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Research article

On oscillatory second order impulsive neutral difference equations

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Abstract: The present paper deals with the problem of oscillation for a class of second order nonlinear neutral impulsive difference equations with fixed moments of impulse effect. The technique employed here is due to the classical impulsive inequalities. Some examples are given to illustrate our results.

Keywords: oscillation; nonoscillation; impulsive difference equation; nonlinear; Krasnoselskii's fixed point theorem
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1. Introduction

The problem of oscillation of solution by imposing proper impulse controls, arises in a wide variety of real world phenomena observed in Sciences and Engineering. Indeed, impulsive differential equations arise in circuit theory, bifurcation analysis, population dynamics, loss less transmission in computer network, biotechnology, mathematical economic, chemical technology, mechanical system with impact, merging of solution, noncontinuity of solution, etc. [4, 10].

With the development of computer techniques, it is essential to formulate discrete dynamical systems while implementing the continuous dynamical systems for computer simulation, for experimental or computational purpose. These discrete time systems, which are described by the difference equations, inherit the similar dynamical characteristics. Because of that, many researchers pay their attentions to dynamical behaviours of difference equations with impulse.

In [9], M. Peng has investigated the oscillation criteria for second order impulsive delay difference equations of the form:

$$(E')\begin{cases}\Delta(a_{n-1}|\Delta x(n-1)|^{\alpha-1}\Delta x(n-1)) + f(n, x(n), x(n-\tau)) = 0, \ n \neq n_k,\\ a_{n_k}|\Delta x(n_k)|^{\alpha-1}\Delta x(n_k) = N_k(a_{n_k-1}|\Delta x(n_k-1)|^{\alpha-1}\Delta x(n_k-1)), \ k \in \mathbb{N}.\end{cases}$$

In another work [8], Peng has extended the work of [9] to the second order impulsive neutral delay difference equations of the form:

$$(E'')\begin{cases} \Delta(r_{n-1}|\Delta(x_{n-1}+p_{n-1}x_{n-\tau-1})|^{\alpha-1}\Delta(x_{n-1}+p_{n-1}x_{n-\tau-1})+f(n,x_n,x_{n-\sigma})=0,\ n\neq n_k,\\ r_{n_k}|\Delta(x_{n_k}+p_{n_k}x_{n_k-\tau})|^{\alpha-1}\Delta(x_{n_k}+p_{n_k}x_{n_k-\tau})\\ =M_k(r_{n_k-1}|\Delta(x_{n_k-1}+p_{n_k-1}x_{n_k-\tau-1})|^{\alpha-1}\Delta(x_{n_k-1}+p_{n_k-1}x_{n_k-\tau-1})),\ k\in\mathbb{N}\end{cases}$$

and obtained the sufficient conditions for oscillation of the system (E'') when $p(n) = -1$.

From the above works [8] and [9], we have a common question:

(Q) Can we find some oscillation criteria for (E') and (E'') when the neutral coefficient $p(n) \in \mathbb{R}$ viz. $-\infty < p(n) < -1$, $-1 < p(n) \le 0$ and $0 \le p(n) < \infty$?

The aim of this paper is to give a positive answar to this question by using the techniques developed in [16], to obtain some oscillation and nonoscillation criteria for a class of second order nonlinear neutral impulsive difference systems of the form:

$$(E)\begin{cases} \Delta[a(n)\Delta(x(n)+p(n)x(n-\tau))]+q(n)G(x(n-\sigma))=0,\ n\neq m_j & (1.1)\\ \underline{\Delta}[a(m_j-1)\Delta(x(m_j-1)+p(m_j-1)x(m_j-\tau-1))] & \\ \qquad\qquad +r(m_j-1)G(x(m_j-\sigma-1))=0,\ j\in\mathbb{N}, & (1.2)\end{cases}$$

where τ, $\sigma > 0$ are integers, p, q, r, a are real valued functions with discrete arguments such that $q(n) > 0$, $r(n) > 0$, $a(n) > 0$, $|p(n)| < \infty$ for $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, \cdots\}$, $G \in C(\mathbb{R},\mathbb{R})$ with the property $xG(x) > 0$ for $x \neq 0$, and Δ is the forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$. Let $m_1, m_2, m_3, \cdots$ be the moments of impulsive effect with the properties $0 < m_1 < m_2 < \cdots$, $\lim_{j\to\infty} m_j = +\infty$. And $\underline{\Delta}$ is the difference operator defined by $\underline{\Delta}u(m_j-1) = u(m_j) - u(m_j-1)$.

We refer the reader to some of the related works [2,3,5–7,11–13,15,17,20] and the references cited there in.

Definition 1.1. *By a solution of (E) we mean a real valued function $x(n)$ defined on $\mathbb{N}(n_0-\rho)$ which satisfy (E) for $n \ge n_0$ with the initial conditions $x(i) = \phi(i)$, $i = n_0-\rho, \cdots, n_0$, where $\phi(i)$, $i = n_0-\rho, \cdots, n_0$ are given and $\rho = \max\{\tau,\sigma\}$.*

Definition 1.2. *A nontrivial solution $x(n)$ of (E) is said to be nonoscillatory, if it is either eventually positive or eventually negative. Otherwise, the solution is said to be oscillatory. The system (E) is said to be oscillatory, if all its solutions are oscillatory.*

Theorem 1.3. *[1](Krasnoselskii's Fixed Point Theorem)*
Let X be a Banach space and S be a bounded closed subset of X. Consider two maps T_1 and T_2 of S into X such that $T_1x + T_2y \in S$ for every pair $x, y \in S$. If T_1 is a contraction and T_2 is completely continuous, then the equation $T_1x + T_2x = x$ has a solution in S.

2. Oscillation properties

In this section, we discuss the oscillation criteria for neutral impulsive difference equations (E). We assume that $a(n)$ satisfies
(A_0) $A(n) = \sum_{s=n_0}^{n} \frac{1}{a(s)}$ *and* $\lim_{n\to\infty} A(n) = \infty$.

Theorem 2.1. *Let* $-1 \le p(n) \le 0$ *and* $\tau < \sigma$. *In addition to* (A_0), *assume that*
(A_1) $G(-u) = -G(u),\ u \in \mathbb{R}$,
(A_2) G *satisfies* $\int_0^{\pm\alpha} \frac{du}{G(u)} < \infty, \alpha > 0$,
(A_3) $\sum_{n=1}^{\infty} q(n) + \sum_{j=1}^{\infty} r(m_j - 1) = \infty$
and
(A_4) $\sum_{n=1}^{\infty} q'(n) + \sum_{j=1}^{\infty} r'(m_j - 1) = \infty$
hold, where $q'(n) = \min\left\{\frac{q(n)}{a(n)}, \frac{q(n)}{a(n+1)}\right\}$ *and* $r'(n) = \min\left\{\frac{r(m_j-1)}{a(m_j-1)}, \frac{r(m_j-1)}{a(m_j)}\right\}$. *Then every solution of* (E) *oscillates.*

Proof. Suppose on the contrary that $x(n)$ is a nonoscillatory solution of (E) for $n \ge n_0$. Without loss of generality and due to (A_1), we may assume that $x(n) > 0$, $x(n-\tau) > 0$ and $x(n-\sigma) > 0$ for $n \ge n_0 > \rho$. Setting

$$\begin{cases} y(n) = x(n) + p(n)x(n-\tau), \\ y(m_j - 1) = x(m_j - 1) + p(m_j - 1)x(m_j - \tau - 1) \end{cases} \tag{2.1}$$

in (E), we have

$$\Delta[a(n)\Delta y(n)] = -q(n)G(x(n-\sigma)) < 0,\ n \ne m_j,\ j \in \mathbb{N},$$
$$\underline{\Delta}[a(m_j - 1)\Delta y(m_j - 1)] = -r(m_j - 1)G(x(m_j - \sigma - 1)) < 0$$

for $n \ge n_1 > n_0 + \sigma$. Therefore, $a(n)\Delta y(n)$ and $y(n)$ are monotonic for $n \ge n_1$. Here, we arise four possible cases, viz.,

$$1.\ a(n)\Delta y(n) > 0,\ y(n) > 0; \quad 2.\ a(n)\Delta y(n) > 0,\ y(n) < 0;$$
$$3.\ a(n)\Delta y(n) < 0,\ y(n) > 0; \quad 4.\ a(n)\Delta y(n) < 0,\ y(n) < 0.$$

Case 1. We can choose $n_2 > n_1 + 1$ and a constant $\beta > 0$ such that $y(n) \ge \beta$ for $n \ge n_2$. Indeed, $y(n) > 0$ and $-1 \le p(n) \le 0$ implies that $y(n) \le x(n)$ and hence $y(m_j - 1) \le x(m_j - 1)$. Now, the impulsive system (E) reduces to

$$(E_1)\begin{cases} \Delta[a(n)\Delta y(n)] + G(\beta)q(n) \le 0,\ n \ne m_j \\ \underline{\Delta}[a(m_j - 1)\Delta y(m_j - 1)] + G(\beta)r(m_j - 1) \le 0,\ j \in \mathbb{N}. \end{cases}$$

Summing (E_1) from n_2 to $n-1$, we get

$$a(n+1)\Delta y(n+1) - a(n)\Delta y(n) - \sum_{n_2 \le m_j - 1 \le n-1} \underline{\Delta}[a(m_j - 1)\Delta y(m_j - 1)] \le -G(\beta)\sum_{s=n_2}^{n-1} q(s),$$

that is,

$$G(\beta)\left[\sum_{s=n_2}^{n-1} q(s) + \sum_{n_2 \le m_j - 1 \le n-1} r(m_j - 1)\right] \le a(n)\Delta y(n) - a(n+1)\Delta y(n+1)$$
$$< a(n)\Delta y(n) < \infty \text{ as } n \to \infty,$$

a contradiction to (A_3) due to $\lim_{n\to\infty} a(n)y(n) < \infty$.
Case 2. Since $y(n) < 0$ for $n \geq n_2$, then we can find $n_3 > n_2$ such that

$$y(n) > p(n)x(n-\tau) \geq -x(n-\tau),$$
$$y(m_j-1) > p(m_j-1)x(m_j-\tau-1) \geq -x(m_j-\tau-1).$$

Therefore, from (E_1) we get

$$\begin{cases} \Delta[a(n)\Delta y(n)] - q(n)G(y(n+\tau-\sigma)) \leq 0,\ n \neq m_j \\ \underline{\Delta}[a(m_j-1)\Delta y(m_j-1)] - r(m_j-1)G(y(m_j+\tau-\sigma-1)) \leq 0,\ j \in \mathbb{N}, \end{cases}$$

that is,

$$\begin{cases} -a(n)\Delta y(n) - q(n)G(y(n+\tau-\sigma)) \leq 0,\ n \neq m_j \\ -a(m_j-1)\Delta y(m_j-1) - r(m_j-1)G(y(m_j+\tau-\sigma-1)) \leq 0,\ j \in \mathbb{N} \end{cases}$$

implies that

$$\begin{cases} -a(n)\Delta y(n) - q(n)G(y(n)) \leq 0,\ n \neq m_j \\ -a(m_j-1)\Delta y(m_j-1) - r(m_j-1)G(y(m_j-1)) \leq 0,\ j \in \mathbb{N} \end{cases}$$

due to the nondecreasing nature of y and $\tau < \sigma$. Clearly,

$$\frac{\Delta y(n)}{G(y(n))} + \frac{q(n)}{a(n)} \leq 0,\ n \neq m_j,$$
$$\frac{\Delta y(m_j-1)}{G(y(m_j-1))} + \frac{r(m_j-1)}{a(m_j-1)} \leq 0,\ j \in \mathbb{N},$$

that is,

$$q'(n) \leq -\frac{\Delta y(n)}{G(y(n))},\ n \neq m_j,$$
$$r'(m_j-1) \leq -\frac{\Delta y(m_j-1)}{G(y(m_j-1))},\ j \in \mathbb{N}.$$

If $y(n) \leq u \leq y(n+1)$ and $y(m_j-1) \leq v \leq y(m_{j+1}-1)$, then $\frac{1}{G(y(n))} \geq \frac{1}{G(u)}$ and $\frac{1}{G(y(m_j-1))} \geq \frac{1}{G(v)}$. Therefore, the preceding inequalities reduce to

$$q'(n) \leq -\int_{y(n)}^{y(n+1)} \frac{du}{G(u)},\ n \neq m_j,$$
$$r'(m_j-1) \leq -\int_{y(m_j-1)}^{y(m_{j+1}-1)} \frac{dv}{G(v)},\ j \in \mathbb{N}.$$

As a result,

$$\sum_{s=n_3}^{n} q'(s) \leq -\sum_{s=n_3}^{n} \int_{y(s)}^{y(s+1)} \frac{du}{G(u)} = -\int_{y(n_3)}^{y(n+1)} \frac{du}{G(u)},$$

$$\sum_{j=1}^{\infty} r'(m_j-1) \le -\lim_{s\to\infty}\sum_{j=1}^{s}\int_{y(m_j-1)}^{y(m_{j+1}-1)}\frac{dv}{G(v)} = -\lim_{s\to\infty}\int_{y(m_1-1)}^{y(m_{s+1}-1)}\frac{dv}{G(v)}.$$

Since for nonimpulsive points $m_j - 1$ and n we have $\lim_{n\to\infty} y(n) < \infty$ and $\lim_{j\to\infty} y(m_j - 1) < \infty$, then

$$\sum_{s=n_3}^{\infty} q'(s) + \sum_{j=1}^{\infty} r'(m_j-1) < \infty,$$

a contradiction to (A_4) due to (A_2).

Case 3. As $a(n)\Delta y(n)$ is nonincreasing for $n \ge n_1$, we can find a constant $\gamma > 0$ and $n_2 > n_1 + 1$ such that $a(n)\Delta y(n) < -\gamma$ for $n \ge n_2$ and hence $a(m_j-1)\Delta y(m_j-1) < -\gamma$ for $n \ge n_2$. Summing $\Delta y(n) < -\frac{\gamma}{a(n)}$ from n_2 to $n-1$, we get

$$y(n) - y(n_2) - \sum_{n_2\le m_j-1\le n-1}\Delta y(m_j-1) \le -\sum_{s=n_2}^{n-1}\frac{\gamma}{a(n)},$$

that is,

$$y(n) \le y(n_2) - \gamma\Big[\sum_{s=n_2}^{n-1}\frac{1}{a(n)} + \sum_{n_2\le m_j-1\le n-1}\frac{1}{a(m_j-1)}\Big],$$

a contradiction to the fact that $y(n) > 0$ for $n \ge n_2$.

Case 4. Here, $\lim_{n\to\infty} y(n) = -\infty$ and so also $\lim_{j\to\infty} y(m_j-1) = -\infty$. By Sandwich theorem, it follows that $\lim_{j\to\infty} y(m_j) = -\infty$. Clearly, $y(n) < 0$ for $n \ge n_1$ implies that

$$x(n) \le x(n-\tau) \le x(n-2\tau) \le x(n-3\tau)\cdots \le x(n_1).$$

Analogously,

$$x(m_j-1) \le x(m_j-\tau-1) \le x(m_j-2\tau-1) \le x(m_j-3\tau-1)\cdots \le x(n_1)$$

due to the nonimpulsive points $m_j - 1, m_j - \tau - 1, m_j - 2\tau - 1, \cdots$. Therefore, $x(n)$ is bounded for all nonimpulsive points. We assert that $x(m_j)$ is bounded. If not, let it be $\lim_{j\to\infty} x(m_j) = +\infty$. Ultimately,

$$\begin{aligned} y(m_j) &= x(m_j) + p(m_j)x(m_j-\tau) \\ &\ge x(m_j) - x(m_j-\tau) \ge x(m_j) - B_1 \end{aligned}$$

implies that $y(m_j) > 0$ as $j \to \infty$, a contradiction, where $x(m_j - \tau) \le B_1$. So, our assertation holds and $y(n)$ is bounded for every n. Again this leads to a contradiction to the fact that $y(n)$ is unbounded. This complete the proof of the theorem. □

Theorem 2.2. *Let $-\infty < b \le p(n) \le c < -1$ and $\tau - \sigma \ge 1$. If (A_0), (A_1), (A_3), (A_4) and*
(A_5) G satisfies $\int_{\pm\alpha}^{\pm\infty}\frac{dv}{G(v)} < \infty, \alpha > 0$
hold, then every solution of (E) either oscillates or satisfies $\lim_{n\to\infty} x(n) = 0$.

Proof. Suppose on the contrary that $x(n)$ is a nonoscillatory solution of (E) for $n \geq n_0 > \rho$. Proceeding as in the proof of Theorem 2.1, we have that $a(n)\Delta y(n)$ and $y(n)$ are of one sign for $n \geq n_1 > n_0$. So, we have following four cases:

1. $a(n)\Delta y(n) > 0,\ y(n) > 0$; 2. $a(n)\Delta y(n) > 0,\ y(n) < 0$;
3. $a(n)\Delta y(n) < 0,\ y(n) > 0$; 4. $a(n)\Delta y(n) < 0,\ y(n) < 0$.

The proofs for Case 1 and Case 3 are similar to that of Theorem 2.1.

Case 2. Let $\lim_{n\to\infty} y(n) = l,\ -\infty < l \leq 0$. We claim that $l = 0$. Otherwise, there exists $n_2 > n_1 + 1$ such that $y(n + \tau - \sigma) \leq l$ and so also, $y(m_j + \tau - \sigma - 1) \leq l$. Indeed, $y(n) < 0$ implies that $y(n) > p(n)x(n - \tau) \geq bx(n - \tau)$ and analogously, $y(m_j - 1) \geq bx(m_j - \tau - 1)$ due to nonimpulsive points $m_j - 1, m_j - \tau - 1, \cdots$. Hence, there exists $n_3 > n_2$ such that (E) takes the form

$$\begin{cases} \Delta[a(n)\Delta y(n)] + G(\frac{l}{b})q(n) \leq 0,\ n \neq m_j \\ \underline{\Delta}[a(m_j - 1)\Delta y(m_j - 1)] + G(\frac{l}{b})r(m_j - 1) \leq 0,\ j \in \mathbb{N} \end{cases}$$

for $n \geq n_3$. Summing the above impulsive system from n_3 to $n - 1$, it follows that

$$a(n)\Delta y(n) - a(n_3)\Delta y(n_3) - \sum_{n_3 \leq m_j - 1 \leq n-1} \underline{\Delta}[a(m_j - 1)\Delta y(m_j - 1)] + G\left(\frac{l}{b}\right)\sum_{s=n_3}^{n-1} q(s) = 0,$$

that is,

$$\begin{aligned} G\left(\frac{l}{b}\right)\left[\sum_{s=n_3}^{n-1} q(s) + \sum_{n_3 \leq m_j - 1 \leq n-1} r(m_j - 1)\right] &= a(n_3)\Delta y(n_3) - a(n)\Delta y(n) \\ &\leq a(n_3)\Delta y(n_3) \\ &< \infty \text{ as } n \to \infty, \end{aligned}$$

a contradiction to (A_3). So, our claim holds and thus $\lim_{n\to\infty} y(n) = 0$, $\lim_{j\to\infty} y(m_j - 1) = 0$. Now,

$$\begin{aligned} 0 = \lim_{n\to\infty} y(n) &= \liminf_{n\to\infty}(x(n) + p(n)x(n - \tau)) \\ &\leq \liminf_{n\to\infty}(x(n) + cx(n - \tau)) \\ &\leq \limsup_{n\to\infty} x(n) + \liminf_{n\to\infty}(cx(n - \tau)) \\ &= (1 + c)\limsup_{n\to\infty} x(n) \end{aligned}$$

implies that $\limsup_{n\to\infty} x(n) = 0$ due to $(1 + c) < 0$ and hence $\lim_{n\to\infty} x(n) = 0$. We encounter that $\lim_{j\to\infty} x(m_j - 1) = 0$ because of nonimpulsive points $m_j - 1, j \in \mathbb{N}$. Since $m_j - 1 < m_j < n$, then an application of the Sandwich theorem implies that $\lim_{j\to\infty} x(m_j) = 0$. Therefore, $\lim_{n\to\infty} x(n) = 0$ for all n and m_j, $j \in \mathbb{N}$.

Case 4. For $y(n) < 0$,

$$y(n) > p(n)x(n - \tau) \geq bx(n - \tau).$$

Analogously,

$$y(m_j-1) > p(m_j-1)x(m_j-\tau-1) \geq bx(m_j-\tau-1)$$

due to the nonimpulsive point $m_j-1, m_j-\tau-1, \cdots$ and so on. Therefore,

$$\begin{aligned} &y(n+\tau-\sigma) \geq bx(n-\sigma),\\ &y(m_j+\tau-\sigma-1) \geq bx(m_j-\sigma-1) \end{aligned}$$

for $n \geq n_2 > n_1+1$. Ultimately, (E) becomes

$$\begin{aligned} &\Delta[a(n)\Delta y(n)] + q(n)G(b^{-1}y(n+\tau-\sigma)) \leq 0,\ n \neq m_j,\\ &\underline{\Delta}[a(m_j-1)\Delta y(m_j-1)] + r(m_j-1)G(b^{-1}y(m_j+\tau-\sigma-1)) \leq 0,\ j \in \mathbb{N}, \end{aligned}$$

that is,

$$\begin{aligned} &a(n+1)\Delta y(n+1) + q(n)G(b^{-1}y(n+\tau-\sigma)) \leq \Delta y(n) < 0,\\ &a(m_j)\Delta y(m_j) + r(m_j-1)G(b^{-1}y(m_j+\tau-\sigma-1)) \leq \Delta y(m_j-1) < 0. \end{aligned}$$

Using the fact that y is nonincreasing for $n \geq n_2$ and $\tau-\sigma \geq 1$, we get

$$\begin{aligned} &a(n+1)\Delta y(n+1) + q(n)G(b^{-1}y(n+1)) \leq 0, n \neq m_j,\\ &a(m_j)\Delta y(m_j) + r(m_j-1)G(b^{-1}y(m_j)) \leq 0, j \in \mathbb{N}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\frac{\Delta y(n+1)}{G(b^{-1}y(n+1))} + \frac{q(n)}{a(n+1)} \leq 0, n \neq m_j,\\ &\frac{\Delta y(m_j)}{G(b^{-1}y(m_j))} + \frac{r(m_j-1)}{a(m_j)} \leq 0, j \in \mathbb{N}. \end{aligned}$$

If $b^{-1}y(n+1) \leq u \leq b^{-1}y(n+2)$ and $b^{-1}y(m_j) \leq v \leq b^{-1}y(m_{j+1})$, then the last two inequalities can be written as

$$\begin{aligned} q'(n) &\leq -\int_{b^{-1}y(n+1)}^{b^{-1}y(n+2)} \frac{bdu}{G(b^{-1}y(n+1))} \leq -\int_{b^{-1}y(n+1)}^{b^{-1}y(n+2)} \frac{bdu}{G(u)},\\ r'(m_j-1) &\leq -\int_{b^{-1}y(m_j)}^{b^{-1}y(m_{j+1})} \frac{bdv}{G(b^{-1}y(m_j))} \leq -\int_{b^{-1}y(m_j)}^{b^{-1}y(m_{j+1})} \frac{bdv}{G(v)}, \end{aligned}$$

that is,

$$\begin{aligned} &\sum_{s=n_2}^{n-1} q'(s) \leq -b\sum_{s=n_2}^{n-1}\int_{b^{-1}y(s+1)}^{b^{-1}y(s+2)} \frac{du}{G(u)} = -b\int_{b^{-1}y(n_2+1)}^{b^{-1}y(n+2)} \frac{du}{G(u)},\\ &\sum_{j=1}^{\infty} r'(m_j-1) \leq -b\lim_{s\to\infty}\sum_{j=1}^{s}\int_{b^{-1}y(m_j)}^{b^{-1}y(m_{j+1})} \frac{dv}{G(v)} = -b\lim_{s\to\infty}\int_{b^{-1}y(m_1)}^{b^{-1}y(m_{s+1})} \frac{dv}{G(v)}. \end{aligned}$$

Since for nonimpulsive points $m_j - 1$ and n we have $\lim_{n\to\infty} y(n) = \infty$ and $\lim_{j\to\infty} y(m_j - 1) = \infty$, then an application of Sandwich theorem shows that $\lim_{j\to\infty} y(m_j) = \infty$. Therefore,

$$\sum_{n=n_4}^{\infty} q'(s) + \sum_{j=1}^{\infty} r'(m_j - 1) < \infty,$$

a contradiction to (A_4) due to (A_5). This completes the proof of the theorem. □

Theorem 2.3. *Let* $0 \le p(n) \le d < \infty$ *and* $\tau \le \sigma$. *In addition to* (A_0) *and* (A_1), *assume that*
(A_6) $G(u)G(v) \ge G(uv)$ *for* $u, v \in \mathbb{R}_+$,
(A_7) *there exists* $\lambda > 0$ *such that* $G(u) + G(v) \ge \lambda G(u + v)$ *for* $u, v \in \mathbb{R}_+$,
(A_8) $\sum_{n=\tau}^{\infty} Q(n) + \sum_{j=1}^{\infty} R(m_j - 1) = \infty$
and
(A_9) $\sum_{n=\tau}^{\infty} Q'(n) + \sum_{j=1}^{\infty} R'(m_j - 1) = \infty$
hold, where $Q(n) = \min\{q(n), q(n-\tau)\}$, $R(m_j - 1) = \min\{r(m_j - 1), r(m_j - \tau - 1)\}$, $Q'(n) = \min\left\{\frac{q(n)}{a(n+1)}, \frac{q(n-\tau)}{a(n+1-\tau)}\right\}$ *for* $n \ge \tau$ *and* $R'(m_j - 1) = \min\left\{\frac{r(m_j-1)}{a(m_j)}, \frac{r(m_j-\tau-1)}{a(m_j-\tau)}\right\}$ *for* $m_j \ge \tau + 1$. *Then every solution of* (E) *oscillates.*

Proof. Proceeding as in the proof of Theorem 2.1, we have following two possible cases:

$$1.\ a(n)\Delta y(n) > 0,\ y(n) > 0; \quad 2.\ a(n)\Delta y(n) < 0,\ y(n) > 0.$$

Case 1. In this case, $y(n)$ is nondecreasing for $n \ge n_1$. So, there exist $n_2 > n_1 + 1$ and a constant $\beta > 0$ such that $y(n) \ge \beta$ for $n \ge n_2$. From (1.1), we have

$$\Delta[a(n)\Delta y(n)] + q(n)G(x(n-\sigma)) = 0 \tag{2.2}$$

and

$$G(d)[\Delta(a(n-\tau)\Delta y(n-\tau)) + q(n-\tau)G(x(n-\tau-\sigma))] = 0. \tag{2.3}$$

Combining (2.2) and (2.3), we have

$$\begin{aligned}\Delta[a(n)\Delta y(n)] + G(d)\Delta[a(n-\tau)\Delta y(n-\tau)] + q(n)G(x(n-\sigma))\\ + G(d)q(n-\tau)G(x(n-\tau-\sigma)) = 0\end{aligned}$$

which on applying (A_6), we obtain

$$\Delta[a(n)\Delta y(n) + G(d)\Delta[a(n)\Delta y(n-\tau)] + Q(n)G(x(n-\sigma)) + G(dx(n-\tau-\sigma)) \le 0,$$

that is,

$$\Delta[a(n)\Delta y(n)] + G(d)\Delta[a(n-\tau)\Delta y(n-\tau)] + \lambda Q(n)G(x(n-\sigma) + dx(n-\tau-\sigma)) \le 0$$

due to (A_7). Since $y(n-\sigma) \le x(n-\sigma) + ax(n-\tau-\sigma)$, then the preceding inequality can be written as

$$\Delta[a(n)\Delta y(n)] + G(d)\Delta[a(n-\tau)\Delta y(n-\tau)] + \lambda Q(n)G(y(n-\sigma)) \le 0. \tag{2.4}$$

By using a similar argument in (1.2), we get

$$\underline{\Delta}[a(m_j-1)\Delta y(m_j-1)] + G(d)\underline{\Delta}[a(m_j-\tau-1)\Delta y(m_j-\tau-1)] + \lambda R(m_j-1)G(y(m_j-\sigma-1)) \le 0. \tag{2.5}$$

Summing (2.4) from n_2 to $n-1$ and then using (2.5), we get

$$\begin{aligned} &a(n)\Delta y(n) - a(n_2)\Delta y(n_2) + G(d)a(n-\tau)\Delta y(n-\tau) - G(d)a(n_2-\tau)\Delta y(n_2-\tau) \\ &- \sum_{n_2\le m_j-1\le n-1}\Big[\underline{\Delta}[a(m_j-1)\Delta y(m_j-1)] + G(d)\underline{\Delta}[a(m_j-\tau-1)\Delta y(m_j-\tau-1)]\Big] \\ &+ \lambda\sum_{s=n_2}^{n-1} Q(n)G(y(n-\sigma)) \le 0, \end{aligned}$$

that is,

$$\begin{aligned} \lambda\sum_{s=n_2}^{n-1} Q(n)G(y(n-\sigma)) + \lambda\sum_{n_2\le m_j-1\le n-1} R(m_j-1)G(y(m_j-\sigma-1)) \\ \le \Delta a(n_2)y(n_2) + G(d)a(n_2-\tau)\Delta y(n_2-\tau). \end{aligned}$$

Therefore,

$$\lambda G(\beta)\left[\sum_{s=n_2}^{n-1} Q(n) + \sum_{n_2\le m_j-1\le n-1} R(m_j-1)\right] < \infty \text{ as } n\to\infty,$$

a contradiction to (A_8).

Case 2. From (2.2) and (2.3), we have

$$\begin{aligned} &a(n+1)\Delta y(n+1) + q(n)G(x(n-\sigma)) = a(n)\Delta y(n) < 0, \\ &G(d)\Delta y(n+1-\tau) + G(d)q(n-\tau)G(x(n-\tau-\sigma)) = G(d)a(n-\tau)\Delta y(n-\tau) < 0. \end{aligned}$$

Consequently, (2.4) reduces to

$$\Delta y(n+1) + G(d)\Delta y(n+1-\tau) + \lambda Q'(n)G(y(n-\sigma)) < 0.$$

By a similar argument to (2.5), we get

$$\Delta y(m_j) + G(d)\Delta y(m_j-\tau) + \lambda R'(m_j-1)G(y(m_j-\sigma-1)) < 0.$$

Hence, the impulsive system (E) reduces to

$$\begin{cases} \Delta y(n+1) + G(d)\Delta y(n+1-\tau) + \lambda Q'(n)G(y(n-\sigma)) < 0, n\ne m_j \\ \Delta y(m_j) + G(d)\Delta y(m_j-\tau) + \lambda R'(m_j-1)G(y(m_j-\sigma-1)) < 0, j\in\mathbb{N}. \end{cases}$$

Using the fact that y is nonincreasing and $\tau\le\sigma$, we can find $n_3 > n_2+1$ such that the above inequality can be written as

$$(E_2)\begin{cases} \frac{\Delta y(n+1)}{G(y(n))} + G(a)\frac{\Delta y(n+1-\tau)}{G(y(n-\tau))} + \lambda Q'(n) < 0, n\ne m_j \\ \frac{\Delta y(m_j)}{G(y(m_j-1))} + G(a)\frac{\Delta y(m_j-\tau)}{G(y(m_j-\tau-1))} + \lambda R'(m_j-1) < 0, j\in\mathbb{N} \end{cases}$$

for $n \geq n_3$. If

$$y(n+2) \leq t \leq y(n+1),\ y(n+2-\tau) \leq z \leq y(n+1-\tau),$$
$$y(m_{j+1}) \leq u \leq y(m_j),\ y(m_{j+1}-\tau) \leq v \leq y(m_j-\tau),$$

then form (E_2) it is easy to verify that

$$\int_{y(n+1)}^{y(n+2)} \frac{dt}{G(t)} + G(a)\int_{y(n+1-\tau)}^{y(n+2-\tau)} \frac{dz}{G(z)} + \lambda Q'(n) \leq 0,\ n \neq m_j,$$
$$\int_{y(m_j)}^{y(m_{j+1})} \frac{du}{G(u)} + G(a)\int_{y(m_j-\tau)}^{y(m_{j+1}-\tau)} \frac{dv}{G(v)} + \lambda R'(m_j-1) \leq 0,\ j \in \mathbb{N},$$

that is,

$$\sum_{s=n_3}^{n}\left[\int_{y(s+1)}^{y(s+2)} \frac{dt}{G(t)} + G(a)\int_{y(s+1-\tau)}^{y(s+2-\tau)} \frac{dz}{G(z)}\right] + \lambda\sum_{s=n_3}^{n} Q'(s) \leq 0,\ n \neq m_j,$$
$$\sum_{j=1}^{\infty}\left[\int_{y(m_j)}^{y(m_{j+1})} \frac{du}{G(u)} + G(a)\int_{y(m_j-\tau)}^{y(m_{j+1}-\tau)} \frac{dv}{G(v)}\right] + \lambda\sum_{j=1}^{\infty} R'(m_j-1) \leq 0,\ j \in \mathbb{N}.$$

As a result,

$$\lambda\sum_{s=n_3}^{\infty} Q'(s) \leq -\lim_{n\to\infty}\left[\int_{y(n_3+1)}^{y(n+2)} \frac{dt}{G(t)} + G(a)\int_{y(n_3+1-\tau)}^{y(n+2-\tau)} \frac{dz}{G(z)}\right],$$
$$\lambda\sum_{j=1}^{\infty} R'(m_j-1) \leq -\lim_{s\to\infty}\left[\int_{y(m_1)}^{y(m_s)} \frac{du}{G(u)} + G(a)\int_{y(m_1-\tau)}^{y(m_s-\tau)} \frac{dv}{G(v)}\right]$$

implies that

$$\sum_{s=n_3}^{\infty} Q'(s) + \sum_{j=1}^{\infty} R'(m_j-1) < \infty,$$

a contradiction to (A_9) due to (A_2). This completes the proof of the theorem. □

Next, we establish the criteria for existence of positive solution of the impulsive system (E) .

Theorem 2.4. *Let* $-1 < p_1 \leq p(n) \leq p_2 \leq 0$. *Assume that*
(A_{10}) $\sum_{s=n}^{\infty} \frac{1}{a(s)}\Big[\sum_{t=n^*}^{s-1} q(s) + \sum_{n^*\leq m_j-1\leq s-1} r(m_j-1)\Big] < \infty$
holds. Then (E) *has a bounded non-oscillatory solution.*

Proof. Let $X = l_\infty^{n_1}$ be the Banach space of all real valued bounded sequence $x(n)$ for $n \geq n_1$ with the norm defined by

$$\|x\| = \sup\{|x(n)| : n \geq n_1\}.$$

Consider a closed subset S of X, where

$$S = \{x \in X : \beta_1 \leq x(n) \leq \beta_2, n \geq n_1\},$$

where $\beta_1 > 0$ and $\beta_2 > 0$ are so chosen that $\beta_1 - p_1\beta_2 < \beta_2$. Due to (A_{10}), we can find $n_2 > n_1$ and $\beta_1 < \gamma < (1 + p_1)\beta_2$ such that

$$\sum_{s=n}^{\infty} \frac{1}{a(s)} \left[\sum_{t=n_2}^{s-1} q(t) + \sum_{n_2 \le m_j - 1 < s-1} r(m_j - 1) \right] < \frac{(1 + p_1)\beta_2 - \gamma}{M}, \tag{2.6}$$

where $M = \max\{G(x) : \beta_1 \le x \le \beta_2\}$. For $x \in S$ and $n \ge n_2$, define two maps

$$(T_1 x)(n) = \begin{cases} T_1 x(n_2), & n_2 - \rho \le n \le n_2, \\ \gamma - p(n)x(n - \tau), & n > n_2 \end{cases}$$

and

$$(T_2 x)(n) = \begin{cases} T_2 x(n_2), & n_2 - \rho \le n \le n_2, \\ \sum_{s=n}^{\infty} \frac{1}{a(s)} \Big[\sum_{t=n_2}^{s-1} q(t)G(x(t - \sigma)) + \sum_{n_2 \le m_j - 1 \le s-1} r(m_j - 1)G(x(m_j - \sigma - 1)) \Big], & n > n_2. \end{cases}$$

Indeed, for $x_1, x_2 \in S$ and using (2.6) for $n \ge n_2$, we have

$$\begin{aligned} T_1 x_1(n) + T_2 x_2(n) &= \gamma - p(n)x_1(n - \tau) + \sum_{s=n}^{\infty} \frac{1}{a(s)} \Big[\sum_{t=n_2}^{s-1} q(t)G(x_2(t - \sigma)) \\ &\quad + \sum_{n_2 \le m_j - 1 < s-1} r(m_j - 1)G(x_2(m_j - \sigma - 1)) \Big] \\ &\le \gamma - p_1\beta_2 + \sum_{s=n}^{\infty} \frac{1}{a(s)} \Big[\sum_{t=n_2}^{s-1} q(t)G(x_2(t - \sigma)) \\ &\quad + \sum_{n_2 \le m_j - 1 < s-1} r(m_j - 1)G(x_2(m_j - \sigma - 1)) \Big] \\ &\le \gamma - p_1\beta_2 + M \sum_{s=n}^{\infty} \frac{1}{a(s)} \left[\sum_{t=n_2}^{s-1} q(t) + \sum_{n_2 \le m_j - 1 < s-1} r(m_j - 1) \right] \\ &\le \beta_2 \end{aligned}$$

and

$$\begin{aligned} T_1 x_1(n) + T_2 x_2(n) &\ge \gamma - p(n)x_1(n - \tau) \\ &\ge \gamma \ge \beta_1. \end{aligned}$$

Therefore, $\beta_1 \le T_1 x_1 + T_2 x_2 \le \beta_2$ for $n \ge n_2$. Also, for $x_1, x_2 \in S$ and $n \ge n_2$, we have

$$|T_1 x_1(n) - T_1 x_2(n)| \le |p(n)||x_1(n - \tau) - x_2(n - \tau)| \le -p_1|x_1(n - \tau) - x_2(n - \tau)|,$$

that is,

$$\|T_1 x_1 - T_1 x_2\| \le -p_1\|x_1 - x_2\|$$

and hence T_1 is a contraction mapping with the contraction constant $-p_1 < 1$.

Next, we show that T_2 is completely continuous. For this, we need to show that T_2x is continuous and relatively compact. Let $x_k \in S$ be such that $x_k(n) \to x(n)$ as $k \to \infty$. Since S is closed, then $x = x(n) \in S$. Now, for $n \geq n_2$

$$|(T_2x_k)(n) - (T_2x)(n)| \leq \sum_{s=n}^{\infty} \frac{1}{a(s)}\Big[\sum_{t=n_2}^{s-1} q(t)|G(x_k(t-\sigma)) - G(x(t-\sigma))| + \sum_{n_2\leq m_j-1\leq s-1} r(m_j-1)|G(x_k(m_j-\sigma-1) - G(x(m_j-\sigma-1))|\Big].$$

Since $|G(x_k(n-\sigma)) - G(x(n-\sigma))| \to 0$ as $k \to \infty$, by applying the Lebesgue's dominated convergence theorem [1], we have that $\lim_{k\to\infty} |(T_2x_k)(n) - (T_2x)(n)| \to 0$. Therefore, T_2x is continuous. To show that T_2x is relatively compact, we show that the family of functions $\{T_2x : x \in S\}$ is uniformly bounded and equicontinuous on $[n_2, \infty)$. It is easy to see that T_2x is uniformly bounded.

Next, we show that T_2x is equicontinuous. For $n_4 > n_3 \geq n_2$ and $x \in S$ such that

$$\begin{aligned}|T_2x(n_4) - T_2x(n_3)| &= \Bigg|\sum_{s=n_4}^{\infty} \frac{1}{a(s)}\left[\sum_{t=n_2}^{s-1} q(t)G(x(t-\sigma)) + \sum_{n_2\leq m_{j-1}\leq s-1} r(m_j-1)G(x(m_j-\sigma-1))\right] \\ &\quad - \sum_{s=n_3}^{\infty} \frac{1}{a(s)}\left[\sum_{t=n_2}^{s-1} q(t)G(x(t-\sigma)) + \sum_{n_2\leq m_{j-1}\leq s-1} r(m_j-1)G(x(m_j-\sigma-1))\right]\Bigg| \\ &\leq M\sum_{s=n_3}^{n_4} \frac{1}{a(s)}\left[\sum_{t=n_2}^{s-1} q(t) + \sum_{n_2\leq m_{j-1}\leq s-1} r(m_j-1)\right].\end{aligned}$$

Therefore, there exists $\epsilon > 0$ and $\delta > 0$ such that for $\epsilon < \frac{(1+p_1)\beta_2-\gamma}{M}$

$$|T_2x(n_4) - T_2x(n_3)| < \epsilon \text{ when ever } 0 < n_4 - n_3 < \delta,$$

and this relation continue to hold for every $n_3, n_4 \in [n_2, \infty)$. Therefore, $\{T_2x : x \in S\}$ is uniformly bounded and equicontinuous on $[n_2, \infty)$ and hence T_2x is relatively compact. By Theorem 1.3, $T_1 + T_2$ has a unique fixed point $x \in S$ such that $T_1x + T_2x = x$ for which

$$x(n) = \begin{cases} x(n), \ n_2 - \rho \leq n \leq n_2, \\ \gamma - p(n)x(n-\tau) \\ + \sum_{s=n}^{\infty} \frac{1}{a(s)}\Big[\sum_{t=n_2}^{s-1} q(t)G(x(t-\sigma)) + \sum_{n_2\leq m_j-1<s-1} r(m_j-1)G(x(m_j-\sigma-1))\Big], \ n > n_2. \end{cases}$$

Indeed, $x(n)$ is a positive solution of the impulsive system (E) . This completes the proof of the theorem. □

3. Conclusion and example

We present some examples to illustrate our main results.

Example 3.1. *Consider the impulsive difference equation*

$$(E_3)\begin{cases}\Delta[n\Delta(x(n) - 2x(n-1))] + q(n)x^{1/3}(n-3) = 0, \ n \neq m_j, \ n \geq 4 \\ \underline{\Delta}[(m_j-1)\Delta(x(m_j-1) - 2x(m_j-2))] + r(m_j-1)x^{1/3}(m_j-4) = 0, \ j \in \mathbb{N},\end{cases}$$

where $\tau = 1$, $\sigma = 3$, $a(n) = n$, $p(n) = -1/2$, $q(n) = 6n + 3$, $r(m_j - 1) = 6m_j - 3$, $G(u) = u^{1/3}$, $m_j = 3j$ *for* $j \in \mathbb{N}$. *Clearly,*

$$\sum_{n=4}^{\infty} q(n) = \sum_{n=4}^{\infty} 6n + 3 \geq \sum_{n=4}^{\infty} 6n = 6 \sum_{n=4}^{\infty} n = \infty$$

and

$$\sum_{n=4}^{\infty} q'(n) = \sum_{n=4}^{\infty} \frac{6n + 3}{n} \geq \sum_{n=4}^{\infty} \frac{6n}{n} = \sum_{n=4}^{\infty} 6 = \infty.$$

Therefore, $(A_2) - (A_4)$ *hold. It is easy to see that all conditions of Theorem 2.1 are satisfied. Hence,* (E_3) *is oscillatory. In particular,* $x(n) = (-1)^n$ *is an oscillatory solution of the first equation of* (E_3) *and* $(-1)^{m_j}$ *is an oscillatory solution of the second equation of* (E_3).

Example 3.2. *Consider the impulsive difference equation*

$$(E_4) \begin{cases} \Delta^2(x(n) - 2x(n-3)) + q(n)x^3(n-1) = 0, \ n \neq m_j, \ n \geq 4 \\ \underline{\Delta}[\Delta(x(m_j - 1) - 2x(m_j - 4))] + r(m_j - 1)x^3(m_j - 2) = 0, \ j \in \mathbb{N}, \end{cases}$$

where $\tau = 3$, $\sigma = 1$, $a(n) = 1$, $p(n) = -2$, $q(n) = (n-1)^3 \left(\frac{1}{n+2} + \frac{2}{n+1} + \frac{1}{n} + \frac{2}{n-3} + \frac{4}{n-2} + \frac{2}{n-1}\right)$, $r(m_j - 1) = (m_j - 2)^3 \left(\frac{1}{m_j+5} + \frac{1}{m_j+4} + \frac{2}{m_j+2} + \frac{2}{m_j+1} + \frac{1}{m_j} + \frac{2}{m_j-1} + \frac{2}{m_j-3} + \frac{2}{m_j-4}\right)$, $m_j = 5j$ *for* $j \in \mathbb{N}$ *and* $G(u) = u^3$. *Clearly,*

$$\sum_{n=4}^{\infty} q(n) = \infty = \sum_{n=4}^{\infty} q'(n).$$

Therefore, (A_3) *and* (A_4) *hold. It is easy to see that all conditions of Theorem 2.2 are satisfied. In particular,* $x(n) = \frac{(-1)^{n+1}}{n}$ *is a solution of the first equation of* (E_4) *and* $\frac{(-1)^{m_j}}{m_j-1}$ *is a solution of the second equation of* (E_4).

Remark 3.3. *In Theorem 2.4, we have obtained the necessary condition for the existence of bounded positive solution of the impulsive system* (E) *by using the Krasnoselskii's fixed point theorem in the range* $-1 < p(n) \leq 0$. *It would be interesting to prove the results in the other ranges of* $p(n)$ *by means of Krasnoselskii's fixed point theorem.*

Remark 3.4. *We may note that, Theorem 2.2 guarantees that every solution of* (E) *either oscillates or converges to zero. Unfortunately, we can not establish sufficient condition that ensure that all solutions of* (E) *are just oscillatory.*

Remark 3.5. *Based on Remark 3.4, we can raise following problems for future research:*
(1) Is it possible to establish sufficient condition that ensure that all solutions of (E) *are oscillatory when* $-\infty < p(n) \leq -1$*?*
(2) Is it possible to suggest a different method to study (E) *and find some sufficient conditions which ensure that all solutions of* (E) *are oscillatory when* $|p(n)| < \infty$*?*
(3) Is it possible to find the necessary and sufficient conditions which ensure that all solutions of (E) *are oscillatory?*

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Conflict of interest

The author declares no conflict of interest.

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