\sqrt{A_2(2\,\alpha^2A_2{k_1}^2l_1-12\,{k_1}^2l_2\epsilon_1(\epsilon_1-2)(A_0-1)-3\,l_2\left(4\,A_0{k_1}^2-A_2{\epsilon_3}^2-4\,{k_1}^2\right))}}{k_1\alpha A_2}t$. The existence of the solution for the constraint condition is as $|\varepsilon_3|<\frac{\alpha^2|k_1|}{|\varepsilon_1-1|}\sqrt{\frac{A_0A_2\Omega}{3l_2}}$.

Based on the Family 3, the exact singular solution can be written as

$$u_3(x,t)=-\frac{1}{\alpha}\ln\left[A_0+\frac{(2\,\Omega^2\alpha^4{A_0}^4{k_1}^4+3\,\Omega\,{A_0}^2l_2-6\,\Omega\,A_0l_2+3\,Rl_2)\varepsilon_3(\varepsilon_1-1)^3}{2\Omega^2\,{A_0}^2k_1\alpha^4\,(\epsilon_2-k_1)^2\,(A_0-1)}\left\{\frac{k_2}{2f}+\frac{C_2}{C_1+C_2\xi}\right\}+\right.\tag{6.10}$$

$$\left.\frac{24\Omega\,{A_0}^2l_2\,(\epsilon_2-k_1)^2\,(A_0-1)}{4\,\Omega^2\alpha^4{A_0}^4{k_1}^4+3\,\Omega\,{A_0}^2l_2-6\,\Omega\,A_0l_2+3\,Rl_2}\left\{\frac{k_2}{2f}+\frac{C_2}{C_1+C_2\xi}\right\}^2\right],$$

in which $\xi=x-\frac{\sqrt{A_2(2\,\alpha^2A_2{k_1}^2l_1-12\,{k_1}^2l_2\epsilon_1(\epsilon_1-2)(A_0-1)-3\,l_2\left(4\,A_0{k_1}^2-A_2{\epsilon_3}^2-4\,{k_1}^2\right))}}{k_1\alpha A_2}t$ and $A_2=\frac{24\Omega\,{A_0}^2l_2(\epsilon_2-k_1)^2(A_0-1)}{4\,\Omega^2\alpha^4{A_0}^4{k_1}^4+3\,\Omega\,{A_0}^2l_2-6\,\Omega\,A_0l_2+3\,Rl_2}$.

Based on the Family 6, the exact kink solution can be written as

$$u_4(x,t) = -\frac{1}{\alpha}\ln\left[\frac{12\varepsilon_3 l_2(\varepsilon_1-1)^3}{k_1\Omega\alpha^4 A_2}\left\{\frac{C_1k_2^2\exp\left(\frac{-k_2}{k_1}\xi\right)}{fk_1+C_1k_1k_2\exp\left(\frac{-k_2}{k_1}\xi\right)}\right\}+A_2\left\{\frac{C_1k_2^2\exp\left(\frac{-k_2}{k_1}\xi\right)}{fk_1+C_1k_1k_2\exp\left(\frac{-k_2}{k_1}\xi\right)}\right\}^2\right], \tag{6.11}$$

in which $\xi = x - \frac{\sqrt{A_2(2\alpha^2A_2{k_1}^2l_1+12{k_1}^2l_2\epsilon_1(\epsilon_1-2)-3l_2(-A_2{\epsilon_3}^2-4{k_1}^2))}}{k_1\alpha A_2}t.$

7. Basic idea of the Exp–function method

We first consider the nonlinear equation of form

$$\mathcal{N}(\mathrm{u},\mathrm{u_t},\mathrm{u_x},\mathrm{u_{xx}},\mathrm{u_{tt}},\mathrm{u_{tx}},...) = 0, \tag{7.1}$$

and introduce a transformation as

$$\mathrm{u(x,t)} = \mathrm{u}(\eta), \quad \xi = \mathrm{x} - \beta\mathrm{t}, \tag{7.2}$$

where β is constant to be determined later. Therefore the Eq. (7.1) is reduced to an ODE as follows

$$\mathcal{M}(\mathrm{u},-\beta\mathrm{u}',\mathrm{u}',\mathrm{u}'',...) = 0. \tag{7.3}$$

The EFM is based on the assumption that the travelling wave solutions can be expressed in the form

$$\mathrm{u}(\xi) = \frac{\sum_{\mathrm{n=-c}}^{\mathrm{d}}\mathrm{a_n}\exp(\mathrm{n}\xi)}{\sum_{\mathrm{m=-p}}^{\mathrm{q}}\mathrm{b_m}\exp(\mathrm{m}\xi)}, \tag{7.4}$$

where c, d, p and q are positive integers which could be freely chosen, $\mathrm{a_n}$'s and $\mathrm{b_m}$'s are unknown constants to be determined.

7.1. Application of EFM for Eq. (2.4)

We apply the Exp-function method to Eq. (2.9). In order to determine values of c and p, we balance the terms $(v')^4$ and v^6 in Eq. (2.9) along with Eq. (7.4), then we get

$$(v')^4 = \frac{c_1\exp(4(c+p)\xi)+...}{c_2\exp(8p\xi)+...}, \qquad v^6 = \frac{c_3\exp((6c+2p)\xi)+...}{c_4\exp(8p\xi)+...}, \tag{7.5}$$

respectively. Balancing highest order of the Exp–function in (7.5) and get $4c+4p = 6c+2p$, which leads to the result $c = p$. Similarly, to find values of d and q, for the terms $(v')^4$ and v^6 in Eq. (2.9) by simple calculation, we attain

$$(v')^4 = \frac{d_1\exp(-4(d+q)\xi)+...}{d_2\exp(-8q\xi)+...}, \qquad v^6 = \frac{d_3\exp(-(6d+2q)\xi)+...}{d_4\exp(-8q\xi)+...}, \tag{7.6}$$

respectively. Balancing lowest order of the Exp–function in (7.6), we achieve $d = q$.

Case I: $p = c = 1$ **and** $q = d = 1$.

For simplicity, we set $a_{-1} = 0, b_1 = 1$, $p = c = 1$ and $d = q = 1$. Then Eq. (7.4) reduces to

$$v(\xi) = \frac{a_1\exp(\xi)+a_0}{\exp(\xi)+b_0+b_{-1}\exp(-\xi)}. \tag{7.7}$$

Substituting (7.7) into Eq. (2.9), we get an equation in the following form

$$\left([b_{-1}\exp(-\xi)+b_0+\exp(\xi)]^4\right)^{-1}\sum_{n=-4}^{4}C_n\exp(n\xi)=0, \tag{7.8}$$

where $C_n(-4\le n\le 4)$ are polynomial expressions in terms of $a_1, a_0, a_{-1}, b_0, b_{-1}$ and β. Thus, solving the resulting system $C_n = 0(-4\le n\le 4)$ simultaneously, we acquire the following set as
(I) The first set is:

$$a_1=0,\ a_0=\frac{b_0(4\alpha^2R+\beta^2-l_1)}{2\alpha^2\Omega},\ b_0=b_0,\ b_{-1}=\frac{1}{4}b_0^2,\ R=\frac{3l_2+2\alpha^2(l_1-\beta^2)}{4\alpha^4},\ \beta=\pm\frac{1}{\alpha}\sqrt{3l_2+\alpha^2(l_1\pm 2\sqrt{3l_2\Omega})}, \tag{7.9}$$

$$v(\xi)=\frac{\frac{2b_0(4\alpha^2R+\beta^2-l_1)}{\alpha^2\Omega}}{(2e^{\xi/2}+b_0e^{-\xi/2})^2},\quad \xi=x\mp\frac{1}{\alpha}\sqrt{3l_2+\alpha^2(l_1\pm 2\sqrt{3l_2\Omega})}t, \tag{7.10}$$

then the solution equation (2.4) will be as

$$u_1(x,t)=-\frac{1}{\alpha}\ln\left[\frac{\frac{2b_0(4\alpha^2R+\beta^2-l_1)}{\alpha^2\Omega}}{\left(2e^{\frac{1}{2}\left[x\mp\frac{1}{\alpha}\sqrt{3l_2+\alpha^2(l_1\pm 2\sqrt{3l_2\Omega})t}\right]}+b_0e^{-\frac{1}{2}\left[x\mp\frac{1}{\alpha}\sqrt{3l_2+\alpha^2(l_1\pm 2\sqrt{3l_2\Omega})t}\right]}\right)^2}\right]. \tag{7.11}$$

If we choose $b_0 = 2$ and $b_0 = -2$, then the solution equation (7.11), respectively, give:

$$u_2(x,t)=-\frac{1}{\alpha}\ln\left[\frac{b_0(4\alpha^2R+\beta^2-l_1)}{8\alpha^2\Omega}sech^2\left(\frac{x}{2}\mp\frac{1}{2\alpha}\sqrt{3l_2+\alpha^2(l_1\pm 2\sqrt{3l_2\Omega})t}\right)\right], \tag{7.12}$$

$$u_3(x,t)=-\frac{1}{\alpha}\ln\left[\frac{b_0(4\alpha^2R+\beta^2-l_1)}{8\alpha^2\Omega}csch^2\left(\frac{x}{2}\mp\frac{1}{2\alpha}\sqrt{3l_2+\alpha^2(l_1\pm 2\sqrt{3l_2\Omega})t}\right)\right], \tag{7.13}$$

(II) The second set is:

$$a_1=\frac{\Omega\pm\sqrt{\Omega^2-\Omega R}}{\Omega},\quad a_0=\frac{b_0(a_1(R-2\Omega)+R)}{R-a_1\Omega},\quad b_0=b_0,\quad b_{-1}=0,\quad R=R,\quad \beta=\beta, \tag{7.14}$$

$$v(\xi)=\frac{a_0+a_1e^{\xi}}{b_0+e^{\xi}},\quad \xi=x-\beta t, \tag{7.15}$$

then the solution equation (2.4) will be as

$$u_4(x,t)=-\frac{1}{\alpha}\ln\left[\frac{\frac{b_0(a_1(R-2\Omega)+R)}{R-a_1\Omega}+\frac{\Omega\pm\sqrt{\Omega^2-\Omega R}}{\Omega}e^{x-\beta t}}{b_0+e^{x-\beta t}}\right]. \tag{7.16}$$

Case II: p = c = 2 **and** q = d = 2.
Since the values of c and d can be freely chosen, we set p = c = 2 and d = q = 2 and then the trial function (7.4) becomes

$$u(\xi)=\frac{a_2\exp(2\xi)+a_1\exp(\xi)+a_0+a_{-1}\exp(-\xi)+a_{-2}\exp(-2\xi)}{b_2\exp(2\xi)+b_1\exp(\xi)+b_0+b_{-1}\exp(-\xi)+b_{-2}\exp(-2\xi)}. \tag{7.17}$$

There are some free parameters in (7.17), we set $b_2 = 1$ and $a_1 = a_{-1} = a_{-2} = b_1 = b_{-1} = 0$ for simplicity, the trial function, (7.17) is simplified as follows

$$u(\xi)=\frac{a_0+a_2\exp(2\xi)}{\exp(2\xi)+b_0+b_{-2}\exp(-2\xi)}. \tag{7.18}$$

Substituting (7.18) into Eq. (2.9), we get an equation in the following form

$$\left([b_{-2}\exp(-2\xi)+b_0+\exp(2\xi)]^8\right)^{-1}\sum_{\frac{n}{2}=-4}^{8}C_n\exp(n\xi)=0, \tag{7.19}$$

where $C_n(-8\le n\le 16)$ are polynomial expressions in terms of $a_2, a_0, a_{-1}, b_0, b_{-2}$ and β. Thus, solving the resulting system $C_n = 0(-8 \le n \le 16)$ simultaneously, we obtain the following set of algebraic equations

(I) The first set is:

$$a_2=0,\ a_0=\frac{2b_0(\alpha^2R+\beta^2-l_1)}{\alpha^2\Omega},\ b_0=b_0,\ b_{-1}=\frac{1}{4}b_0^2,\ R=\frac{6l_2+\alpha^2(l_1-\beta^2)}{\alpha^4},\ \beta=\pm\frac{1}{\alpha^3\Omega}\sqrt{\Omega(36l_2^2+\alpha^4\Omega(\alpha^2l_1-6l_2))}, \tag{7.20}$$

$$v(\xi)=\frac{\frac{8b_0(\alpha^2R+\beta^2-l_1)}{\alpha^2\Omega}}{(2e^{\xi}+b_0e^{-\xi})^2},\quad \xi=x\mp\frac{1}{\alpha^3\Omega}\sqrt{\Omega(36l_2^2+\alpha^4\Omega(\alpha^2l_1-6l_2))}t, \tag{7.21}$$

then the solution equation (2.4) will be as

$$u_5(x,t)=-\frac{1}{\alpha}\ln\left[\frac{\frac{8b_0(\alpha^2R+\beta^2-l_1)}{\alpha^2\Omega}}{\left(2e^{\left[x\mp\frac{1}{\alpha^3\Omega}\sqrt{\Omega(36l_2^2+\alpha^4\Omega(\alpha^2l_1-6l_2))}t\right]}+b_0e^{-\left[x\mp\frac{1}{\alpha^3\Omega}\sqrt{\Omega(36l_2^2+\alpha^4\Omega(\alpha^2l_1-6l_2))}t\right]}\right)^2}\right]. \tag{7.22}$$

If we choose $b_0 = 2$ and $b_0 = -2$, then the solution equation (7.22), respectively, give:

$$u_6(x,t)=-\frac{1}{\alpha}\ln\left[\frac{b_0(\alpha^2R+\beta^2-l_1)}{2\alpha^2\Omega}sech^2\left(x\mp\frac{1}{\alpha^3\Omega}\sqrt{\Omega(36l_2^2+\alpha^4\Omega(\alpha^2l_1-6l_2))}t\right)\right], \tag{7.23}$$

$$u_7(x,t)=-\frac{1}{\alpha}\ln\left[\frac{b_0(\alpha^2R+\beta^2-l_1)}{2\alpha^2\Omega}csch^2\left(x\mp\frac{1}{\alpha^3\Omega}\sqrt{\Omega(36l_2^2+\alpha^4\Omega(\alpha^2l_1-6l_2))}t\right)\right], \tag{7.24}$$

(II) The second set is:

$$a_2=\frac{\Omega\pm\sqrt{\Omega^2-\Omega R}}{\Omega},\qquad a_0=\frac{b_0(a_1(R-2\Omega)+R)}{R-a_1\Omega},\qquad b_0=b_0,\qquad b_{-2}=0,\qquad R=R,\qquad \beta=\beta, \tag{7.25}$$

$$v(\xi)=\frac{a_0+a_2e^{2\xi}}{b_0+e^{2\xi}},\qquad \xi=x-\beta t, \tag{7.26}$$

then the solution equation (2.4) will be as

$$u_8(x,t)=-\frac{1}{\alpha}\ln\left[\frac{\frac{b_0(a_2(R-2\Omega)+R)}{R-a_2\Omega}+\frac{\Omega\pm\sqrt{\Omega^2-\Omega R}}{\Omega}e^{2x-2\beta t}}{b_0+e^{2x-2\beta t}}\right]. \tag{7.27}$$

8. Discussion and remark

This paper finds many novel hyperbolic, trigonometric, kink, and kink-singular soliton solutions to governing model. With the help of some calculations, surfaces of results reported have been observed in Figures 1–5. These figures are depended on the family conditions which are of important physically. It has been investigated that all figures plotted have symbolized the nonlinear DNA

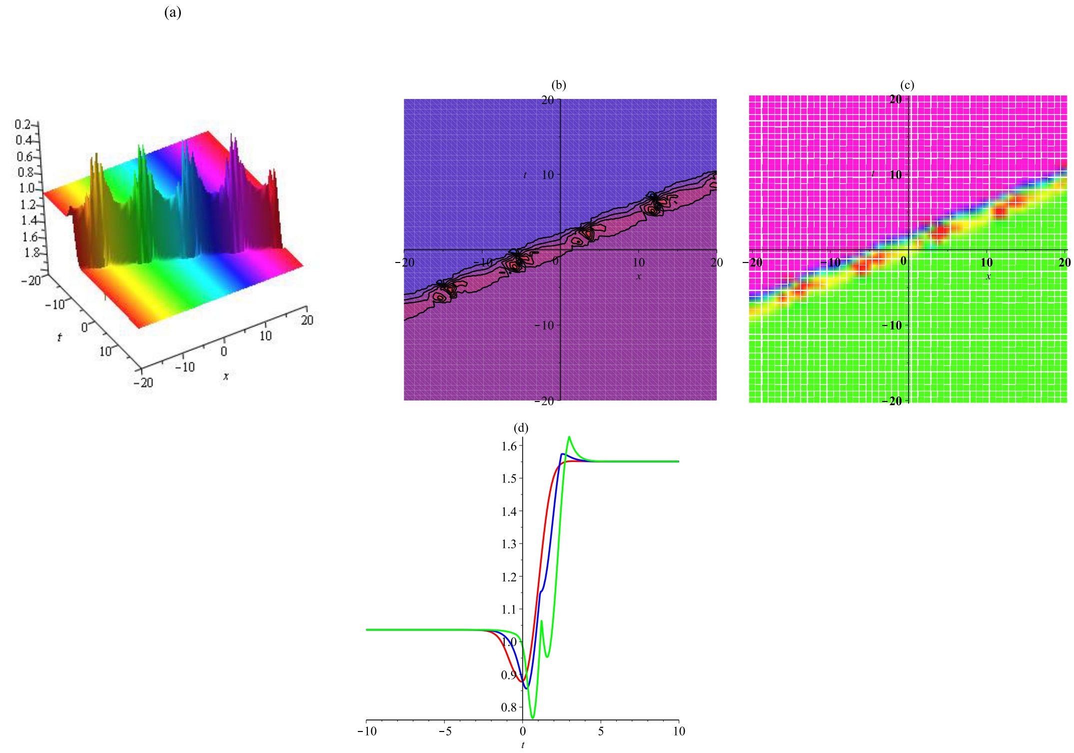

Figure 1. **Plot of DNA dynamics (5.8) by taking $l_1 = 1, l_2 = -1, \alpha = 3, \Omega = 1, R = -1, \mu = 1.5$ and (a) 3D plot, (b) density plot, (c) contour plot and (d) 2D plot with at space (a) red $x = -1$, blue $x = 0$, and green $x = 1$.**

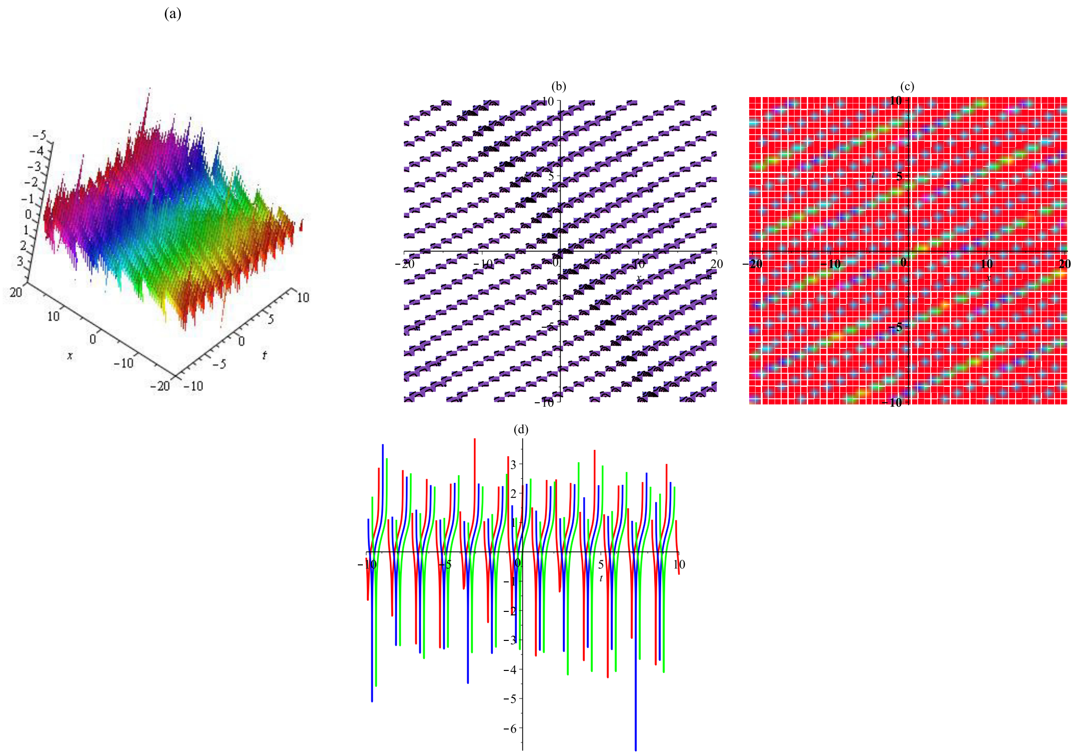

Figure 2. **Plot of DNA dynamics (5.9) by taking $l_1 = 1, l_2 = 2, \alpha = 3, \Omega = 10, R = 5, \mu = 1.5$ and (a) 3D plot, (b) density plot, (c) contour plot and (d) 2D plot with at space (a) red $x = -1$, blue $x = 0$, and green $x = 1$.**

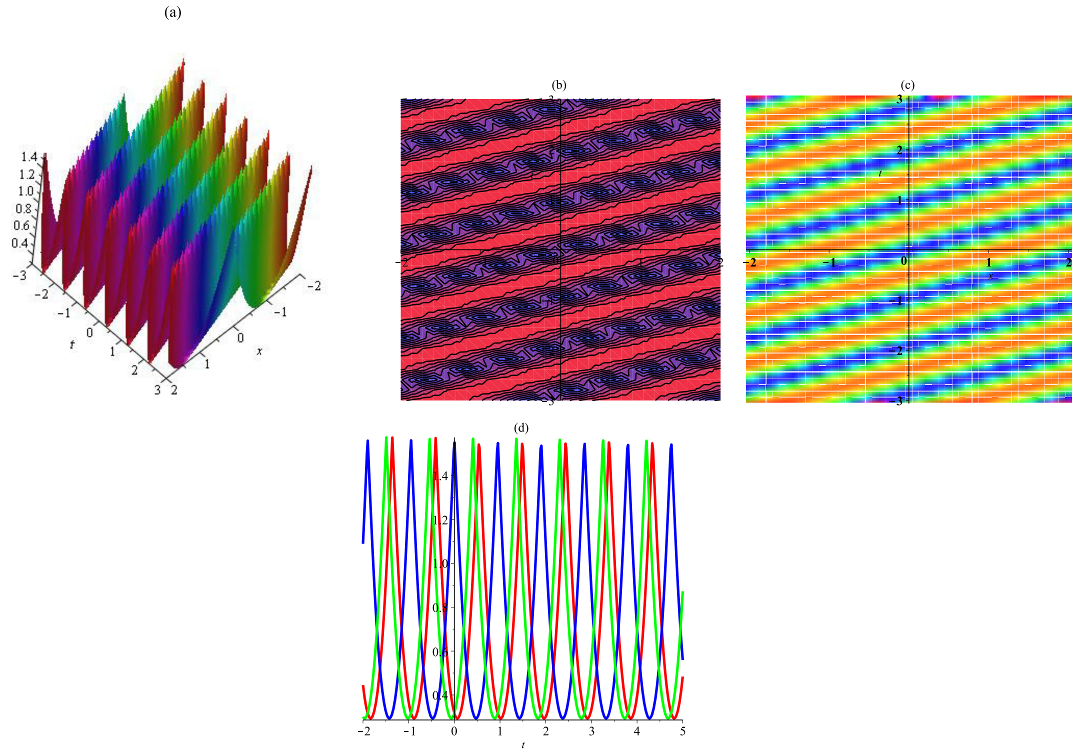

Figure 3. **Plot of DNA dynamics (6.9) by taking** $A_0 = 1, A_2 = 2, k_1 = 2, l_1 = 3, l_2 = 2, \alpha = 2, \Omega = 5, C_1 = 2, C_2 = 3, R = 5$ **and (a) 3D plot, (b) density plot, (c) contour plot and (d) 2D plot with at space (a) red** $x = -1$**, blue** $x = 0$**, and green** $x = 1$**.**

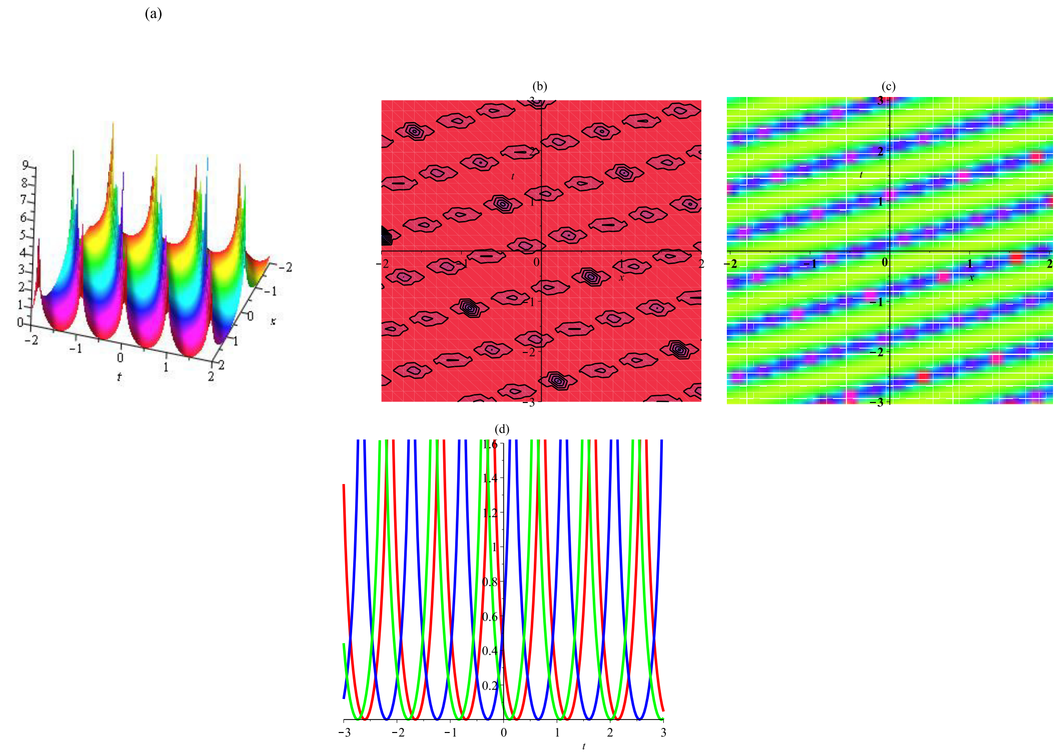

Figure 4. **Plot of DNA dynamics (6.10) by taking** $A_0 = 1, A_2 = 1, k_1 = 2, l_1 = 3, l_2 = 2, \alpha = 2, \Omega = 5, C_1 = 2, C_2 = 3, R = 5$**, and (a) 3D plot, (b) density plot, (c) contour plot and (d) 2D plot with at space (a) red** $x = -1$**, blue** $x = 0$**, and green** $x = 1$**.**

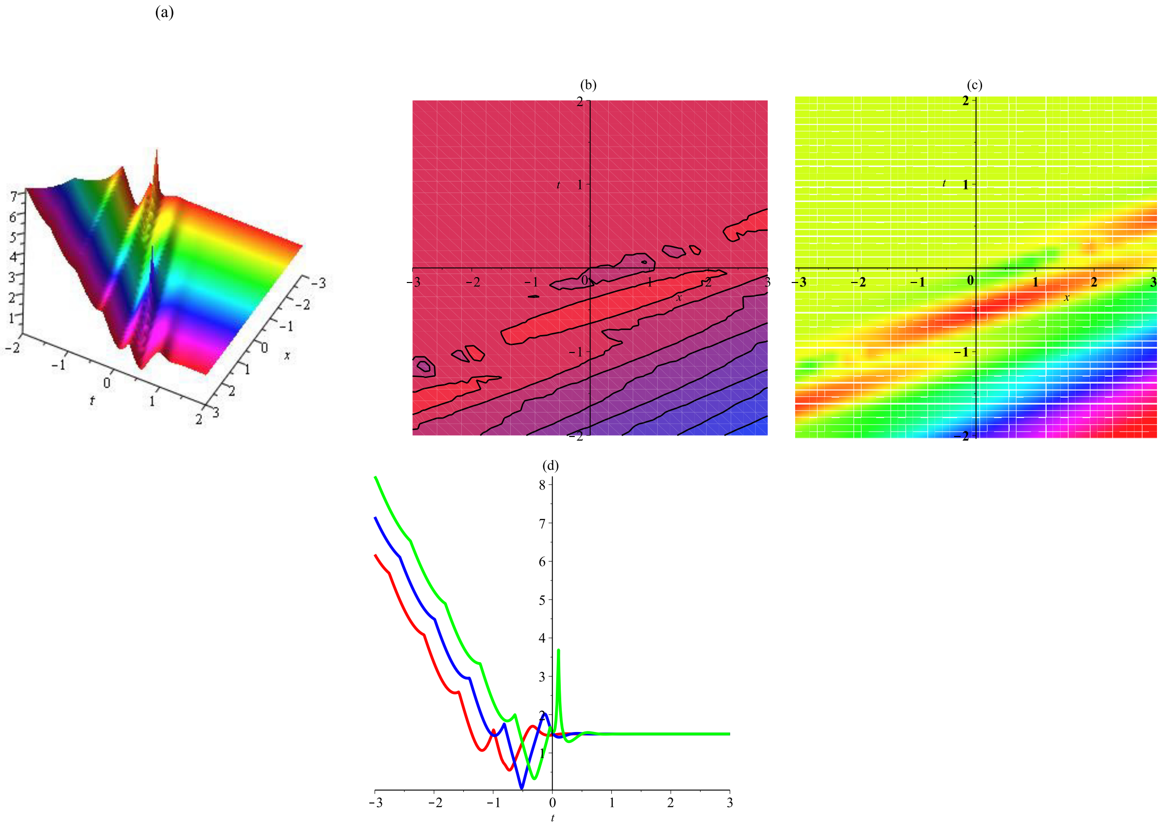

Figure 5. **Plot of DNA dynamics (6.11) by taking $A_0 = 0, A_2 = 1, k_1 = 2, l_1 = 3, l_2 = 2, \alpha = 2, \Omega = 5, R = 5$, and (a) 3D plot, (b) density plot, (c) contour plot and (d) 2D plot with at space (a) red $x = -1$, blue $x = 0$, and green $x = 1$.**

dynamics. These mathematical properties come from trigonometric and hyperbolic function properties. In this sense, from the mathematical and physical points of views, these results take play an important role in explaining waves propagation in nonlinear dispersion. Hence, we consider surfaces plotted in this paper have proved such physical meaning of the solutions. In Figure 1 and Figure 2, we have depicted the 3D, 2D, contour, and density schematic representation of the analytical and numerical solutions at few space positions for three different waves at $x = -1$, $x = 0$, and $x = 1$ by taking $l_1 = 1, l_2 = -1, \alpha = 3, \Omega = 1, R = -1, \mu = 1.5$ for (5.8) and $l_1 = 1, l_2 = 2, \alpha = 3, \Omega = 10, R = 5, \mu = 1.5$ for (5.9). We observe that the breathe soliton wave move in direction (x, t) and increases with move of negative (x, t) to positive (x, t). Also, the periodic wave solution for (6.9) by taking $A_0 = 1, A_2 = 2, k_1 = 2, l_1 = 3, l_2 = 2, \alpha = 2, \Omega = 5, C_1 = 2, C_2 = 3, R = 5$ is presented in Figure 3. Moreover, the rational kink wave solution for the DNA dynamics (6.10) by taking $A_0 = 1, A_2 = 1, k_1 = 2, l_1 = 3, l_2 = 2, \alpha = 2, \Omega = 5, C_1 = 2, C_2 = 3, R = 5$, is offered in Figure 4. Likewise, the DNA dynamics for (6.11) by taking $A_0 = 0, A_2 = 1, k_1 = 2, l_1 = 3, l_2 = 2, \alpha = 2, \Omega = 5, R = 5$, along with 3D plot, density plot, contour plot, and 2D plot with at spaces at $x = -1$, $x = 0$, and $x = 1$ are plotted in Figure 5.

9. Conclusion

The article obtains, the traveling wave solutions of different kinds, which are solitary, topological, singular, periodic and rational solutions to the model for DNA dynamics. The integration mechanisms that are adopted, are improved $\tan(\phi/2)$-expansion scheme, $\exp(-\Omega(\eta))$-expansion scheme, improved $\exp(-\Omega(\eta))$-expansion scheme, generalized (G'/G)-expansion scheme, and exp-function scheme. It is quite visible that these integration schemes has its limitations. Thus, this paper are provides a lot of encouragement for future research in DNA dynamics. Afterwards extra solution methods will be applied to obtain lump and singular soliton solutions to the nonlinear model. In addition to, this model will be considered with other forms of nonlinear media. The constructed results may be helpful in explaining the physical meaning of the studied models and other related nonlinear phenomena models. Results are beneficial to the study of the nonlinear DNA dynamics. All calculations in this paper have been made quickly with the aid of the Maple.

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Conflict of interest

The authors have declared no conflict of interest.

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