Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(3): 2412–2423.
DOI: 10.3934/math.2020159
Received: 30 November 2019
Accepted: 14 February 2020
Published: 05 March 2020

Research article

Int-soft ideals over the soft sets in ordered semigroups

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Abstract: In this paper, the notions of int-soft left (right) ideals, int-soft interior ideals and int-soft bi-ideals over the soft sets are introduced and several related properties of these notions are investigated. Characterizations of int-soft ideals over the soft sets are considered. Moreover, for any soft set (η, U) over S, the notion of a soft set over the soft sets $(\chi_{\eta(u)}, V)$ is introduced. It is prove that a soft set (η, S) is a soft left ideal (resp. right ideal, interior ideal, bi-ideal) over S if and only if $(\chi_{\eta(x)}, S)$ is an int-soft left ideal (resp. right ideal, interior ideal, bi-ideal) over the soft sets.

Keywords: ordered semigroups; soft sets; soft ideals over the soft sets; int-soft left (right) ideals; int-soft interior ideals; int-soft bi-ideals
Mathematics Subject Classification: 06F05, 06D72, 20M12

1. Introduction

To solve complicated problems in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as the probability theory, the theory of (intuitionistic) fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [1]. Molodtsov [1] and Maji et al. [2] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [1] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. In the theory of soft sets to discuss the theory of uncertainty, the parameters are used that makes it more appropriate theory than the theory of rough sets and fuzzy sets. The decision making problem in soft sets had been considered by Maji et al. [3]. In [2], Maji et al. investigated several operations on soft sets. The notions of soft sets introduced in different algebraic structures had been applied and studied by several authors; for example, Aktaş and Çağman [4] for soft groups, Feng et al. [5] for soft semirings and Naz et al. [6, 7] for soft semihypergroups.

In [8], Song initiated the study of int-soft semigroups, int-soft left (resp. right) ideals, and int-soft products. In [9], Dudek et al. introduced and characterized the concept of a soft interior ideals of semigroups. The concept of union-soft semigroups, union-soft l-ideals, union-soft r-ideals, and union-soft semiprime soft sets have been considered by Jun et al. [10]. Furthermore, Jun et al. and Muhiuddin et al. have applied the fuzzy set theory, soft set theory and related notions to the semigroup theory on different aspects (see for e.g., [11–16]). In addition, Muhiuddin et al. and other researchers studied the soft set theory on various algebraic structures (see for e.g., [17–26]).

As a generalization of the concept of soft semigroups, soft left (right) ideals and left (right) idealistic soft semigroups, in 2010, Jun et al. [27] initiated the study of soft ordered semigroups, soft left (right) ideals and left (right) idealistic soft ordered semigroups. Shabir et al. [28] introduced the concept of union soft version of semiprime quasi-ideals of ordered semigroups and characterized completely regular ordered semigroup in terms of semiprime uni-soft quasi-ideals. Ali [29] introduced the notion of soft filters in ordered semigroup and discussed the related properties of soft filters and soft prime ideals.

The notions of fuzzy soft semigroups and fuzzy soft left(right) ideals were introduced by Yang [30]. He characterized some properties of these notions by the concerned level set. The concept of fuzzy soft ordered semigroups, fuzzy soft left(right) ideals and fuzzy interior ideals over an ordered semigroup have been defined and their related properties have been considered by Khan et al. [31].

In this paper, by using the technique of soft sets over an ordered semigroup, the concept of a soft set over the soft sets is introduced. In fact, we defined the int-soft left (right) ideals, int-soft interior ideals and int-soft bi-ideals over the soft sets. Moreover, for any soft set over the soft sets (η, U), the soft set (η^u, U) over an ordered semigroup is defined. Then the fact that (η, U) is an int-soft semigroup over the soft sets if and only if for each $u \in U$, (η^u, U) is an int-soft semigroup over S has been proved. In the sequel, we prove that int-soft ideals over the soft set are int-soft interior ideals (or int-soft bi-ideals) over the soft sets but the converses are not true in general, it is discussed by constructed new examples for both cases. Finally, coincidence relations among int-soft left (right) ideals, int-soft interior ideals and int-soft bi-ideals over the soft sets are discussed.

2. Preliminaries

An *ordered semigroup* $(S, \cdot, \leq)$ is an algebraic structure which is a semigroup endowed with a partial order $\leq$ such that

$$r \leq s \Rightarrow rt \leq st \text{ and } tr \leq ts$$

for all $r, s, t \in S$.

Throughout the paper, for the simplicity, S will denote an ordered semigroup.

For a subset T of S, let

$$(T] = \{s \in S \mid s \leq t \text{ for some } t \in T\}.$$

A subset $T(\neq \emptyset)$ of S is said to be a *subsemigroup* of S if $TT \subseteq T$ and is said to be a *left (resp. right) ideal* of S if $ST \subseteq T$ ($TS \subseteq T$) and $(T] \subseteq T$. If T is both a left and a right ideal of S, then T is said to be an *ideal* of S. A subsemigroup B of S is said to be a *bi-ideal* of S if $BSB \subseteq B$ and $(B] \subseteq B$. A subsemigroup I of S is said to be an *interior ideal* of S if $SIS \subseteq I$ and $(I] \subseteq I$. A subset $Q(\neq \emptyset)$ of S is said to be a *quasi-ideal* of S if $(QS] \cap (SQ] \subseteq Q$ and $(Q] \subseteq Q$.

In S, if for each $x \in S$, there exists an element $a \in S$ such that $x \leq xax$, then we say that S is regular; and if, for each $x \in S$, there exist $a, b \in S$ such that $x \leq ax^2b$, then we say that S is intra-regular; and if, for each $x \in S$, there exist $a, b \in S$ such that $x \leq xaxb$, then we that S is right weakly regular.

Let U be a universal set and let E be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of U and let $A \subseteq E$. A pair (F, A) is called a *soft set (over U)* [27] if $F : A \to \mathcal{P}(U)$ is a mapping.

Now we recall the basic definitions of the soft ideals and int-soft ideals on ordered semigroups, according with [8, 27, 29, 31]. A soft set (F, A) over an ordered semigroup S is called a

(1) *soft ordered semigroup over S* if and only if $F(e)$ is a subsemigroup of S, for all $e \in A$.
(2) *soft left (right) ideal over S* if and only if $F(e)$ is a left (right) ideal of S, for all $e \in A$.
(3) *soft interior ideal over S* if and only if $F(e)$ is an interior ideal of S, for all $e \in A$.
(4) *soft bi-ideal over S* if and only if $F(e)$ is a bi-ideal of S, for all $e \in A$.

Let (F, A) and (G, B) be soft sets over U. Then (G, B) is called a *soft subset* of (F, A) if $B \subseteq A$ and $G(b) \subseteq F(b)$ for all $b \in B$.

Let (F, A) and (G, A) be two soft sets. Then, for each $a \in A$, the union " $\Cup$ " and intersection " $\Cap$ " are defined as

$$(F \Cup G)(a) = F(a) \cup G(a),$$
$$(F \Cap G)(a) = F(a) \cap G(a).$$

A soft set (F, S) over U is called an *int-soft left (resp. right) ideal over U* if $F(rs) \supseteq F(r)$ (resp. $F(rs) \supseteq F(s)$) for all $r, s \in S$, and for any $r, s \in S$, $r \leq s$ implies $F(r) \supseteq F(s)$. It is called an *int-soft ideal over U* if it is both an int-soft left ideal and an int-soft right ideal over S. A soft set (F, S) over U is called an *int-soft interior ideal over U* if $F(ras) \supseteq F(y)$ for all $r, a, s \in S$ and for any $r, s \in S$, $r \leq s$ implies $F(r) \supseteq F(s)$. A soft set (F, S) over U is called an *int-soft bi-ideal over U* if $F(ras) \supseteq F(r) \cap F(s)$ for all $r, a, s \in S$; and for any $r, s \in S$, $r \leq s$ implies $F(r) \supseteq F(s)$.

In whatever follows, $L_S(S), R_S(S), J_S(S), I_S(S)$ and $B_S(S)$ denote the set of all int-soft left ideals, int-soft right ideals, int-soft ideals, int-soft interior ideals and int-soft bi-ideals over S.

For more details of soft sets the reader is referred to [32–37].

3. Int-soft ideals, int-soft interior ideals and int-soft bi-ideals over the soft sets

In this section, the notions of int-soft semigroups, int-soft ideals, int-soft interior ideals and int-soft bi-ideals over the soft sets are introduced.

A pair (η, U) is called a *soft set over the soft sets*, where $\eta : U \to P_S(S)$ is a mapping, and $P_S(S)$ is the set of all soft sets over S with a parameter set S.

Definition 3.1. A soft set over the soft sets (η, U) is said to be *an int-soft semigroup over the soft sets* if and only if for each $u \in U$, $(\eta(u), S)$ is an int-soft ordered semigroup over S.

Definition 3.2. Let (η_1, U) and (η_2, U) be two soft sets over the soft sets. The intersection of (η_1, U) and (η_2, U), denoted by $(\eta_1, U)\widetilde{\Cap}(\eta_2, U)$, is defined as $(\eta_1\widetilde{\Cap}\eta_2)(u) = \eta_1(u) \Cap \eta_2(u)$ for each $x \in U$.

Lemma 3.3. *Let S be an ordered semigroup and $(\eta_1, U), (\eta_2, U)$ be two int-soft ordered semigroups over the soft sets. Then $(\eta_1, U)\widetilde{\Cap}(\eta_2, U)$ is an int-soft ordered semigroup over the soft sets.*

Proof. Straightforward. □

Definition 3.4. Let (η_1, U) and (η_2, U) be two soft sets over the soft sets. The union of (η_1, U) and (η_2, U), denoted by $(\eta_1, S)\widetilde{\Cup}(\eta_2, S)$, is defined as $(\eta_1\widetilde{\Cup}\eta_2)(u) = \eta_1(u) \Cup \eta_2(u)$ for each $u \in U$.

Remark 3.5. In general, union of two int-soft ordered semigroups over the soft sets need not be an int-soft ordered semigroup over the soft sets as shown by the following example.

Example 3.6. Let $S = \{0, y, z\}$. The binary operation "·" and an order "≤" on S are defined as follows:

$\cdot$	0	y	z
0	0	z	z
y	z	y	z
z	z	z	z

$$\leq := \{(0, 0), (y, y), (z, z), (y, z)\}.$$

Then S is an ordered semigroup. Define soft sets (η_1, U) and (η_2, U) over the soft sets defined as follows:

$$\eta_1(e_1)(s) = \begin{cases} \{y\} & \text{if } s = 0, \\ \{y, z\} & \text{if } s = y, \\ \{z\} & \text{if } s = z, \end{cases} \text{ and } \eta_2(e_1)(s) = \begin{cases} \{0\} & \text{if } s = 0, \\ S & \text{if } s = y, \\ \{z\} & \text{if } s = z. \end{cases}$$

It is a routine to verify that (η_1, U) and (η_2, U) are int-soft ordered semigroups over the soft sets, but $(\eta_1, U)\widetilde{\Cap}(\eta_2, U)$ is not an int-soft ordered semigroup over the soft sets because $\eta(e_1)(0y) = \{z\} \not\supseteq \eta(e_1)(0) \cap \eta(e_1)(y) = \{0, y\}$.

Definition 3.7. A soft set over the soft sets (η, U) is called an *int-soft left (resp. right) ideal over the soft sets* iff $\eta(u) \in L_S(S)$ (resp. $\eta(u) \in R_S(S)$) for each $u \in U$.

We denote the set of all int-soft left (resp. right) ideals over the soft sets by $\mathcal{L}_S(S)$ (resp. $\mathcal{R}_S(S)$).

Example 3.8. Let $S = \{0, y, z\}$. The binary operation '·' and an order " ≤ " on S are defined as follows:

$\cdot$	0	y	z
0	0	0	0
y	0	y	z
z	0	z	y

$$\leq := \{(0, 0), (y, y), (z, z), (0, y), (0, z)\}.$$

Then S is an ordered semigroup. Define a soft set (η, U) over the soft sets as follows:

$$\eta(e_1)(a) = \begin{cases} S & \text{if } a = 0, \\ \{0, y\} & \text{if } a = y, \\ \{0\} & \text{if } a = z. \end{cases}$$

It is a routine to verify that (η, U) is a soft left ideal over the soft sets.

Definition 3.9. A soft set over the soft sets (η, U) of S is called an *int-soft ideal over the soft sets* iff $\eta(u) \in J_S(S)$ for each $u \in U$.

We denote the set of all int-soft ideals over the soft sets by $\mathcal{J}_S(S)$.

Definition 3.10. Let S be an ordered semigroup and (η, U) be a soft set over the soft sets. For each $u \in U$, define the set (η^u, U) as follows:

$$\eta^u(x) = \{y \in S \mid \eta(x)(y) \supseteq \eta(x)(u)\}$$

for each $x \in S$. Clearly (η^u, U) is a soft set over S.

Theorem 3.11. *Let S be an ordered semigroup and let (η, S) be a soft set over the soft sets. Then (η, S) is an int-soft semigroup over the soft sets $\Leftrightarrow$ for each $u \in S$, (η^u, S) is an int-soft semigroup over S.*

Proof. ($\Rightarrow$) Take any $a, b \in S$ and $t \in \eta^u(a) \cap \eta^u(b)$. Then $\eta(a)(t) \supseteq \eta(a)(u)$ and $\eta(a)(t) \supseteq \eta(a)(u)$. By hypothesis, $\eta(ab)(t) \supseteq \eta(b)(t) \cap \eta(a)(t)$ and, so, $\eta(ab)(t) \supseteq \eta(a)(u)$. It, then, follows that $t \in \eta^u(ab)$. Thus $\eta^u(ab) \supseteq \eta^u(a) \cap \eta^u(b)$.

($\Leftarrow$) Assume that, for each $u \in S$, (η^u, S) is an int-soft semigroup over S. Take any $u, x \in S$. If r, s exist in S such that $\eta(x)(r) \cap \eta(x)(s) \supset \eta(x)(rs)$, then there exists $v \in S$ such that $\eta(x)(rs) \subset \eta(x)(v) \subseteq \eta(x)(r) \cap \eta(x)(s)$. This implies that $s \in \eta^v(x)$. As $rs \notin \eta^v(x)$, we get a contradiction. Thus, for each $r, s \in S$, $\eta(x)(rs) \supseteq \eta(x)(r) \cap \eta(x)(s)$, as required. □

Theorem 3.12. *Let (η, S) be a soft set over the soft sets. Then $\eta \in \mathcal{L}_S(S)$ (resp. $\eta \in \mathcal{R}_S(S)$) $\Leftrightarrow$ $\eta^u \in L_S(S)$ (resp. $\eta^u \in R_S(S)$) for each $u \in S$.*

Proof. ($\Rightarrow$) Take any $r, s \in S$ such that $r \leq s$. Now, for any $t \in \eta^u(s)$, $\eta(s)(t) \supseteq \eta(s)(u)$. As (η, S) is an int-soft left ideal over the soft sets, $\eta(r)(t) \supseteq \eta(s)(t)$ which implies $\eta(r)(t) \supseteq \eta(s)(u)$. Therefore $t \in \eta^u(r)$, and, so, $\eta^u(r) \supseteq \eta^u(s)$. Now take any $r, s \in S$. Let $t \in \eta^u(r)$. Then $\eta(r)(u) \subseteq \eta(r)(t)$. By hypothesis, $\eta(r)(t) \subseteq \eta(rs)(t)$ and, so, $\eta(r)(u) \subseteq \eta(rs)(t)$. It follows that $t \in \eta^u(rs)$. Thus $\eta^u(rs) \supseteq \eta^u(r)$.

($\Leftarrow$) Suppose for each $u \in S$, $\eta^u \in L_S(S)$. If there exist $x, a, b \in S$ with $a \leq b$ such that $\eta(x)(b) \supset \eta(x)(a)$, then there exists $t \in S$ such that $\eta(x)(a) \subset \eta(x)(t) \subseteq \eta(x)(b)$. This implies $b \in \eta^t(x)$. But, as $a \notin \eta^t(x)$, we get a contradiction. Thus, for each $a, b \in S$ such that $a \leq b$, we have $\eta(x)(a) \supseteq \eta(x)(b)$. Now, again, if r, s exist in S such that $\eta(x)(s) \supset \eta(x)(rs)$, then there exists $v \in U$ such that $\eta(x)(rs) \subset \eta(x)(v) \subseteq \eta(x)(s)$. This implies that $s \in \eta^v(x)$. But, as $rs \notin \eta^v(x)$, we get a contradiction. Thus for each $r, s \in S$, $\eta(x)(rs) \supseteq \eta(x)(s)$, as required. □

Theorem 3.13. *Let (η, S) be a soft set over the soft sets. Then $\eta \in \mathcal{J}_S(S) \Leftrightarrow \eta^u \in J_S(S)$ for each $u \in S$.*

Proof. On the similar lines to the proof of theorem 3.12. □

Definition 3.14. Let (η, U) be a soft set over S with a parameter set S and $u \in U$. Then $(\chi_{\eta(u)}, V)$ is a soft set over the soft sets and, for each $v \in V$, $\chi_{\eta(u)}$ is defined as:

$$\chi_{\eta(u)}(v)(a) = \begin{cases} S & \text{if } a \in \eta(u), \\ \emptyset & \text{if } a \notin \eta(u). \end{cases}$$

Theorem 3.15. *A soft set (η, S) is a soft left(right) ideal over S $\Leftrightarrow \chi_{\eta(x)} \in \mathcal{L}_S(S)$ (resp. $\chi_{\eta(x)} \in \mathcal{R}_S(S)$).*

Proof. ($\Rightarrow$) Suppose that (η, S) is a soft left ideal over S. Now we show that, for each $y \in S$, $(\chi_{\eta(x)}(y), S) \in \mathcal{L}_S(S)$. Take any $r, s \in S$ such that $r \leq s$. If $\chi_{\eta(x)}(y)(s) = \emptyset$, then $\chi_{\eta(x)}(y)(r) \supseteq \chi_{\eta(x)}(y)(s)$. If $\chi_{\eta(x)}(y)(s) = S$, then $s \in \eta(x)$. As $\eta(x)$ is a left ideal, $r \in \eta(x)$. Thus $\chi_{\eta(x)}(y)(r) \supseteq \chi_{\eta(x)}(y)(s)$. Next take any $u, v \in S$. If $\chi_{\eta(x)}(y)(uv) = \emptyset$, then $uv \notin \eta(x)$. Infact $v \notin \eta(x)$, otherwise $uv \in \eta(x)$ which is a contradiction. So $\chi_{\eta(x)}(y)(v) \subseteq \chi_{\eta(x)}(y)(uv)$. If $\chi_{\eta(x)}(y)(uv) = S$, then consequently $\chi_{\eta(x)}(y)(uv) \supseteq \chi_{\eta(x)}(y)(v)$.

($\Leftarrow$) Assume that $\chi_{\eta(x)} \in \mathcal{L}_S(S)$ and $r, s \in S$ such that $r \leq s$. If $s \in \eta(x)$ for each $x \in S$, then $\chi_{\eta(x)}(y)(s) = S$. By hypothesis, $\chi_{\eta(x)}(y)(r) \supseteq \chi_{\eta(x)}(y)(s) = S$. It follows that $\chi_{\eta(x)}(y)(r) = S$. Thus $r \in \eta(x)$. Now take any $u \in S$ and $v \in \eta(x)$. Then $\chi_{\eta(x)}(y)(v) = S$. By hypothesis, $\chi_{\eta(x)}(y)(uv) \supseteq \chi_{\eta(x)}(y)(v) = S$. This implies $\chi_{\eta(x)}(y)(uv) = S$. So $uv \in \eta(x)$. Thus $\eta(x)$ is a left ideal of S for each $x \in S$. Hence (η, S) is a soft left ideal over S. □

Theorem 3.16. *A soft set (η, S) is a soft ideal over S $\Leftrightarrow \chi_{\eta(x)} \in \mathcal{J}_S(S)$.*

Proof. On the similar lines to the proof of theorem 3.15. □

Definition 3.17. A soft set over the soft sets (η, U) is said to be an *int-soft interior ideal over the soft sets* iff $\eta(u) \in I_S(S)$ for each $u \in U$.

We denote the set of all int-soft interior ideals over the soft sets by $\mathcal{I}_S(S)$.

Example 3.18. On the set $S = \{0, x, y, z\}$, define a binary operation $'\cdot'$ and order " $\leq$ " by following way:

$\cdot$	0	x	y	z
0	0	0	0	0
x	x	x	x	x
y	y	y	y	y
z	0	0	x	0

$$\leq := \{(0,0), (x,x), (y,y), (z,z), (0,x), (0,y)\}.$$

It follows immediately that $(S, \cdot, \leq)$ is an ordered semigroup. Let (η, U) be the soft set over the soft sets which is defined as

$$\eta(e_1)(a) = \begin{cases} S & \text{if } a \in \{0, x, y\}, \\ \{0, x, y\} & \text{if } a = d. \end{cases}$$

It is a routine to verify that (η, U) is a soft interior ideal over the soft set.

Theorem 3.19. *Let (η, S) a soft set over the soft sets. Then $\eta \in \mathcal{I}_S(S) \Leftrightarrow \eta^u \in I_S(S)$ for each $u \in S$.*

Proof. (⇒) Take $a, b \in S$, $a \leq b$ such that $b \in \eta^u(x)$ for each $u, x \in S$. Then $\eta(x)(b) \supseteq \eta(x)(u)$. As $(\eta, S) \in \mathcal{I}_S(S)$, $\eta(x)(a) \supseteq \eta(x)(b) \supseteq \eta(x)(u)$. Therefore $a \in \eta^u(x)$. Now take any $r, s \in S$ and $p \in \eta^u(x)$. Then $\eta(x)(p) \supseteq \eta(x)(u)$. By hypothesis $\eta(x)(rps) \supseteq \eta(x)(p)$, and so $\eta(x)(rps) \supseteq \eta(x)(u)$. It follows that $rps \in \eta^u(x)$.

(⇐) Suppose for each $u \in S$, $\eta^u \in I_S(S)$. If there exist $a, b \in S$ with $a \leq b$ such that $\eta(x)(b) \supset \eta(x)(a)$, then there exists $t \in S$ such that $\eta(x)(a) \subset \eta(x)(t) \subseteq \eta(x)(b)$ implies $b \in \eta^t(x)$ but $a \notin \eta^t(x)$ a contradiction. Thus for each and $a, b \in S$ such that $a \leq b$ implies $\eta(x)(a) \supseteq \eta(x)(b)$. Now again, if there exist $r, s, p \in S$ such that $\eta(x)(p) \supset \eta(x)(rps)$, then there exist $v \in S$ such that $\eta(x)(rps) \subset \eta(x)(v) \subseteq \eta(x)(p)$ implies $p \in \eta^v(x)$ but $rps \notin \eta^v(x)$ a contradiction. Thus for each $r, s, p \in S$, $\eta(x)(rps) \supseteq \eta(x)(p)$, as required. □

Theorem 3.20. *A soft set (η, S) is a soft interior ideal over S ⇔ $\chi_{\eta(x)} \in \mathcal{I}_S(S)$.*

Proof. (⇒) To show that for each $y \in S$, $\chi_{\eta(x)}(y) \in I_S(S)$. Take any $r, s \in S$ such that $r \leq s$. If $\chi_{\eta(x)}(y)(s) = \emptyset$, the $\chi_{\eta(x)}(y)(r) \supseteq \chi_{\eta(x)}(y)(s)$. If $\chi_{\eta(x)}(y)(s) = S$, then $s \in \eta(x)$. As $\eta(x)$ is an interior ideal, $r \in \eta(x)$. Thus $\chi_{\eta(x)}(y)(s) \subseteq \chi_{\eta(x)}(y)(r)$. Next take any $u, v, w \in S$. If $\chi_{\eta(x)}(y)(uvw) = \emptyset$, then $uvw \notin \eta(x)$. Infact $v \notin \eta(x)$ otherwise $uvw \in \eta(x)$ which is not possible. So $\chi_{\eta(x)}(y)(uvw) \supseteq \chi_{\eta(x)}(y)(v)$. If $\chi_{\eta(x)}(y)(uvw) = S$, then consequently $\chi_{\eta(x)}(y)(uvw) \supseteq \chi_{\eta(x)}(y)(v)$.

(⇐) Assume that $\chi_{\eta(x)} \in \mathcal{I}_S(S)$ and $r, s \in S$ such that $r \leq s$. If $s \in \eta_x$ for each $x \in S$. Then $\chi_{\eta(x)}(y)(s) = S$. By hypothesis $\chi_{\eta(x)}(y)$ is an int-soft interior ideal, so $\chi_{\eta(x)}(y)(r) \supseteq \chi_{\eta(x)}(y)(s) = S$. It follows that $\chi_{\eta(x)}(y)(r) = S$. Thus $r \in \eta(x)$. Now take any $u, v, w \in S$ such that $v \in \eta(x)$. Then $\chi_{\eta(x)}(y)(v) = S$. By hypothesis $\chi_{\eta(x)}(y)(uvw) \supseteq \chi_{\eta(x)}(y)(v) = S$ implies $\chi_{\eta(x)}(y)(uvw) = S$. So $uvw \in \eta_x$. Thus $\eta(x)$ is interior ideal of S for each $x \in S$, as required. □

Definition 3.21. A soft set over the soft sets (η, U) is said to be an *int-soft bi-deal over the soft sets* iff $\eta(u) \in B_S(S)$ for each $u \in U$.

We denote the set of all int-soft bi-ideals over the soft sets by $\mathcal{B}_S(S)$.

Example 3.22. Let S be the ordered semigroup of example 3.18. Let (η, U) be the soft set over the soft sets which is defined as

$$\eta(e_1)(a) = \begin{cases} \{0, x, y\} & \text{if } a = 0, \\ \{x, y\} & \text{if } a = x, \\ \{y\} & \text{if } a = y, \\ \{0\} & \text{if } a = z. \end{cases}$$

It is a routine to verify that (η, U) is a soft bi-ideal over the soft sets.

Theorem 3.23. *Let S be an ordered semigroup and (η, S) a soft set over the soft sets. Then $\eta \in \mathcal{B}_S(S)$ ⇔ $\eta^u \in B_S(S)$ for each $u \in S$.*

Proof. On similar lines to the proof of theorem 3.19. □

Theorem 3.24. *A soft set (η, S) is a soft interior ideal of S ⇔ $\chi_{\eta(x)} \in \mathcal{B}_S(S)$.*

Proof. On the similar lines to the proof of theorem 3.20. □

4. Coincidence of int-soft ideals, int-soft interior ideals and int-soft bi-ideals over the soft sets

In this section, we discuss the coincidence relations among these notions.

Lemma 4.1. *In an ordered semigroup,* $\eta \in \mathcal{J}_S(S) \Rightarrow \eta \in \mathcal{I}_S(S)$.

Proof. Straightforward. □

Remark 4.2. In general, in an ordered semigroup, $\eta \in \mathcal{I}_S(S) \nRightarrow \eta \in \mathcal{J}_S(S)$.

Example 4.3. On the set $S = \{0, y, z\}$, define a binary operation $'\cdot'$ and order " $\leq$ " by following way:

$\cdot$	0	y	z
0	0	0	0
y	0	z	0
z	0	0	0

$$\leq := \{(0,0), (y,y), (z,z), (0,y), (0,z)\}.$$

It follows immediately that $(S, \cdot, \leq)$ is an ordered semigroup. Let (η, U) be the soft set over the soft sets which is defined as follows:

$$\eta(e_1)(a) = \begin{cases} \{0, y\} & \text{if } a \in \{0, y\}, \\ \{0\} & \text{if } a = z. \end{cases}$$

It is a routine to verify that $\eta \in \mathcal{I}_S(S)$. Since $\eta(e_1)(yy) = \eta(e_1)(z) = \{0\} \nsupseteq \eta(e_1)(y) = \{0, y\}$, $\eta \in \mathcal{J}_S(S)$

Lemma 4.4. *If* S *is regular ordered semigroup, then* $\eta \in \mathcal{I}_S(S) \Rightarrow \eta \in \mathcal{J}_S(S)$.

Proof. Suppose $\eta \in \mathcal{I}_S(S)$ and $r, s \in S$. As S is regular there exits $x \in S$ such that $r \leq rxr$. Now for each $u \in U$, we have

$$\eta(u)(rs) \supseteq \eta(u)(rxrs) \supseteq \eta(u)(r).$$

Similarly for each $r, s \in S$, $\eta(u)(rs) \supseteq \eta(u)(s)$. □

Lemma 4.5. *If* S *is right weakly regular ordered semigroup, then* $\eta \in \mathcal{I}_S(S) \Rightarrow \eta \in \mathcal{J}_S(S)$.

Proof. Suppose $\eta \in \mathcal{I}_S(S)$ and take any $x, y \in S$. As S is right weakly regular, p, q, r, s exist in S such that $x \leq xpxq$ and $y \leq yrys$. Now for each $u \in U$, we have

$$\eta(u)(xy) \supseteq \eta(u)(xpxqy) = \eta(u)((xp)x(qy)) \supseteq \eta(u)(x)$$

and

$$\eta(u)(xy) \supseteq \eta(u)(xyrys) = \eta(u)((x)y(rys)) \supseteq \eta(u)(y).$$

Thus the proof is completed. □

Lemma 4.6. *If* S *is intra-regular ordered semigroup, then* $\eta \in \mathcal{I}_S(S) \Rightarrow \eta \in \mathcal{J}_S(S)$.

Proof. Suppose $\eta \in \mathcal{I}_S(S)$ and take any $x, y \in S$. As S is right weakly regular, p, q, r, s exist in S such that $x \leq px^2q$ and $y \leq ry^2s$. Now for each $u \in U$, we have

$$\eta(u)(xy) \supseteq \eta(u)(px^2qy) = \eta(u)((px)x(qy)) \supseteq \eta(u)(x)$$

and

$$\eta(u)(xy) \supseteq \eta(u)(xry^2s) = \eta(u)((xr)y(ys)) \supseteq \eta(u)(y),$$

as required. □

Lemma 4.7. *In a simple ordered semigroup S, $\eta \in \mathcal{I}_S(S) \Rightarrow \eta \in \mathcal{J}_S(S)$.*

Proof. Suppose that $\eta \in \mathcal{I}_S(S)$ and take any $a, b \in S$. As S is simple ordered semigroup, p, q, r, s exist in S such that $a \leq pbq$ and $b \leq ras$. Now for each $u \in U$, we have

$$\eta(u)(ab) \supseteq \eta(u)(pbqb) = \eta(u)((pb(qb)) \supseteq \eta(u)(b)$$

and

$$\eta(u)(ab) \supseteq \eta(u)(aras) = \eta(u)((ar)as) \supseteq \eta(u)(a),$$

which completes the proof. □

Corollary 4.8. *In a regular (right weakly regular, intra-regular, simple) ordered semigroup, $\eta \in \mathcal{I}_S(S) \Leftrightarrow \eta \in \mathcal{J}_S(S)$.*

Lemma 4.9. *In an ordered semigroup, $\eta \in \mathcal{J}_S(S) \Rightarrow \eta \in \mathcal{B}_S(S)$.*

Proof. Straightforward. □

Remark 4.10. In general, in an ordered semigroup, $\eta \in \mathcal{B}_S(S) \not\Rightarrow \eta \in \mathcal{J}_S(S)$.

Example 4.11. Let $S = \{0, y, z\}$. The binary operation " $\cdot$ " and order " $\leq$ " on S are defined by following way:

$\cdot$	0	y	z
0	0	0	0
y	0	0	0
z	0	0	y

$$\leq := \{(0,0), (y,y), (z,z), (0,y), (0,z)\}.$$

Then $(S, \cdot, \leq)$ is an ordered semigroup. Let (η, U) be the soft set over the soft set which is defined as:

$$\eta(e_1)(a) = \begin{cases} S & \text{if } a = 0, \\ \{0, y\} & \text{if } a = b, \\ \{0, z\} & \text{if } a = c. \end{cases}$$

It is a routine to verify that $\eta \in \mathcal{B}_S(S)$. Since $\eta(e_1)(xz) = \{x, y\} \not\supseteq \eta(e_1)(z) = \{x, z\}$. Therefore $\eta \notin \mathcal{J}_S(S)$.

Lemma 4.12. *If S is duo regular ordered semigroup, $\eta \in \mathcal{B}_S(S) \Rightarrow \eta \in \mathcal{J}_S(S)$.*

Proof. Suppose that S is duo regular ordered semigroup, $\eta \in \mathcal{B}_S(S)$ and $x, y \in S$. As S is duo, $(Sx]$ is an ideal of S. Since S is regular, $x \in (xSx]$. Now we have $xy \in (xSy]b \subseteq (x(Sx]]S \subseteq (x(Sx]S] \subseteq (xSx]$ implies that there exists $z \in S$ such that $xy \leq xzx$. Therefore

$$\eta(u)(xy) \supseteq \eta(u)(xzx) \supseteq \eta(u)(x) \cap \eta(u)(x) = \eta(u)(x).$$

Similarly for each $x, y \in S$, $\eta(u)(xy) \supseteq \eta(u)(y)$, as required.

□

Lemma 4.13. *If S is duo right weakly regular ordered semigroup, then $\eta \in \mathcal{B}_S(S) \Rightarrow \eta \in \mathcal{J}_S(S)$.*

Proof. Similar to the Proof of lemma 4.12. □

Lemma 4.14. *If S is left simple and right simple ordered semigroup, then $\eta \in \mathcal{B}_S(S) \Rightarrow \eta \in \mathcal{J}_S(S)$.*

Proof. Suppose $\eta \in \mathcal{B}_S(S)$ and $a, b \in S$. As S is left simple and right simple ordered semigroup there exits $p, q, r, s \in S$ such that $b \leq pa$ and $a \leq bq$. Now for each $u \in U$, we have

$$\eta(u)(ab) \supseteq \eta(u)(bqb) \supseteq \eta(u)(b) \cap \eta(u)(b) = \eta(u)(b)$$

and

$$\eta(u)(ab) \supseteq \eta(u)(apa) \supseteq \eta(u)(a) \cap \eta(u)(a) = \eta(u)(a).$$

Which completes the required proof. □

Corollary 4.15. *In a duo regular (duo right weakly regular, left simple and right simple) ordered semigroup, $\eta \in \mathcal{B}_S(S) \Leftrightarrow \eta \in \mathcal{J}_S(S)$.*

5. Conclusions and future work

Theory of soft sets plays an important role to discuss the uncertainty. Due to parameterize nature of soft set theory, it is able to solve those problems which were not easily handled by the previous defined mathematical tools probability theory, the theory of (intuitionistic) fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. The algebraic theory of soft sets has been extensively studied by many authors. Nowadays, the theory of soft sets has a rapid growth as mathematical tools to solve the uncertainty problems.

In the present paper, we introduce the int-soft left (right) ideals over the soft sets, int-soft interior ideals over the soft sets and int-soft bi-ideals over the soft sets. For any soft set over the soft sets (η, U), the soft set (η^u, U) is defined. Then we discuss the related properties of soft sets over the soft sets (η, U) and soft set (η^u, U) which is defined with the help of (η, U). Also, for any soft set (η, U) over S, we introduce a soft set over the soft sets $(\chi_{\eta(u)}, V)$, and we discuss their related properties.

The presented notions in this paper can be applied to the theory of hyperstructures, ordered hyperstructures, semirings, hemirings, groups and BCI/BCK algebras. One may also apply this concept to study some applications in many fields like decision making, knowledge base systems, data analysis, etc.

Acknowledgments

The authors are grateful to the anonymous referee(s) for a careful checking of the details and for helpful comments that improved the overall presentation of this paper.

Conflict of interest

The authors declare no conflict of interest.

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