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Research article

Branchwise solid generalized BCH-algebras

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Abstract: In this paper, we investigate an equivalent condition for a branchwise strongly solid $gBCH$-algebra to be branchwise commutative. Moreover, we show that a branchwise strongly solid $gBCH$-algebra is both branchwise commutative and branchwise positive implicative if and only if it is branchwise implicative.

Keywords: generalized BCH-algebra; branchwise solid generalized BCH-algebras; branchwise strongly solid generalized BCH-algebras; branchwise implicativeness
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1. Introduction

Some implicational logics have contributed to give rise to the notions of a few abstract algebras such as BCK-algebras and BCI-algebras, see [1] and [2]. The recent developments in the field of artificial intelligence has contributed greatly in many aspects of daily life. The basic tools of artificial intelligence which assist in decision making are logical systems. The fundamental axioms of the implicational calculus are the motivation behind the introduction and development of BCK-algebra and BCI-algebra, see [1] and [2]. Keeping in view the strong relationships between these algebras and corresponding logics, translation procedures have been developed to relate theorems and formulas of a logic and corresponding algebra. Hence, the study of abstract algebras, which have been motivated by logical systems, and their generalization has remained a topic of interest for those who are working in the areas of artificial intelligence, logical systems and algebraic structures. Consequently, another class of algebras known as the class of BCH-algebras has been introduced in [3,4]. It has been shown [5] that the class of BCK/BCI-algebras is a proper subclass of BCH-algebras. Several aspects of this algebra have been studied in [5–9]. Recently, Chaudhry et al. [10] introduced the notion of a $gBCH$-algebra. They showed that $gBCH$-algebra is a generalization of $BCK/BCI/BCH$-algebras. Consequently, this algebra carries some connections with the BCK/BCI positive logics and corresponding systems which are used in the process of decision making in the field of artificial intelligence. The study of $gBCH$-algebra is a desirable topic for researchers working in the relevant areas.

In this paper we study the class of $gBCH$-algebras. We define the notions of a branchwise solid $gBCH$-algebra and a branchwise strongly solid $gBCH$-algebra. We also introduce the notions of branchwise commutative, branchwise implicative and branchwise positive implicative $gBCH$-algebras, and we investigate relations between these classes.

2. Preliminaries

Definition 1. *A BCH-algebra* [3, 4] *is a non-empty set X with a constant* 0 *and a binary operation* "$*$" *satisfying the following axioms:*

(I) $x * x = 0$,
(II) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$,
(III) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$.

From now onward, we denote $x * y$ by xy, $x * (y * z)$ by $x(yz)$. It is well known that the class of all BCK/BCI-algebras is a proper subclass of the class of all BCH-algebras, see [5].

Definition 2. *A generalized BCH-algebra (shortly, $gBCH$-algebra) [10] is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the conditions (I), (II) and the following conditions:*

(IV) $(x(xy))y = 0$,
(V) $(xy)x = 0y$

for all x, y *in* X.

Note that the condition (V) is equivalent to $(xy)x = (xx)y$ in $gBCH$-algebras. Every BCH-algebra is a $gBCH$-algebra, since the condition (III) implies that $(x(xy))y = (xy)(xy) = 0$ and $(xy)x = (xx)y = 0y$ for all $x, y \in X$.

A BCH-algebra (or $gBCH$-algebra) is said to be *proper* if it does not satisfy the condition:

(VI) $((xy)(xz))(zy) = 0$

for all $x, y, z \in X$.

Example 1. [10] *Let $X := \{0, w, x, y, z\}$ be a set with the following table:*

$*$	0	w	x	y	z
0	0	0	0	y	y
w	w	0	0	y	y
x	x	x	0	y	z
y	y	y	y	0	0
z	z	y	y	w	0

Then it is easy to see that $(X, *, 0)$ *is a gBCH-algebra, but not a BCH-algebra, since* $(xy)z = 0 \neq w = (xz)y$.

This research is a continuation of Chaudhry et al. [10], and so we refer several definitions and theorems discussed in [10]. Let $(X, *, 0)$ be a $gBCH$-algebra. We define a binary relation "$\leq$" on X by $x \leq y$ if and only if $xy = 0$. An element x_0 is said to be a *minimal element* of X if $x \leq x_0$ implies $x = x_0$. We denote $Min(X)$ the set of all minimal elements of X. The set $Med(X)$ consists of all elements $x \in X$ satisfying $0(0x) = x$, and we call it a *medial part* of X. A set $B(x_0) := \{x \in X | x_0 \leq x\}$, where $x_0 \in Min(X)$, is called a *branch* of X. It is known that $Min(X) = Med(X)$ in generalized BCH-algebras (see [10]).

Proposition 1. [10] *Let* $(X, *, 0)$ *be a gBCH-algebra. Then*

(i) *0 is a minimal element of X,*
(ii) $x0 = x$ *for all* $x \in X$.

Theorem 1. [10] *Let* $(X, *, 0)$ *be a gBCH-algebra. If* $x \in X$*, then there exists a unique* $x_0 \in Med(X)$ *such that* $x \in B(x_0)$.

Theorem 2. [10] *Let* $(X, *, 0)$ *be a gBCH-algebra and* $x_0, x_1 \in Med(X)$. *Then*

(i) $0(xy) = (0x)(0y)$ *for all* $x, y \in X$,
(ii) $x, y \in B(x_0)$ *if and only if* $xy \in B(0)$,
(iii) *if* $y \in B(x_0)$ *and* $x \leq y, y \leq z$, *then* $x, z \in B(x_0)$,
(iv) *if* $x_0 \neq x_1$, *then* $B(x_0) \cap B(x_1) = \emptyset$.

3. Branchwise solid $gBCH$-algebras

Definition 3. *A gBCH-algebra* $(X, *, 0)$ *is said to be branchwise solid if, for any* $x, y, z \in B(a)$, *where* $a \in Med(X)$,

$$((xy)(xz))(zy) = 0, \tag{3.1}$$

i.e., $(xy)(xz) \leq zy$.

Note that if $(X, *, 0)$, $(Y, \bullet, 0)$ are two $gBCH$-algebras then the cartesian product $X \times Y$ is also a $gBCH$-algebra with constant $(0, 0)$, where the binary operation "$\odot$" on $X \times Y$ is defined by component-wise from the operations on X and Y, respectively, i.e., $(x, a) \odot (y, b) := (x * y, a \bullet b)$ for any $(x, a), (y, b) \in X \times Y$.

Definition 4. *A branchwise solid gBCH-algebra is said to be proper branchwise solid gBCH-algebra if it is not a BCH-algebra.*

Now, we give two examples of a proper branchwise solid $gBCH$-algebra as follows.

Example 2. *Let X be a gBCH-algebras as in Example 1 and* $Y := \{0, p\}$ *with the binary operation* $\bullet$ *defined by*

$\bullet$	0	p
0	0	0
p	p	0

Then it is easy to see that $Y \times X$ is a branchwise solid gBCH-algebra, but not a BCH-algebra, since $[(0, x)(0, y)](0, z) = (0, 0) \neq (0, w) = [(0, x)(0, z)](0, y)$*. Hence it is a proper branchwise solid gBCH-algebra.*

Example 3. *Let $(X, *, 0)$ be a gBCH-algebras as in Example 1. Then $(X \times X, \odot, (0, 0))$ is a proper gBCH-algebra, since $[(x, w)(y, x)](z, z) = (y, 0)(z, z) = (0, y)$ and $[(x, w) \odot (z, z)](y, x) = (z, y)(y, x) = (w, y)$. Moreover, $[[(w, w)(z, z)][(x, x)(y, y)]]\,[(y, y)(z, z)] = [(z, z)(y, y)][(y, y)(z, z)] = (w, w)(0, 0) = (w, w) \neq 0$. It shows that $(X \times X, \odot, (0, 0))$ does not satisfy the condition (1) in Definition 3. However, routine calculations give that every branch $B(a)$, $a \in Med(X)$, satisfies the condition (1), hence it is a proper branchwise solid gBCH-algebra.*

Theorem 3. *Let $(X, *, 0)$ be a branchwise solid gBCH-algebra and let $x, y, z \in B(a)$ where $a \in Med(X)$. Then*

- (i) *if $x \leq y$, then $xz \leq yz$,*
- (ii) *if $x \leq y$, then $zy \leq zx$,*
- (iii) *if $x \leq y$, $y \leq z$, then $x \leq z$,*
- (iv) *$xy = x(x(xy))$,*
- (v) *if $x \leq y$, then $z(zx) \leq y$.*

Proof. (i) Let $x \leq y$. By Proposition 1 (ii), we have $x0 = x$ for all $x \in X$. It follows from (1) of Definition 3 that $(xz)(yz) = ((xz)0)(yz) = ((xz)(xy))(yz) = 0$. Hence $xz \leq yz$.

(ii) If $x \leq y$, then $xy = 0$ and hence $(zy)(zx) = ((zy)(zx))0 = ((zy)(zx))(xy) = 0$ by Proposition 1 (ii). Hence $zy \leq zx$.

(iii) Let $x \leq y$, $y \leq z$. Then $xy = 0$ and $yz = 0$. By Proposition 1 (ii) and (1), we obtain $xz = (xz)0 = (xz)(xy) = ((xz)(xy))0 = ((xz)(xy))(yz) = 0$, proving that $x \leq z$.

(iv) Given $x, y \in B(a)$. Since $x(xy) \leq y$, so by Theorem 2 (iii), we obtain $x(xy) \in B(a)$. Since X is a branchwise solid $gBCH$-algebra, so by Definition 3 and (IV) we have

$$(xy)(x(x(xy))) \leq (x(xy))y = 0.$$

On the other hand, (IV) gives $(x(x(xy)))(xy) = 0$. So by (II), we have $x(x(xy)) = xy$.

(v) Let $x \leq y$. Since $z(zx) \leq x$ and $x \in B(a)$, by Theorem 2 (iii), we have $z(zx) \in B(a)$ and $z(zx) \leq x \leq y$. By Theorem 3 (iii), we obtain $z(zx) \leq y$.

□

4. Branchwise commutativity and branchwise (positive) implicativity

Definition 5. *A branchwise solid gBCH-algebra $(X, *, 0)$ is said to be branchwise strongly solid if, for any $x, y, z \in B(a)$, where $a \in Med(X)$,*

$$(xy)z = (xz)y. \tag{4.1}$$

To demonstrate the significance of this notion, we provide some examples. It is easy to see that Examples 1, 2 and 3 are branchwise strongly solid $gBCH$-algebras.

Example 4. *Let $Z := \{0, a, b, c, d\}$ be a set with the following table:*

$\bullet$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	0	a	d
b	b	b	0	0	d
c	c	c	c	0	d
d	d	d	d	d	0

Then $(Z, \bullet, 0)$ is a gBCH-algebra. Let $(X, *, 0)$ be a gBCH-algebra as in Example 1. Then $(Z \times X, \odot, (0, 0))$ is a proper gBCH-algebra, where $\odot$ is defined component wise as described earlier. But, it is is not branchwise solid, since $[[(a, 0)(c, w)][(a, 0)(b, w)]][(b, w) \odot (c, w)] = [(a, 0)(0, 0)](0, 0) = (a, 0)(0, 0) = (a, 0) \neq (0, 0)$.

Definition 6. *A gBCH-algebra $(X, *, 0)$ is said to be branchwise commutative if, for any $x, y \in B(a)$, $a \in Med(X)$, $x(xy) = y(yx)$.*

The following theorem provides an equivalent condition for a branchwise strongly solid $gBCH$-algebra $(X, *, 0)$ to be branchwise commutative.

Theorem 4. *Let $(X, *, 0)$ be a branchwise strongly solid gBCH-algebra. Then it is branchwise commutative if and only if $y(yx) = x(x(y(yx)))$, for all $x, y \in B(a)$, $a \in Med(X)$.*

Proof. Assume $(X, *, 0)$ is branchwise commutative. Given $x, y \in B(a), a \in Med(X)$, since $y(yx) \leq x$, by Theorem 1 and Theorem 2 (*iii*), there exists uniquely $x_0 \in Min(X)$ such that $y(yx), x \in B(x_0)$. By Theorem 2 (*iv*), we can show that $a = x_0$. Since $(X, *, 0)$ is branchwise commutative, by Proposition 1 (*ii*), we obtain

$$\begin{aligned} x(x(y(yx))) &= (y(yx))[(y(yx))x] \\ &= (y(yx))0 \\ &= y(yx). \end{aligned}$$

Conversely, assume $y(yx) = x(x(y(yx)))$, for all $x, y \in B(a)$, for all $a \in Med(X)$. By Theorem 2 (*ii*), we obtain $yx \in B(0)$. It follows that $0(yx) = 0$. Since $(X, *, 0)$ is a branchwise strongly solid $gBCH$-algebra, we have $(y(yx))y = (yy)(yx) = 0(yx) = 0$. Since $x(xy) \leq y$, both $x(xy)$ and y belong to $B(a)$. Since $(X, *, 0)$ is a branchwise strongly solid $gBCH$-algebra, by applying Theorem 3 (*iv*), we obtain

$$[x(x(y(yx)))](x(xy)) = [x(x(xy))][x(y(yx))] = (xy)[x(y(yx))]. \tag{4.2}$$

Since $(X, *, 0)$ is a branchwise solid $gBCH$-algebra, by applying (4.2), we obtain

$$[y(yx)][x(xy)] = [x(x(y(yx)))](x(xy))$$

$$
\begin{aligned}
&= (xy)[x(y(yx))] \\
&\leq (y(yx))y \\
&= 0.
\end{aligned}
$$

This proves that $y(yx) \leq x(xy)$. If we interchange the role of x and y, we obtain $x(xy) \leq y(yx)$, proving that $(X, *, 0)$ is branchwise commutative. □

Now, we introduce the notions of a branchwise implicative $gBCH$-algebra as well as of a branchwise positive implicative $gBCH$-algebra.

Definition 7. *A $gBCH$-algebra $(X, *, 0)$ is said to be branchwise implicative if, for any $x, y \in B(a)$, $a \in Med(X)$, $x(yx) = x$.*

Definition 8. *A $gBCH$-algebra $(X, *, 0)$ is said to be branchwise positive implicative if $xy = (xy)(y(0(0y)))$, for any $x, y \in B(a)$, $a \in Med(X)$.*

Theorem 5. *Let $(X, *, 0)$ be a branchwise strongly solid $gBCH$-algebra. If X is branchwise implicative, then it is branchwise commutative.*

Proof. Let $x, y \in B(a), a \in Med(X)$. Since $x(xy) \leq y$, by Theorem 2 (*iii*), we have $x(xy) \in B(a)$. Moreover, since $y(y(x(xy))) \leq x(xy)$, we have $y(y(x(xy))) \in B(a)$. Since $(X, *, 0)$ is branchwise implicative, we obtain

$$x(xy) = [x(xy)][y(x(xy))]. \tag{4.3}$$

for any $x, y \in B(a), a \in Med(X)$. Since $(X, *, 0)$ is branchwise strongly solid, we have

$$[[x(xy)][y(x(xy))]][y(y(x(xy)))] = [[x(xy)][y(y(x(xy)))]][y(x(xy))]. \tag{4.4}$$

Since $x(xy) \leq y$, by Theorem 3 (*i*) and (IV), we obtain

$$
\begin{aligned}
[x(xy)][y(y(x(xy)))] &\leq y[y(y(x(xy)))] \\
&\leq y(x(xy)).
\end{aligned}
$$

It follows that

$$[[x(xy)][y(y(x(xy)))]][y(x(xy))] = 0. \tag{4.5}$$

By (4.4) and (4.5), we obtain $[[x(xy)][y(x(xy))]][y(y(x(xy)))] = 0$, i.e.,

$$[x(xy)][y(x(xy))] \leq y(y(x(xy))). \tag{4.6}$$

By (4.3) and (4.6), we obtain $x(xy) \leq y(y(x(xy)))$. By (IV), we have $y(y(x(xy))) \leq x(xy)$. This proves that $x(xy) = y(y(x(xy)))$. By applying Theorem 4, $(X, *, 0)$ is branchwise commutative. □

Theorem 6. *Let $(X, *, 0)$ be a branchwise strongly solid $gBCH$-algebra. If X is branchwise implicative, then it is branchwise positive implicative.*

Proof. Given $x, y \in B(a), a \in Med(X)$, since X is branchwise strongly solid, by applying Theorem 2 (i) and Theorem 3 (iv), we obtain

$$\begin{aligned}[(xy)(y(0(0y)))](xy) &= [(xy)(xy)][y(0(0y))] \\ &= 0(y(0(0y))) \\ &= (0y)(0(0(0y))) \\ &= (0y)(0y) \\ &= 0.\end{aligned}$$

It follows that

$$(xy)(y(0(0y))) \leq xy \tag{4.7}$$

for any $x, y \in B(a), a \in Med(X)$. Now, since $0(0y) \leq y$ and $y \in B(a)$, we have $0(0y) \in B(a)$. Since X is branchwise strongly solid, we obtain

$$(y(xy))(0(0y)) = [y(0(0y))](xy) \tag{4.8}$$

By Theorem 5, X is also branchwise commutative. It follows that

$$\begin{aligned}(xy)[(xy)(y(0(0y)))] &= [y(0(0y))][[y(0(0y))](xy)] \\ &= [y(0(0y))][(y(xy))(0(0y))] \qquad [\because (4.8)] \\ &= [y(0(0y))][y(0(0y))] \qquad [\because X : \text{branch. imp.}] \\ &= 0.\end{aligned}$$

Hence we obtain $xy \leq (xy)(y(0(0y)))$. By combining it with (4.7) and using (II), we get $xy = (xy)(y(0(0y)))$. □

The following theorem yields sufficient conditions for a branchwise strongly solid $gBCH$-algebra $(X, *, 0)$ to be branchwise implicative.

Theorem 7. *Let* $(X, *, 0)$ *be a branchwise strongly solid, branchwise commutative and branchwise positive implicative* $gBCH$*-algebra. Then it is branchwise implicative.*

Proof. Given $x, y \in B(a), a \in Med(X)$, by Theorem 2 ($ii$), we obtain $xy, yx \in B(0)$, i.e., $0(xy) = 0(yx) = 0$. Since X is branchwise strongly solid, we have

$$(x(yx))x = (xx)(yx) = 0(yx) = 0.$$

This shows that $x(yx) \leq x$. Since $x \in B(a)$, by applying Theorem 2 (iii), we obtain $x(yx) \in B(a)$. If we take $x_0 := 0(0x)$, then $x_0 \leq x$ and $x_0 \in Min(X) = Med(X)$. Since $x \in B(a)$, by Theorem 2 (iii), we have $x_0 \in B(a)$. Since $x_0, a \in Med(X)$, by Theorem 2 (iv), we obtain $a = x_0$. Hence

$$\begin{aligned}(0(0x))(x(yx)) &= x_0(x(yx)) \\ &= a(x(yx)) \\ &= 0.\end{aligned}$$

Since X is branchwise commutative, by applying Theorem 4, we obtain

$$\begin{aligned} x(x(yx)) &= (yx)[(yx)(x(x(yx)))] \\ &= [(yx)(x(0(0x)))][(yx)(x(x(yx)))] \ [\because X : \text{branch.pos.imp.}] \\ &\leq [x(x(yx))][x(0(0x))] \qquad [\because \text{Definition 5}] \\ &\leq (0(0x))(x(yx)) \qquad [\because \text{Definition 5}] \\ &= 0. \end{aligned}$$

It follows that $x \leq x(yx)$. Thus we proved that $x = x(yx)$. □

The following is a consequence of the above three theorems.

Corollary 1. *A branchwise strongly solid $gBCH$-algebra $(X, *, 0)$ is both branchwise commutative and branchwise positive implicative if and only if it is branchwise implicative.*

5. Conclusion

The theory of a $gBCH$-algebra is one of recent topics in the field of algebraic structures, which has attraction to many mathematicians and computer scientists. In this article, several notions such as branchwise solid $gBCH$-algebras, branchwise strongly solid $gBCH$-algebras, branchwise commutativity, branchwise implicativity and branchwise positive implicativity have been studied. Moreover, we investigated necessary and sufficient conditions for a branchwise strongly solid $gBCH$-algebra to be branchwise commutative. We also developed some relationships among the branchwise implicativity and the branchwise commutativity and the branchwise positive implicativity. The sufficient condition for a $gBCH$-algebra to be branchwise implicative has also been proved. The notion of a $gBCH$-algebra provides some possibility to open the doors of BCH-algebras into the area of $gBCH$-algebras. The areas of the categorical aspects, graph algebras, ideals and filters in $gBCH$-algebras will be discussed in sequel. It will also be interesting to investigate: (i) which parts of the Theorem 3 can be proved for a gBCH-algebra and whether the condition that x, y, z are from the same branch $B(a)$ is necessary or it may be relaxed for some parts? and (ii) an example of a branchwise solid gBCH-algebra which is not branchwise strongly solid gBCH-algebra.

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Conflict of interest

The authors hereby declare that there are no conflicts of interest regarding the publication of this paper.

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