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Research article

Subclass of Bazilevič functions of complex order

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Abstract: In this paper, we define a class of analytic functions by using the concept of complex order. This class of analytic functions generalizes the class of Bazilevič functions. In the present work, we derive various useful properties and characteristics of this class such as coefficient bounds, Fekete-Szegö type inequality, arclength, integral preserving property, radius problem and some other interesting properties. Relevant connections of the results presented here with those obtained in earlier works are pointed.

Keywords: Bazilevič functions; complex order
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1. Introduction and preliminaries

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$. If $f(z)$ and $g(z)$ are analytic in E, we say $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$. If there exists a Schwarz function $w(z)$, $w(0) = 0$ and $|w(z)| < 1$ in E then $f(z) = g(w(z))$. Let $P(b)$, $b \neq 0$ (complex) denote the class of analytic functions

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{1.2}$$

such that $1 + \frac{1}{b}\{p(z) - 1\} \in P$, where P is the well-known class of analytic functions with positive real part. The class $P(b)$ is defined by Nasr and Aouf [15].

Let $S^*(\gamma)$, $0 \le \gamma < 1$ is the class of starlike univalent functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.3}$$

of order γ such that $Re\frac{zg'(z)}{g(z)} > \gamma,\ z \in E$. This class was introduced by Robertson, for details, see [8].

The class of Bazilevič functions in the open unit disc E was first introduced by Bazilevič [3] in 1955. He defined Bazilevič functions by the relation

$$f(z) = \left\{ (\alpha + i\beta) \int_0^z g^{\alpha}(t)\, p(t)\, t^{i\beta-1} dt \right\}^{\frac{1}{\alpha+i\beta}}, \tag{1.4}$$

where $p \in P,\ g \in S^*,\ \alpha > 0$ and β is any real. The class of Bazilevič function is the largest family of univalent function. Bazilevič showed that the class of Bazilevič function is univalent in E. Except this, a very little is known regarding the family as a whole. Indeed, it is easy to verify that, with special choices of the parameters and and the function $g(z)$, the class of Bazilevič functions reduces to some well-known subclasses of univalent functions. By choosing $g(z) = z$ and $\beta = 0$, Singh [21] studied the class $B_1(\alpha)$ of Bazilevič functions. Recently, some authors have find coefficient bounds for this class of functions. In particular, Cho et al. [6] have studied coefficient difference for the class of Bazilevič functions. Fifth and sixth coefficient bounds for the subclass $B_1(\alpha)$ have been found by Cho and Kumar [5], and Marjono et al. [14]. For some more work, see [1, 7, 9, 11, 12, 19, 20]. In 1979 Campbell and Pearce [4] generalized the class of Bazilevič functions by means of differential equation

$$1 + \frac{zf''(z)}{f'(z)} + (\alpha + i\beta - 1)\frac{zf'(z)}{f(z)} = \alpha\frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)} + i\beta. \tag{1.5}$$

They associate each generalized Bazilevič function f with the quadruple (α, β, g, p), where $g \in S^*$ and $p \in P,\ \alpha > 0$ and β any real.

Definition 1.1. Let $S^*(\gamma)$ be the class of functions g of the form (1.3) and let $P(b)$, $b \neq 0$(complex) be the class of normalized functions p defined by (1.2). Then a function f of the form (1.1), analytic in E, belongs to the generalized Bazilevič functions associated with the quadruple (α, β, g, p) if and only if f satisfies (1.5) or

$$\frac{zf'(z)}{f(z)} = \left(\frac{g(z)}{z}\right)^{\alpha} \left(\frac{z}{f(z)}\right)^{\alpha+i\beta} p(z),\quad z \in E.$$

The above differential equation can be written as

$$\frac{z^{1-i\beta} f'(z)}{f^{1-(\alpha+i\beta)}(z)\, g^{\alpha}(z)} = p(z). \tag{1.6}$$

Since $p \in P(b)$, therefore we can write

$$1 + \frac{1}{b}\left\{ \frac{z^{1-i\beta} f'(z)}{f^{1-(\alpha+i\beta)}(z)\, g^{\alpha}(z)} - 1 \right\} \in P,$$

where $g \in S^*(\gamma)$, $0 \le \gamma < 1$, $\alpha > 0$ and β is any real.

We have the following special cases.
(i) For $\gamma = 0$, we have the class of Bazilevič functions of complex order, defined by Noor [16].
(ii) For $\gamma = 0,\ b = 1$, we obtain the generalized Bazilevič functions defined in [4].

In this paper, we study the class of functions (α, β, g, p). We study coefficient bounds, inclusion result, arc length problem and radii problems. Our results generalize some previously proven results.

We need the following lemmas which will be used in our main results.

Lemma 1.1. *[8] Let $g \in S^*(\gamma)$, $0 \le \gamma < 1$. Then*
(*i*) $|b_n| \le \frac{1}{(n-1)!} \prod\limits_{k=2}^{n} (k - 2\gamma)$.
(*ii*) $\frac{r}{(1+r)^{2(1-\gamma)}} \le |g(z)| \le \frac{r}{(1-r)^{2(1-\gamma)}}$, $z = re^{i\theta}$.
These inequalities are sharp for the function $g_0(z) = \frac{z}{(1-z)^{2(1-\gamma)}}$.

Lemma 1.2. *Let $p \in P(b)$. Then, for $z = re^{i\theta}$*
(*i*) $\frac{1}{2\pi} \int\limits_0^{2\pi} \left| p\left(re^{i\theta}\right)\right|^2 d\theta \le \frac{1+\left(4|b|^2-1\right)r^2}{1-r^2}$, *see [18],*
(*ii*) $\frac{1-2|b|r+(2Re\ b-1)r^2}{1-r^2} \le |p(z)| \le \frac{1+2|b|r+(2Re\ b-1)r^2}{1-r^2}$.
This result is the special case of the one, proved in [2].

Now we introduce the Hypergeometric function. Let a_1, b_1 and c_1 be complex numbers with $c_1 \ne 0, -1, -2, \cdots$. The function

$$_2F_1(a_1, b_1, c; z) = 1 + \frac{a_1 b_1}{c}\frac{z}{1!} + \frac{a_1(a_1+1)\,b_1(b_1+1)}{c_1(c_1+1)}\frac{z^2}{2!} + \dots.$$

called, the confluent Gaussian hypergeometric, is analytic in E and satisfies hypergeometric differential equation

$$z(1-z)\,w''(z) + [c_1 - (a_1 + b_1 + 1)\,z]\,w'(z) - a_1 b_1 w(z) = 0.$$

This can be written as

$$_2F_1(a_1, b_1, c_1; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k\,(b_1)_k}{(c_1)_k}\frac{z^k}{k!}.$$

Some properties of Gaussian hypergeometric are given in the following lemma.

Lemma 1.3. *[22] Let a_1, b_1 and $c_1 \ne 0, -1, -2 \cdots$ be complex numbers. Then, for $Re\ c_1 > Re\ b_1 > 0$*

$$\begin{aligned}
(i)\quad {}_2F_1(a_1, b_1, c_1; z) &= \frac{\Gamma(c_1)}{\Gamma(c_1 - b_1)\,\Gamma(b_1)} \int_0^1 t^{b_1-1}(1-t)^{c_1-b_1-1}(1-tz)^{-a_1}\,dt,\\
(ii)\quad {}_2F_1(a_1, b_1, c_1; z) &= {}_2F_1(b_1, a_1, c_1; z),\\
(iii)\quad {}_2F_1(a_1, b_1, c_1; z) &= (1-z)^{-a_1}\ {}_2F_1\left(a_1, c_1 - b_1, c_1; \frac{z}{z-1}\right).
\end{aligned}$$

Lemma 1.4. *[10, inequality 7, p.10] Let Ω be the class of analytic functions w, normalized by $w(0) = 0$, satisfying the condition $|w(z)| < 1$. If $w \in \Omega$ and $w(z) = w_1 z + w_2 z^2 + \cdots$, $z \in E$, then*

$$\left|w_2 - t w_1^2\right| \le \max\{1; |t|\},$$

for any complex number t. The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

Lemma 1.5. *Let* $p \in P(b)$, $b \neq 0$ *(complex) and of the form* (1.2). *Then for* μ *a complex number*

$$\left|p_2 - \mu p_1^2\right| \leq 2|b| \max\{1; |2\mu b - 1|\}.$$

This result is sharp.

Proof. Since $p \in P(b)$, therefore we can write

$$p(z) \prec \frac{1 + (2b-1)z}{1-z} = 1 + 2bz + 2bz^2 + \cdots.$$

Thus

$$1 + \sum_{n=1}^{\infty} p_n z^n = 1 + 2b\left(w_1 z + w_2 z^2 + \ldots\right) + 2b\left(w_1 z + w_2 z^2 + \cdots\right)^2 + \cdots.$$

Comparing the coefficients of z and z^2, we have

$$\begin{aligned} p_1 &= 2bw_1 \\ p_2 &= 2bw_2 + 2bw_1^2 \end{aligned}$$

$$\left|p_2 - \mu p_1^2\right| = 2|b|\left|w_2 - (2\mu b - 1) w_1^2\right|.$$

Now using Lemma 1.4 we have the required result. This result is sharp for the functions

$$p_0(z) = \frac{1 + (2b-1)z}{1-z} = 1 + 2bz + 2bz^2 + \cdots, \tag{1.7}$$

or

$$p_1(z) = \frac{1 + (2b-1)z^2}{1-z^2} = 1 + 2bz^2 + 2bz^4 + \cdots. \tag{1.8}$$

□

Lemma 1.6. *Let* $g \in S^*(\gamma)$, $0 \leq \gamma < 1$ *and of the form* (1.3). *Then for* μ *complex*

$$\left|b_3 - \mu b_2^2\right| \leq (1-\gamma) \max\{1; |2(1-\gamma)(2\mu - 1) - 1|\}.$$

This result is best possible.

This result is a special case of the one, proved in [10].

Lemma 1.7. *[16] If N and D are analytic in E,* $N(0) = D(0) = 0$, *D maps E onto a many sheeted region which is starlike with respect to the origin, then*

$$\frac{N'(z)}{D'(z)} \in P(b) \text{ implies } \frac{N(z)}{D(z)} \in P(b).$$

Lemma 1.8. *[13, p. 109] Let* $\beta_1, \gamma_1, A \in \mathbb{C}$, *with* $Re[\beta_1 + \gamma_1] > 0$, *and let* $B \in [-1, 0]$ *satisfy*

$$Re\left[\beta_1(1 + AB) + \gamma_1\left(1 + B^2\right)\right] \geq \left|\beta_1 A + \bar{\beta_1} B + B(\gamma_1 + \bar{\gamma_1})\right|, \quad B \in (-1, 0],$$

or

$$Re\beta_1(1 + A) > 0 \text{ and } Re[\beta_1(1 - A) + 2\gamma_1] \geq 0, \quad B = -1.$$

If $h(z)=1+\sum_{n=1}^{\infty}c_nz^n$ *satisfies*

$$h(z)+\frac{zh'(z)}{\beta_1h(z)+\gamma_1}\prec\frac{1+Az}{1+Bz},$$

then $h(z)\prec q(z)\prec\frac{1+Az}{1+Bz}$, *where* $q(z)$ *is the best dominant and*

$$q(z)=\frac{1}{\beta_1}\left\{\frac{1}{g(z)}-\gamma_1\right\},$$

with

$$g(z)=\int_0^1\left[\frac{1+Btz}{1+Bz}\right]^{\beta_1\left(\frac{A}{B}-1\right)}t^{\beta_1+\gamma_1-1}dt,\quad B\neq 0.$$

Lemma 1.9. *Let* $g\in S^*(\gamma),\ 0\leq\gamma<1$. *Then*

$$G^{\alpha}(z)=\frac{\alpha+i\beta+c}{z^{c+i\beta}}\int_0^z t^{c+i\beta-1}g^{\alpha}(t)\,dt, \tag{1.9}$$

$c>0,\ \alpha>0$ *and* β *any real, belongs to* $S^*(\delta)$, *where* $\delta=\min_{|z|=1}Re\,q(z)$ *and*

$$q(z)=\frac{1}{\alpha}\left\{\frac{\alpha+i\beta+c}{{}_2F_1\left(1,2\alpha(1-\gamma),\alpha+i\beta+c+1;\frac{z}{z-1}\right)}-(c+i\beta)\right\}.$$

Proof. From (1.9), we have

$$(\alpha+i\beta+c)\left[\frac{g(z)}{G(z)}\right]^{\alpha}=\alpha p(z)+(c+i\beta), \tag{1.10}$$

where $\frac{zG'(z)}{G(z)}=p(z)$. Differentiating (1.10) logarithmically and using the fact that $g\in S^*(\gamma)$, we have

$$p(z)+\frac{zp'(z)}{\alpha p(z)+(c+i\beta)}\prec\frac{1+(1-2\gamma)z}{1-z}.$$

Now using the Lemma 1.8 for $A=1-2\gamma,\ B=-1,\ \beta_1=\alpha,\gamma_1=c+i\beta$ and then Lemma 1.3 we have the required result. □

Throughout the main results we assume that g belongs to the class of starlike functions of order γ and $p\in P(b)$ unless otherwise stated.

2. Main results

Theorem 2.1. *Let the generalized Bazilevič function* f *be represented by the quadruple* $(\alpha,0,g,p)$. *Then, for* $\alpha>0$

$$|f(z)|^{\alpha}\leq(Re\,b+|b|)\,r^{\alpha}\,{}_2F_1\left(2\alpha(1-\gamma)+1,\alpha,\alpha+1;r\right),$$

where ${}_2F_1$ *is the Gauss hypergeometric function.*

Proof. Since f is represented by $(\alpha, 0, g, h)$, therefore from (1.6), we have

$$\frac{zf'(z)}{f^{1-\alpha}(z)\, g^{\alpha}(z)} = p(z).$$

This implies that

$$f^{\alpha}(z) = \alpha \int_0^z t^{-1} g^{\alpha}(t)\, p(t)\, dt.$$

Thus

$$|f(z)|^{\alpha} \leq \alpha \int_0^r t^{-1} |g(t)|^{\alpha} |p(t)|\, dt.$$

Using the Lemma 1.1 (ii) and Lemma 1.2 (ii), we obtain

$$\begin{aligned} |f(z)|^{\alpha} &\leq \alpha \int_0^r t^{-1} \left(\frac{t}{(1-t)^{2(1-\gamma)}}\right)^{\alpha} \left(\frac{1 + 2|b|t + (2Re\, b - 1)t^2}{1-t^2}\right) dt \\ &\leq \alpha (Re\, b + |b|) \int_0^r t^{\alpha-1} (1-t)^{-\{2\alpha(1-\gamma)+1\}}\, dt. \end{aligned}$$

Now for $t = ru$ and using Lemma 1.3, we have

$$\begin{aligned} |f(z)|^{\alpha} &\leq \alpha (Re\, b + |b|)\, r^{\alpha} \int_0^1 u^{\alpha-1} (1-ru)^{-\{2\alpha(1-\gamma)+1\}}\, du \\ &= (Re\, b + |b|)\, r^{\alpha}\, {}_2F_1(2\alpha(1-\gamma)+1, \alpha, \alpha+1; r). \end{aligned}$$

Hence the proof is completed. □

Theorem 2.2. *Let f be generalized Bazilevič function represented by the quadruple $(\alpha, 0, g, p)$. Then*

$$L_r f(z) \leq \begin{cases} C(b)\, M^{1-\alpha}(r) \left(\frac{1}{1-r}\right)^{2\alpha(1-\gamma)}, & 0 < \alpha \leq 1, \\ C(b)\, m^{\alpha-1}(r) \left(\frac{1}{1-r}\right)^{2\alpha(1-\gamma)}, & \alpha > 1, \end{cases} \tag{2.1}$$

where $m(r) = \min\limits_{|z|=r} |f(z)|$, $M(r) = \max\limits_{|z|=r} |f(z)|$ and $C(b)$ is constant depending upon b only.

Proof. We know that

$$L_r f(z) = \int_0^{2\pi} |zf'(z)|\, d\theta, \quad z = re^{i\theta},\ 0 < r < 1,\ 0 \leq \theta \leq 2\pi.$$

Since f is generalized Bazilevič function represented by the quadruple $(\alpha, 0, g, h)$, therefore

$$zf'(z) = f^{1-\alpha}(z)\, g^{\alpha}(z)\, p(z).$$

This implies that

$$\begin{aligned} L_r f(z) &\leq \int_0^{2\pi} |f(z)|^{1-\alpha} |g(z)|^{\alpha} |p(z)| \, d\theta, \\ &\leq M^{1-\alpha}(r) \int_0^{2\pi} |g(z)|^{\alpha} |p(z)| \, d\theta. \end{aligned}$$

Now using Cauchy Schwarz inequality, we have

$$L_r f(z) \leq 2\pi M^{1-\alpha}(r) \left(\frac{1}{2\pi} \int_0^{2\pi} |g(z)|^{2\alpha} \, d\theta \right)^{\frac{1}{2}} \left(\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 \, d\theta \right)^{\frac{1}{2}}. \tag{2.2}$$

By Lemma 1.1(ii), Lemma 1.2(i) and a subordination result, we obtain

$$\begin{aligned} L_r f(z) &\leq 2\pi M^{1-\alpha}(r) \left(\frac{1}{1-r} \right)^{2\alpha(1-\gamma)-\frac{1}{2}} \left(\frac{1 + \left(4|b|^2 - 1\right) r^2}{1 - r^2} \right)^{\frac{1}{2}} \\ &\leq C(b) M^{1-\alpha}(r) \left(\frac{1}{1-r} \right)^{2\alpha(1-\gamma)}. \end{aligned}$$

Similarly for $\alpha > 1$, we have

$$L_r f(z) \leq C(b) m^{\alpha-1}(r) \left(\frac{1}{1-r} \right)^{2\alpha(1-\gamma)}.$$

□

For $\gamma = 1$, we have the following result, proved by Noor [16].

Corollary 2.3. *Let $g \in S^*$ and $p \in P(b)$. Then, for $0 < \alpha \leq 1$*

$$L_r f(z) \leq C(b) M^{1-\alpha}(r) \left(\frac{1}{1-r} \right)^{2\alpha}.$$

Theorem 2.4. *Let f be generalized Bazilevič function represented by the quadruple $(\alpha, 0, g, p)$. Then*

$$|a_n| \leq \begin{cases} C_1(b) M^{1-\alpha}(n) (n)^{2\alpha(1-\gamma)-1}, & 0 < \alpha \leq 1, \\ C_1(b) m^{\alpha-1}(n) (n)^{2\alpha(1-\gamma)-1}, & \alpha > 1, \end{cases}$$

where m, M are the same as in Theorem 2.2 and $C_1(b)$ is a constant depending upon b only.

Proof. Since with $z = re^{i\theta}$, Cauchy theorem gives

$$na_n = \frac{1}{2\pi r^n} \int_0^{2\pi} z f'(z) e^{-in\theta} d\theta.$$

Therefore

$$n\left|a_n\right| \leq \frac{1}{2\pi r^n} L_r f(z).$$

Now using Theorem 2.2 for $0 < \alpha \leq 1$, we have

$$n\left|a_n\right| \leq \frac{1}{2\pi r^n} C(b) M^{1-\alpha}(r) \left(\frac{1}{1-r}\right)^{2\alpha(1-\gamma)}.$$

Putting $r = 1 - \frac{1}{n}$, we have

$$\left|a_n\right| \leq C_1(b) M^{1-\alpha}(n)(n)^{2\alpha(1-\gamma)-1}.$$

Similarly we have for $\alpha > 1$. □

For $\gamma = 1$, we have the following result, proved by Noor [16].

Corollary 2.5. *Let $g \in S^*$ and $p \in P(b)$. Then, for $0 < \alpha \leq 1$*

$$\left|a_n\right| \leq C_1(b) M^{1-\alpha}(n)(n)^{2\alpha-1}.$$

Theorem 2.6. *Let the generalized Bazilevič function be represented by the quadruple (α, β, g, p). Then*

$$\left|a_3 - \frac{3+\alpha+i\beta}{2(2+\alpha+i\beta)} a_2^2\right| \leq \frac{\alpha(1-\gamma) + 2|b| \max\{1; |b-1|\}}{|2+\alpha+i\beta|}.$$

This result is best possible.

Proof. Since f is generalized Bazilevič function, therefore we have

$$1 + \frac{zf''(z)}{f'(z)} + (\alpha + i\beta - 1)\frac{zf'(z)}{f(z)} = \alpha\frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)} + i\beta.$$

Multiplying both sides by $f(z) f'(z) g(z) p(z)$, we obtain

$$\begin{aligned} & (1-i\beta) f(z) f'(z) g(z) p(z) + zf(z) f''(z) g(z) p(z) \\ & + (\alpha + i\beta - 1) z (f'(z))^2 g(z) p(z) \\ = \; & \alpha z f(z) f'(z) g'(z) p(z) + zf(z) f'(z) g(z) p'(z). \end{aligned}$$

Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, therefore comparing the coefficients of z^3, we have

$$\begin{aligned} & (1-i\beta)(3a_2 + b_2 + p_1) + 2a_2 + (\alpha + i\beta - 1)(4a_2 + b_2 + p_1) \\ = \; & \alpha(3a_2 + 2b_2 + p_1) + p_1. \end{aligned}$$

Thus

$$(1 + \alpha + i\beta) a_2 = \alpha b_2 + p_1. \tag{2.3}$$

Similarly comparing the coefficients of z^4 and using above inequality, we obtain

$$2(2 + \alpha + i\beta) a_3 = \alpha\left(2b_3 - b_2^2\right) + 2p_2 - p_1^2 + a_2^2(3 + \alpha + i\beta). \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\left|a_3 - \frac{3+\alpha+i\beta}{2(2+\alpha+i\beta)}a_2^2\right| = \left|\frac{\alpha\left(b_3 - \frac{1}{2}b_2^2\right) + \left(p_2 - \frac{1}{2}p_1^2\right)}{2+\alpha+i\beta}\right|.$$

Now using Lemma 1.5 and Lemma 1.6 for $\mu = \frac{1}{2}$, we have

$$\left|a_3 - \frac{3+\alpha+i\beta}{2(2+\alpha+i\beta)}a_2^2\right| \leq \frac{\alpha(1-\gamma) + 2|b|\max\{1; |b-1|\}}{|2+\alpha+i\beta|}.$$

Result is sharp for the function f_0 represented by the quadruple $\left(\alpha, \beta, \frac{z}{(1-z)^{2(1-\gamma)}}, \frac{1+(2b-1)z}{1-z}\right)$ or f_1 represented by the quadruple $\left(\alpha, \beta, \frac{z}{(1-z)^{2(1-\gamma)}}, \frac{1+(2b-1)z^2}{1-z^2}\right)$. □

For $\gamma = 0,\ b = 1$, we have the result proved by Campbell and Pearce [4].

Corollary 2.7. *Let $g \in S^*$ and $p \in P$. Then*

$$\left|a_3 - \frac{3+\alpha+i\beta}{2(2+\alpha+i\beta)}a_2^2\right| \leq \frac{\alpha+2}{|2+\alpha+i\beta|}.$$

Theorem 2.8. *(i) If f is generalized Bazilevič function with representation (α, β, g, p), then for $\alpha \geq 0$*

$$|a_2| \leq \frac{2[\alpha(1-\gamma) + |b|]}{|1+\alpha+i\beta|}.$$

(ii) If f is in $(0, \beta, g, p)$, then

$$|a_3| \leq \frac{2|b|}{|2+i\beta|}\max\left\{1, \left|\left(1 - \frac{3+i\beta}{(1+i\beta)^2}\right)b - 1\right|\right\}.$$

Both the inequalities are sharp.

Proof. (i) From (2.3), we have

$$(1+\alpha+i\beta)a_2 = \alpha b_2 + p_1.$$

This implies that

$$|a_2| \leq \frac{\alpha|b_2| + |p_1|}{|1+\alpha+i\beta|}.$$

Using Lemma 1.1(i), we have $|b_2| \leq 2(1-\gamma)$ and using the fact that $|p_1| \leq 2|b|$, we obtain

$$|a_2| \leq \frac{2[\alpha(1-\gamma) + |b|]}{|1+\alpha+i\beta|}.$$

Result is sharp for the functions f_0 defined by the quadruple $\left(\alpha, \beta, \frac{z}{(1-z)^{2(1-\gamma)}}, \frac{1+(2b-1)z}{1-z}\right)$.
(ii) Since f is represented by the quadruple $(0, \beta, g, p)$, therefore (2.3) and (2.4) yield

$$(1+i\beta)a_2 = p_1,$$

and

$$2(2+i\beta)a_3 = 2p_2 - p_1^2 + (3+i\beta)a_2^2.$$

Therefore

$$|a_3| = \frac{1}{|2+i\beta|}\left|p_2 - \frac{1}{2}\left(1 - \frac{3+i\beta}{(1+i\beta)^2}\right)p_1^2\right|$$
$$= \frac{1}{|2+i\beta|}\left|p_2 - \mu p_1^2\right|.$$

Using Lemma 1.5 for $\mu = \frac{1}{2}\left(1 - \frac{3+i\beta}{(1+i\beta)^2}\right)$, we have the required result. This result is best possible for the function f_0 represented by the quadruple $\left(0,\beta,g,\frac{1+(2b-1)z}{1-z}\right)$ or f_1 represented by the quadruple $\left(0,\beta,g,\frac{1+(2b-1)z^2}{1-z^2}\right)$. □

For $\gamma = 0$ and $b = 1$, we have the following result proved in [4].

Corollary 2.9. *Let $g \in S^*$ and $p \in P$. Then*

$$|a_2| \le \frac{2(\alpha+1)}{|1+\alpha+i\beta|},$$

and for $\alpha = 0$

$$|a_3| \le \frac{2}{|2+i\beta|}\max\left\{1, \left|\frac{3+i\beta}{(1+i\beta)^2}\right|\right\}.$$

Theorem 2.10. *Let f be a generalized Bazilevič function represented by (α,β,g,p). Then*

$$F(z) = \left[\frac{\alpha+i\beta+c}{z^c}\int_0^z t^{c-1}f^{\alpha+i\beta}(t)\,dt\right]^{\frac{1}{\alpha+i\beta}} \tag{2.5}$$

belongs to the class of generalized Bazilevič functions represented by (α,β,G,p), where $G \in S^*(\delta)$, defined by (1.9) with $c > 0$, $\alpha > 0$ and β is any real.

Proof. From (2.5), we have

$$z^c F^{\alpha+i\beta}(z) = (\alpha+i\beta+c)\int_0^z t^{c-1}f^{\alpha+i\beta}(t)\,dt.$$

This implies that

$$\frac{z^{1-i\beta}F'(z)}{(F(z))^{1-(\alpha+i\beta)}} = \frac{1}{\alpha+i\beta}\left\{(c+\alpha+i\beta)z^{-i\beta}(f(z))^{\alpha+i\beta} - cz^{-i\beta}(F(z))^{\alpha+i\beta}\right\}.$$

Using (1.9), we obtain

$$\frac{z^{1-i\beta}F'(z)}{(F(z))^{1-(\alpha+i\beta)}G^{\alpha}(z)} = \frac{\frac{1}{\alpha+i\beta}\left\{z^c(f(z))^{\alpha+i\beta} - c\int_0^z t^{c-1}f^{\alpha+i\beta}(t)\,dt\right\}}{\int_0^z t^{c+i\beta-1}g^{\alpha}(t)\,dt}$$

$$= \frac{N(z)}{D(z)}.$$

Therefore

$$\begin{aligned} & \frac{N'(z)}{D'(z)} \\ = & \frac{\frac{1}{\alpha+i\beta}\left\{cz^{c-1}(f(z))^{\alpha+i\beta}+(\alpha+i\beta)z^{c}(f(z))^{\alpha+i\beta-1}f'(z)-cz^{c-1}f^{\alpha+i\beta}(z)\right\}}{z^{c+i\beta-1}g^{\alpha}(z)} \\ = & \frac{z^{1-i\beta}f'(z)}{f^{1-(\alpha+i\beta)}(z)\,g^{\alpha}(z)} \in P(b). \end{aligned}$$

Since $D(z) = \int_0^z t^{c+i\beta-1} g^{\alpha}(t)\,dt$ is $(\alpha + c)$ valent starlike function therefore using Lemma 1.7, we have the required result. □

Corollary 2.11. *Let $\beta = 0$ and $\gamma = 0$. Then*

$$G^{\alpha}(z) = \frac{\alpha + c}{z^{c}} \int_0^z t^{c-1} f^{\alpha}(t)\,dt$$

belongs to $S^*(\delta_1)$, where $\delta_1 = \frac{-(1+2c)+\sqrt{(1+2c)^2+8\alpha}}{4\alpha}$. Hence $G(z)$ is starlike, when $g \in S^*$, therefore

$$F^{\alpha}(z) = \frac{\alpha + c}{z^{c}} \int_0^z t^{c-1} f^{\alpha}(t)\,dt$$

belongs to the class of Bazilevič function represented by the quadruple $(\alpha, 0, g, p)$. This result is proved by Noor [16].

Theorem 2.12. *Let f is generalized Bazilevič function represented by the quadruple $(\alpha, 0, g, p)$. Then f is $\frac{1}{\alpha}$-convex for $r_0 \in (0, 1)$, where r_0 is the least positive root of the equation*

$$Ar^4 + Br^3 + Cr^2 + Dr + \alpha = 0,$$

where

$$\begin{aligned} A &= \alpha(1 - 2\gamma)(2Re\, b - 1), \\ B &= \alpha\{2|b|(1 - 2\gamma) - 2(1 - \gamma)(2Re\, b - 1)\} - 2|b|, \\ C &= \alpha\{(2Re\, b - 1) - 4|b|(1 - \gamma) + (1 - 2\gamma)\} - 4Re\, b, \\ D &= 2\alpha\{|b| - (1 - \gamma)\} - 2|b|. \end{aligned}$$

Proof. Since f is generalized Bazilevič function, therefore

$$\frac{1}{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right) + \left(1 - \frac{1}{\alpha}\right)\frac{zf'(z)}{f(z)} = p_1(z) + \frac{1}{\alpha}\frac{zp'(z)}{p(z)},$$

where $p_1 \in P(\gamma)$, $0 \leq \gamma < 1$ and $p \in P(b)$, $b \neq 0$ (complex). Therefore

$$Re\left\{\frac{1}{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)+\left(1-\frac{1}{\alpha}\right)\frac{zf'(z)}{f(z)}\right\} \geq Re p_1(z) - \frac{1}{\alpha}\left|\frac{zp'(z)}{p(z)}\right|.$$

Using Lemma 1.2(ii) and well-known distortion result for the class $P(\gamma)$, we have

$$Re\left\{\frac{1}{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)+\left(1-\frac{1}{\alpha}\right)\frac{zf'(z)}{f(z)}\right\} \geq \frac{Ar^4+Br^3+Cr^2+Dr+\alpha}{\alpha(1-r^2)(1+2|b|r+(2Re\, b-1)r^2)}.$$

Since $\alpha\left(1-r^2\right)\left(1+2|b|r+(2Re\, b-1)r^2\right) > 0$, for $Re\, b \geq 1$, therefore we must have $Ar^4+Br^3+Cr^2+Dr+\alpha > 0$. Now $Q(0) = \alpha > 0$ and $Q(1) = -4(Re\, b + |b|) < 0$. Hence f is $\frac{1}{\alpha}$-convex. □

The following result is proved in [17].

Corollary 2.13. *For* $b = 1, \gamma = 0$ *and* $\alpha > 0$, f *is* $\frac{1}{\alpha}$*-convex for* $r_0 = \frac{(\alpha+1)-\sqrt{2\alpha+1}}{\alpha}$.

3. Conclusion

In this paper, we have generalized the class of Bazilevič functions associated with the quadruple (α, β, g, p) by taking the generalized versions of functions g and p. This generalization unifies and generalizes certain already known classes of Bazilevič functions. We have explored certain aspects of this generalized class which includes coefficient bounds, radius problem, inclusion of Bernardi integral operator and arc length problem. Our results generalize various results in the literature.

There is still more to explore about these functions which includes coefficient bounds, Hankel determinants and Toeplitz determinants with multiple orders. Moreover, several generalizations can also be introduced and studied by suitable variation in quadruple (α, β, g, p). In particular, the assumption of generalized versions of starlike and caratheodory functions can result the proposed generalizations.

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Conflict of interest

Authors declare that they have no conflict of interest.

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