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*Research article*

# On generalized inverse sum indeg index and energy of graphs

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**Abstract:** Topological indices are used to predict certain phsio-chemical properties of the chemical compounds. Among all indices, degree based indices are of vital importance. In this paper, we introduce generalized inverse sum indeg index and generalized inverse sum indeg energy of graphs. We study the generalized inverse sum indeg index and energy from an algebraic point of view. Extremal values of this index for some graph classes are determined. Some spectral properties of generalized inverse sum indeg matrix are studied. We also find Nordhaus-Gaddum-type results for generalized inverse sum indeg index spectral radius and energy.

**Keywords:** generalized ISI index; extremal graphs; generalized ISI energy; generalized ISI spread of graphs; Nordhaus-Gaddum-type results
**Mathematics Subject Classification:** 05C07, 05C35, 05C50

## 1. Introduction

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of neighbours of a vertex $w$ in $G$ is called the degree of $w$, denoted by $d_G(w)$. If vertices $w$ and $z$ are connected by an edge, we denote it by $wz$. The order $n(G)$ of a graph $G$ is given by $n(G) = |V(G)|$. The size $e(G)$ of a graph $G$ is defined by $e(G) = |E(G)|$. For any $w \in V(G)$, $N_G(w)$ is the set of all vertices adjacent to $w$ in graph $G$. The graph $G - \{w\}$ is a graph formed from $G$ by removing the vertex $w$ of $G$ and all edges incident with $w$. The largest (smallest) degree of $G$ is the largest (smallest) vertex degree in $G$, represented as $\Delta_G$ ($\delta_G$). A graph of order $n(G)$, size $e(z)$, maximum degree $\Delta_G$ and minimum degree $\delta_G$ is denoted by $G(n(G), e(G), \Delta_G, \delta_G)$ and a graph of order $n$ and size $m$ is denoted by $G_n^m$. Throughout this paper, we consider simple and connected graphs.

A star graph $S_n$ on $n$ vertices is a tree consisting of a central vertex adjacent to $n - 1$ pendant vertices. An $n$-vertex cycle $C_n$ ($n \geq 3$) is a graph with $V(C_n) = \{v_1, \ldots, v_n\}$ and $E(C_n) = \{v_j v_{j+1} \mid j = 1, 2, \ldots, n-1\} \cup \{v_n v_1\}$. A simple graph of order $n$ in which every vertex is joined by an edge to other

$n-1$ vertices is said to be a complete graph represented by $K_n$. If we can split $V(G)$ of $G$ into two disjoint sets $X_1$ and $X_2$ with the property that no two vertices of the same set are adjacent is called a bipartite graph. A complete bipartite graph $K_{m,n}$ is a bipartite graph with $|X_1| = m$, $|X_2| = n$ and each vertex in $X_1$ is adjacent to eaxh vertex in $X_2$.

A topological index TI($G$) of a graph $G$ is a molecular descriptor which is a conversion of a molecular structure into some real number. In theoretical chemistry, many of the molecular descriptors are considered and have found applications, see [1–18].

A degree based topological index of a graph $G$ can be represented as

$$\mathrm{TI}(G) = \sum_{wz\in E(G)} \mathscr{F}(d_G(w), d_G(z)),$$

where $\mathscr{F}$ is a function with the property $\mathscr{F}(x, y) = \mathscr{F}(y, x)$.

The inverse sum indeg (henceforth, ISI) index of a graph $G$ was introduced by Vukičević and Gašperov [19] and defined as

$$\mathrm{ISI}(G) = \sum_{wz\in E(G)} \frac{d_G(w)\, d_G(z)}{d_G(w) + d_G(z)}.$$

In this paper, we introduce generalized inverse sum indeg (henceforth, ISI) index and generalized ISI energy of graphs. Our strong motivation to define generalized ISI index and energy is that a lot of the degree based topological indices and energies are derived from it by giving the specific values to the parameters $\alpha$, $\beta$. We now define generalized inverse sum indeg index as

$$\mathrm{S}_{\alpha,\beta}(G) = \sum_{wz\in E(G)} \frac{(d_G(w)\, d_G(z))^{\alpha}}{(d_G(w) + d_G(z))^{\beta}}, \tag{1.1}$$

where $\alpha$ and $\beta$ are real numbers.

The adjacency matrix $A(G) = [a_{ij}]_{n\times n}$ of an $n$-vertex graph $G$ is defined as

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

The $A$-characteristic polynomial of $G$ is the polynomial of the form:

$$\begin{aligned} \Phi(G, \lambda) &= \det(A(G) - \lambda I_n) \\ &= \lambda^n + \sum_{i=1}^{n} a_i\, \lambda^{n-i}, \end{aligned}$$

where $I_n$ is the identity matrix of order $n$. The $A$-eigenvalues of $G$ are the $A$-eigenvalues of $A(G)$.

Let $\lambda_1, \ldots, \lambda_n$ be the $A$-eigenvalues of a graph $G$. Gutman [20] defined the energy of $G$ as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Zangi et al. [21] defined the ISI matrix $S(G) = [s_{ij}]_{n\times n}$ of an $n$-vertex graph $G$ as:

$$s_{ij} = \begin{cases} \frac{d_G(v_i)\, d_G(v_j)}{d_G(v_i)+d_G(v_j)} & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

The $S$-characteristic polynomial of $G$ is given by:

$$\begin{aligned}\Phi_S(G,\rho) &= \det(S(G)-\rho I_n)\\ &= \rho^n+\sum_{i=1}^{n} b_i\,\rho^{n-i}.\end{aligned}$$

The $S$-eigenvalues of $G$ are the $S$-eigenvalues of $S(G)$.

Let $\rho_1,\ldots,\rho_n$ be the $S$-eigenvalues of $G$. Zangi et al. [21] defined the ISI energy of $G$ as

$$E_{\mathrm{ISI}}(G)=\sum_{i=1}^{n}|\rho_i|.$$

We can now define a generalized ISI matrix $A_{\alpha,\beta}(G)=[b_{ij}]_{n\times n}$ of an $n$-vertex graph $G$ as

$$b_{ij}=\begin{cases}\frac{(d_G(v_i)\,d_G(v_j))^{\alpha}}{(d_G(v_i)+d_G(v_j))^{\beta}} & \text{if } v_iv_j\in E(G),\\ 0 & \text{otherwise.}\end{cases}$$

The $A_{\alpha,\beta}$-characteristic polynomial of $G$ is given by:

$$\begin{aligned}\psi(G,\sigma) &= \det(A_{\alpha,\beta}(G)-\sigma I_n)\\ &= \sigma^n+\sum_{i=1}^{n} c_i\,\sigma^{n-i}.\end{aligned}$$

The $A_{\alpha,\beta}$-eigenvalues of $G$ are the $A_{\alpha,\beta}$-eigenvalues of $A_{\alpha,\beta}(G)$.

Let $\sigma_1,\ldots,\sigma_n$ be the $A_{\alpha,\beta}$-eigenvalues of $G$. Then we define the generalized ISI energy of graph $G$ as

$$E_{\alpha,\beta}(G)=\sum_{i=1}^{n}|\sigma_i|. \tag{1.2}$$

We list here some of the degree based indices and energies of a graph $G$ that can be obtained from the generalized ISI index and energy by only giving specific values to the parameters $\alpha,\beta$.

1. If $\alpha=0$ and $\beta=-1$, then $\mathrm{S}_{\alpha,\beta}(G)=\mathrm{M}_1(G)$ is the first Zagreb index [3] and matrix $A_{0,-1}(G)$ is the first Zagreb matrix [12]. The energy corresponding to $A_{0,-1}(G)$ is the first Zagreb energy $ZE_1(G)$ also introduced in [12]. Note that $ZE_1(G)=E_{0,-1}(G)$.
2. If $\alpha=0$ and $\beta=1/2$, then $\mathrm{S}_{\alpha,\beta}(G)=\mathrm{SCI}(G)$ is the sum-connectivity index [22] and matrix $A_{0,1/2}(G)$ is the sum-connectivity matrix [23]. The energy corresponding to $A_{0,1/2}(G)$ is the sum-connectivity energy $SE(G)$, introduced in [23]. It is easy to see that $SE(G)=E_{0,\,1/2}(G)$.
3. If $\alpha=0$ and $\beta=-\alpha$ then $\mathrm{S}_{\alpha,\beta}(G)=\chi_\alpha(G)$ is the general sum connectivity index [11] and matrix $A_{0,-\alpha}(G)$ is the general sum-connectivity matrix [24]. The energy corresponding to $A_{0,-\alpha}(G)$ is the general sum-connectivity energy $GSE(G)$, defined in [24]. See that $GSE(G)=E_{0,-\alpha}(G)$.
4. If $\alpha=1$ and $\beta=0$ then $\mathrm{S}_{\alpha,\beta}(G)=\mathrm{M}_2(G)$ is the second Zagreb index [11] and matrix $A_{1,0}(G)$ is the second Zagreb matrix [12]. The energy corresponding to $A_{1,0}(G)$ is the second Zagreb energy $ZE_2(G)$, introduced in [12]. Note $ZE_2(G)=E_{1,0}(G)$.

5. If $\alpha = -1/2$ and $\beta = 0$ then $\mathrm{S}_{\alpha,\beta}(G) = \mathrm{R}(G)$ is the Randić index [13] and matrix $A_{-1/2,0}(G)$ is the Randić matrix. The energy corresponding to $A_{-1/2,0}(G)$ is the Randić energy $RE(G)$, defined in [25, 26]. See that $RE(G) = E_{-1/2,0}(G)$.
6. If $\beta = 0$ then $\mathrm{S}_{\alpha,\beta}(G) = \mathrm{R}_\alpha(G)$ is the generalized form of Randić index (also known as general product-connectivity index) [27] and matrix $A_{\alpha,0}(G)$ is the general Randić matrix. The energy corresponding to $A_{\alpha,0}(G)$ is the general Randić energy $R_\alpha E(G)$, introduced in [28]. It is easy to see that $R_\alpha E(G) = E_{\alpha,0}(G)$.
7. If $\alpha = 1/2$ and $\beta = 1$ then $2\,\mathrm{S}_{\alpha,\beta}(G) = \mathrm{GA}(G)$ is the geometric-arithmetic index [14] and matrix $A_{1/2,1}(G)$ is the geometric-arithmetic matrix. The energy corresponding to $A_{1/2,\,1}(G)$ is the geometric-arithmetic energy $GAE(G)$, defined in [15]. Note that $GAE(G) = 2E_{1/2,\,1}(G)$.
8. If $\alpha = 1$ and $\beta = 1$ then $\mathrm{S}_{\alpha,\beta}(G) = \mathrm{ISI}(G)$ is the inverse sum indeg index [19] and matrix $A_{1,1}(G)$ is the inverse sum indeg matrix. The energy corresponding to $A_{1,1}(G)$ is the inverse sum indeg energy $ISIE(G)$ introduced in [21]. See that $ISIE(G) = E_{1,1}(G)$.

For study of more degree-based topological indices, see [29] and references therein.

In this paper, we study the generalized inverse sum indeg index and energy from an algebraic point of view. Extremal values of this index for some graph classes are determined. Some spectral properties of generalized inverse sum indeg matrix are studied. We also find Nordhaus-Gaddum-type results for generalized inverse sum indeg index spectral radius and energy.

## 2. Basic results

Under certain conditions, we now determine the monotonicity of the generalized ISI index of a graph $G$ when new edges are added in the graph.

**Lemma 2.1.** *Let $w$ and $z$ be two non-adjacent vertices of a graph $G$. Also let $G+wz$ is the graph formed from $G$ by joining $w$ and $z$ by an edge $wz$. If $\alpha,\beta \in \mathbb{R}$ with $\alpha \geq 0$ and $\alpha \geq \beta$, then $\mathrm{S}_{\alpha,\beta}(G+wz) > \mathrm{S}_{\alpha,\beta}(G)$.*

*Proof.* If $\alpha,\beta \in \mathbb{R}$ with $\alpha \geq 0$ and $\alpha \geq \beta$, then for any real numbers $x, y \geq 1$, we have $\left(1 + \frac{1}{x}\right)^\alpha \geq \left(1 + \frac{1}{x+y}\right)^\beta$. This implies $\frac{(x+1)^\alpha}{(x+y+1)^\beta} \geq \frac{x^\alpha}{(x+y)^\beta}$. Hence $\frac{((x+1)y)^\alpha}{(x+y+1)^\beta} \geq \frac{(xy)^\alpha}{(x+y)^\beta}$.

Let $N_G(w) = \{w_1, \ldots, w_r\}$ and $N_G(z) = \{z_1, \ldots, z_t\}$. Then

$$\begin{aligned}
\mathrm{S}_{\alpha,\beta}(G+wz) - \mathrm{S}_{\alpha,\beta}(G) &= \left[\frac{((d_G(w)+1)\,(d_G(z)+1))^\alpha}{(d_G(w)+d_G(z)+2)^\beta}\right] \\
&\quad + \sum_{i=1}^{r}\left[\frac{((d_G(w)+1)\,d(_G(w_i)))^\alpha}{(d_G(w)+d_G(w_i)+1)^\beta} - \frac{(d_G(w)\,d_G(w_i))^\alpha}{(d_G(w)+(d_G(w_i))^\beta}\right] \\
&\quad + \sum_{j=1}^{t}\left[\frac{((d_G(z)+1)\,d_G(z_j))^\alpha}{(d_G(z)+d_G(z_j)+1)^\beta} - \frac{(d_G(z)\,d_G(z_j))^\alpha}{(d_G(z)+(d_G(z_j))^\beta}\right] \\
&> 0,
\end{aligned}$$

where $\left[\frac{((d_G(w)+1)\,(d_G(z)+1))^\alpha}{(d_G(w)+d_G(z)+2)^\beta}\right] > 0$. Therefore $\mathrm{S}_{\alpha,\beta}(G+wz) > \mathrm{S}_{\alpha,\beta}(G)$. □

Two simple graphs $G_1$ and $G_2$ are said to be isomorphic if there exists a bijection $\phi : V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. We write $G_1 \cong G_2$ if $G_1$ and $G_2$ are isomorphic.

Next corollary is obtained from Lemma 2.1.

**Corollary 2.2.** *Let $\alpha,\beta \in \mathbb{R}$ with $\alpha \geq 0$ and $\alpha \geq \beta$. Suppose $T$ is a spanning tree of graph $G$ with $n(G) = n$ and $G \ncong T$. Then $\mathrm{S}_{\alpha,\beta}(G) > \mathrm{S}_{\alpha,\beta}(T)$.*

Next theorem relates $\mathrm{S}_{\alpha,\beta}(G)$ with $\chi_\beta(G)$.

**Theorem 2.3.** *Suppose $\mathcal{G} = G(n, m, \Delta_G, \delta_G)$ is a graph.*

(1). *If $\alpha \geq 0$, then $\mathrm{S}_{\alpha,\beta}(\mathcal{G}) \geq \frac{m^2\delta_{\mathcal{G}}^{2\alpha}}{\chi_\beta(\mathcal{G})}$.*

(2). *If $\alpha \leq 0$, then $\mathrm{S}_{\alpha,\beta}(\mathcal{G}) \geq \frac{m^2\Delta_{\mathcal{G}}^{2\alpha}}{\chi_\beta(\mathcal{G})}$.*

*In both cases, the inequality becomes equality if $\mathcal{G}$ is a regular graph.*

*Proof.* (1). By Arithmetic mean-Harmonic mean inequality, we have

$$\frac{m}{\mathrm{S}_{\alpha,\beta}(\mathcal{G})} = \frac{m}{\sum\limits_{wz\in E(\mathcal{G})} \frac{(d_{\mathcal{G}}(w)\, d_{\mathcal{G}}(z))^\alpha}{(d_{\mathcal{G}}(w)+d_{\mathcal{G}}(z))^\beta}} \leq \frac{1}{m} \sum_{wz\in E(\mathcal{G})} \frac{(d_{\mathcal{G}}(w) + d_{\mathcal{G}}(z))^\beta}{(d_{\mathcal{G}}(w)\, d_{\mathcal{G}}(z))^\alpha}.$$

Now if $\alpha \geq 0$, then $(d_{\mathcal{G}}(w)\, d_{\mathcal{G}}(z))^\alpha \geq \delta_{\mathcal{G}}^{2\alpha}$. Therefore

$$\frac{1}{m} \sum_{wz\in E(\mathcal{G})} \frac{(d_{\mathcal{G}}(w) + d_{\mathcal{G}}(z))^\beta}{(d_{\mathcal{G}}(w)\, d_{\mathcal{G}}(z))^\alpha} \leq \frac{1}{m} \sum_{wz\in E(\mathcal{G})} \frac{(d_{\mathcal{G}}(w) + d_{\mathcal{G}}(z))^\beta}{\delta_{\mathcal{G}}^{2\alpha}} = \frac{\chi_\beta(\mathcal{G})}{m\delta_{\mathcal{G}}^{2\alpha}}.$$

Hence $\mathrm{S}_{\alpha,\beta}(\mathcal{G}) \geq \frac{m^2\delta_{\mathcal{G}}^{2\alpha}}{\chi_\beta(\mathcal{G})}$. Now the above inequality becomes equality if and only if for every $wz \in E(\mathcal{G})$, $\frac{(d_{\mathcal{G}}(w)\, d_{\mathcal{G}}(z))^\alpha}{(d_{\mathcal{G}}(w)+d_{\mathcal{G}}(z))^\beta} = \frac{b^{2\alpha-\beta}}{2^\beta}$, where $b$ is some positive constant. This is possible if and only if $G$ is a $b$-regular graph.

Similarly one can prove (2). □

Now we give relationship between $\mathrm{S}_{\alpha,\beta}(G)$ and $\mathrm{R}_\alpha(G)$.

**Theorem 2.4.** *Suppose $\mathcal{G} = G(n, m, \Delta_G, \delta_G)$ is a graph.*

(1). *If $\beta \geq 0$, then $\frac{\mathrm{R}_\alpha(\mathcal{G})}{2^\beta\Delta_{\mathcal{G}}^\beta} \leq \mathrm{S}_{\alpha,\beta}(\mathcal{G}) \leq \frac{\mathrm{R}_\alpha(\mathcal{G})}{2^\beta\delta_{\mathcal{G}}^\beta}$.*

(2). *If $\beta \leq 0$, then $\frac{\mathrm{R}_\alpha(\mathcal{G})}{2^\beta\delta_{\mathcal{G}}^\beta} \leq \mathrm{S}_{\alpha,\beta}(\mathcal{G}) \leq \frac{\mathrm{R}_\alpha(\mathcal{G})}{2^\beta\Delta_{\mathcal{G}}^\beta}$.*

*In both cases, the inequality becomes equality if $G$ is a regular graph.*

*Proof.* (1). If $\beta \geq 0$, then $(d_{\mathcal{G}}(w) + d_{\mathcal{G}}(z))^\beta \leq (2\Delta_{\mathcal{G}})^\beta$ and $(d_{\mathcal{G}}(w) + d_{\mathcal{G}}(z))^\beta \geq (2\delta_{\mathcal{G}})^\beta$. Hence

$$\mathrm{S}_{\alpha,\beta}(\mathcal{G}) = \sum_{wz\in E(\mathcal{G})} \frac{(d_{\mathcal{G}}(w)\, d_{\mathcal{G}}(z))^\alpha}{(d_{\mathcal{G}}(w) + d_{\mathcal{G}}(z))^\beta} \geq \frac{\sum\limits_{wz\in E(\mathcal{G})} (d_{\mathcal{G}}(w)\, d_{\mathcal{G}}(z))^\alpha}{2^\beta\Delta_{\mathcal{G}}^\beta} = \frac{\mathrm{R}_\alpha(\mathcal{G})}{2^\beta\Delta_{\mathcal{G}}^\beta}.$$

$$\mathrm{S}_{\alpha,\beta}(\mathcal{G}) = \sum_{wz\in E(\mathcal{G})} \frac{(d_{\mathcal{G}}(w)\, d_{\mathcal{G}}(z))^\alpha}{(d_{\mathcal{G}}(w) + d_{\mathcal{G}}(z))^\beta} \leq \frac{\sum\limits_{wz\in E(\mathcal{G})} (d_{\mathcal{G}}(w)\, d_{\mathcal{G}}(z))^\alpha}{2^\beta\delta_{\mathcal{G}}^\beta} = \frac{\mathrm{R}_\alpha(\mathcal{G})}{2^\beta\delta_{\mathcal{G}}^\beta}.$$

Clearly the equality holds if and only if $\mathcal{G}$ is a regular graph.

Part (2) can be proved analogously. □

By direct computation, we obtain the following results.

**Theorem 2.5.** *Suppose $G_1, G_2, \ldots, G_r$ are components of a graph $G$. Then $E_{\alpha,\beta}(G) = \sum\limits_{j=1}^{r} E_{\alpha,\beta}(G_j)$.*

**Theorem 2.6.** *Suppose $G$ is an $n$-vertex and $k$-regular graph. Then $E_{\alpha,\beta}(G) = \frac{k^{2\alpha}}{2^\beta\, k^\beta}\, E(G)$.*

**Theorem 2.7.** $E_{\alpha,\beta}(K_{m,n}) = \frac{2(mn)^{\frac{1}{2}+\alpha}}{(m+n)^\beta}$.

*Proof.* Since $A_{\alpha,\beta}(K_{m,n}) = \frac{(mn)^\alpha}{(m+n)^\beta} A(K_{m,n})$, we have $E_{\alpha,\beta}(K_{m,n}) = \frac{(mn)^\alpha}{(m+n)^\beta} E(K_{m,n}) = \frac{2(mn)^{\frac{1}{2}+\alpha}}{(m+n)^\beta}$. □

Using Theorem 2.6, we get the following two results.

**Theorem 2.8.** $E_{\alpha,\beta}(C_n) = 4^{\alpha-\beta} E(C_n)$.

**Theorem 2.9.** $E_{\alpha,\beta}(K_n) = 2^{1-\beta}\,(n-1)^{2\alpha-\beta+1}$.

## 3. Extremal values of generalized ISI index

In this section, we find extremal values of graphs and bounds with respect to generalized ISI index in some graph classes.

**Theorem 3.1.** *Suppose $T$ is a tree with $n(T) = n$. If $\alpha = \beta$ and $0 \le \alpha \le 1$, then*

$$\mathrm{S}_{\alpha,\beta}(T) \ge \frac{(n-1)(n-1)^\alpha}{n^\alpha},$$

*where the inequality becomes equality if $T \cong S_n$.*

*Proof.* We prove the result by induction on $n$.

For $n = 1, 2, 3$, the only tree is the star graph $S_n$. So the statement follows trivially for $n \le 3$. Now assume that the statement holds true for $n \ge 4$.

Suppose $T$ is a tree with $n(T) = n$. Let $vw$ be a pendent edge of $T$ with $d_T(w) = 1$ and $d_T(v) = t$. As $n \ge 4$, we have $2 \le t \le n$. Further, since $T$ is not isomorphic to a star, we have that there exists at least one neighbor $u$ of $v$ in $T$ with $d_T(u) \ge 2$. Let $N_T(v) \backslash \{w, u\} = \{v_1, \ldots, v_{t-2}\}$.

Let $\widetilde{T} = T - w$. Then $n(\widetilde{T}) = n - 1$. Now

$$\begin{aligned} \mathrm{S}_{\alpha,\beta}(T) - \mathrm{S}_{\alpha,\beta}(\widetilde{T}) \;&=\; \left(\frac{t}{t+1}\right)^\alpha + \left[\left(\frac{d_T(u)\, t}{d_T(u)+t}\right)^\alpha - \left(\frac{d_T(u)(t-1)}{d_T(u)+t-1}\right)^\alpha\right] \\ &\quad + \sum_{i=1}^{t-2}\left[\left(\frac{d_T(v_i)\, t}{d_T(v_i)+t}\right)^\alpha - \left(\frac{d_T(v_i)(t-1)}{d_T(v_i)+t-1}\right)^\alpha\right]. \end{aligned}$$

Let $y > 0$ and define

$$g(y) = \left(\frac{yt}{y+t}\right)^\alpha - \left(\frac{y(t-1)}{y+t-1}\right)^\alpha.$$

Then

$$g'(y) = \alpha y^{\alpha-1}\left[\left(\frac{t}{y+t}\right)^{\alpha+1} - \left(\frac{t-1}{y+t-1}\right)^{\alpha+1}\right] = \alpha y^{\alpha-1}\left[\frac{(yt+t^2-t)^{\alpha+1} - (yt+t^2-t-y)^{\alpha+1}}{(y+t)^{\alpha+1}(y+t-1)^{\alpha+1}}\right].$$

As $t \geq 2$, $0 \leq \alpha \leq 1$ and $y > 0$, we have $(y+t)^{\alpha+1} > 0$ and $(y+t-1)^{\alpha+1} > 0$. Also $(yt+t^2-t) > (yt+t^2-t-y)$. Therefore $(yt+t^2-t)^{\alpha+1} > (yt+t^2-t-y)^{\alpha+1}$. Hence $g'(y) > 0$ and thus $g(y)$ is strictly increasing for $y > 0$. Also $2^\alpha > 1$ for $0 \leq \alpha \leq 1$, $d_T(v_i) \geq 1$ and $d_T(u) \geq 2$, we have

$$\begin{aligned}
\mathrm{S}_{\alpha,\beta}(T) - \mathrm{S}_{\alpha,\beta}(\widetilde{T}) &\geq \left(\frac{t}{t+1}\right)^{\alpha} + \left[\left(\frac{2\,t}{2+t}\right)^{\alpha} - \left(\frac{2(t-1)}{t+1}\right)^{\alpha}\right] + \sum_{i=1}^{t-1}\left[\left(\frac{t}{t+1}\right)^{\alpha} - \left(\frac{t-1}{t}\right)^{\alpha}\right] \\
&\geq \left(\frac{t}{t+1}\right)^{\alpha} + \left[\left(\frac{2\,t}{2+t}\right)^{\alpha} - \left(\frac{2(t-1)}{t+1}\right)^{\alpha}\right] > \left(\frac{t}{t+1}\right)^{\alpha} + 2^{\alpha}\left(\frac{t}{2+t}\right)^{\alpha} \\
&> \left(\frac{t}{t+2}\right)^{\alpha} + 2^{\alpha}\left(\frac{t}{t+2}\right)^{\alpha} = \left(\frac{t}{t+2}\right)^{\alpha}(1+2^{\alpha}) > 2\left(\frac{t}{t+2}\right)^{\alpha}.
\end{aligned}$$

Since $t \geq 2$, we have

$$\begin{aligned}
\mathrm{S}_{\alpha,\beta}(T) - \mathrm{S}_{\alpha,\beta}(\widetilde{T}) &> 2\left(\frac{t}{t+2}\right)^{\alpha} > 1 \\
&\geq \frac{(n-1)(n-1)^{\alpha}}{n^{\alpha}} - \frac{(n-2)(n-2)^{\alpha}}{(n-1)^{\alpha}} = \mathrm{S}_{\alpha,\beta}(S_n) - \mathrm{S}_{\alpha,\beta}(S_{n-1})
\end{aligned}$$

Therefore by induction hypothesis $\mathrm{S}_{\alpha,\beta}(T) - \mathrm{S}_{\alpha,\beta}(S_n) > \mathrm{S}_{\alpha,\beta}(\widetilde{T}) - \mathrm{S}_{\alpha,\beta}(S_{n-1}) \geq 0$. This concludes the proof by induction and clearly equality holds if $T \cong S_n$. □

Next theorem gives the minimal graph with respect to generalized ISI index in class of all connected graphs with smallest degree 2.

**Theorem 3.2.** *Among all connected graphs $G_n^m$ with smallest degree 2, we have*

(1). *If $\alpha \geq 0$ and $\beta \leq 0$, then $\mathrm{S}_{\alpha,\beta}(G_n^m) \geq m\,4^{\alpha-\beta}$.*
(2). *If $\alpha = \beta \geq 0$, then $\mathrm{S}_{\alpha,\beta}(G_n^m) \geq m$.*

*In both cases, the inequality becomes equality for $G_n^m \cong C_n$.*

*Proof.* (1). If $\alpha \geq 0$ and $\beta \leq 0$, then for any $w, z \in V(G_n^m)$, we have $(d_{G_n^m}(w)\, d_{G_n^m}(z))^{\alpha} \geq 4^{\alpha}$ and $(d_{G_n^m}(w) + d_{G_n^m}(z))^{\beta} \leq 4^{\beta}$. Therefore

$$\mathrm{S}_{\alpha,\beta}(G_n^m) = \sum_{wz \in E(G_n^m)} \frac{(d_{G_n^m}(w)\, d_{G_n^m}(z))^{\alpha}}{(d_{G_n^m}(w) + d_{G_n^m}(z))^{\beta}} \geq m\,4^{\alpha-\beta}.$$

Now $\mathrm{S}_{\alpha,\beta}(G_n^m) = m\,4^{\alpha-\beta}$ if and only if $d_{G_n^m}(w) = d_{G_n^m}(v) = 2$ for every edge $wz \in E(G_n^m)$. Therefore the inequality becomes equality for $G_n^m \cong C_n$.

(2). Since $\delta_{G_n^m} = 2$, therefore $(d_{G_n^m}(w)\, d_{G_n^m}(z)) \geq (d_{G_n^m}(w) + d_{G_n^m}(z))$. Hence

$$\mathrm{S}_{\alpha,\beta}(G_n^m) = \sum_{wz \in E(G_n^m)} \frac{(d_{G_n^m}(w)\, d_{G_n^m}(z))^{\alpha}}{(d_{G_n^m}(w) + d_{G_n^m}(z))^{\beta}} \geq m.$$

Similar to the proof of Part (1), the inequality becomes equality for $G_n^m \cong C_n$. □

Any subset of pairwise non-adjacent vertices of a graph $G$ is called an independent set of a graph $G$. The maximum size of an independent set of a graph $G$ is called the independence number of $G$. The join $G \vee H$ of two graphs $G$ and $H$ is formed by making every vertex of $G$ adjacent to every vertex of $H$.

The proof of next theorem is similar to the proof of Theorem 3.2 [30] and thus omitted.

**Theorem 3.3.** *Let $\alpha \geq 0$ is a real number and $\alpha \geq \beta$ when $\beta \in \mathbb{R}$. Also let $n \geq 4$ and $G$ be a connected graph with $n(G) = n \geq 4$ and independence number $\xi$. Then*

$$\mathrm{S}_{\alpha,\beta}(G) \leq \frac{(n-\xi)(n-\xi-1)(n-1)^{2\alpha-\beta}}{2^{\beta+1}} + \xi(n-\xi)\frac{((n-\xi)(n-1))^{\alpha}}{(2n-\xi-1)^{\beta}},$$

*where the inequality becomes equality when $G \cong \overline{K}_{\xi} \vee K_{n-\xi}$.*

## 4. Spectral radius and spread of the generalized ISI matrix

In this section, we give lower and upper bounds on spectral radius and spread of graphs with respect to generalized ISI matrix. For any complex $n \times n$ matrix $M$ with eigenvalues $\mu_1, \ldots, \mu_n$, the spread $s(M)$ of $M$ is introduced in [10] and is defined as $s(M) = \max_{i,j} |\mu_i - \mu_j|$.

Let $\sigma_1 \geq \cdots \geq \sigma_n$ be the $A_{\alpha,\beta}$ -eigenvalues of a simple graph $G$. Then spread $A_{\alpha,\beta}(G)$ is defined as $s(A_{\alpha,\beta}(G)) = \sigma_1 - \sigma_n$, since the eigenvalues $\sigma_1, \ldots, \sigma_n$ are all real.

For convenience, we define some notations. We denote determinant of $A_{\alpha,\beta}(G)$ by $\det(A_{\alpha,\beta}(G))$. Let

$$Q = \sum_{1 \leq i < j \leq n} \frac{(d_G(v_i)\, d_G(v_j))^{2\alpha}}{(d_G(v_i) + d_G(v_j))^{2\beta}}, \quad \Omega = \det(A_{\alpha,\beta}(G)).$$

We first give some lemmas that are used to prove our main results. The proof is straight forward.

**Lemma 4.1.** *Let $G$ be an $n$-vertex graph and $\sigma_1, \ldots, \sigma_n$ be its $A_{\alpha,\beta}$ -eigenvalues. Then*

(1). $\sum_{i=1}^{n} \sigma_i = 0,$

(2). $\sum_{i=1}^{n} \sigma_i^2 = 2Q.$

**Lemma 4.2** (Horn and Johnson [5])**.** *Let $A_1 = [a_{ij}]_{n\times n}$ and $A_2 = [b_{ij}]_{n\times n}$ be $n \times n$ symmetric and non-negative matrices. If $A_1 \geq A_2$, that is, $a_{ij} \geq b_{ij}$ for all $i, j = 1, \ldots, n$, then $\eta_1(A_1) \geq \eta_1(A_2)$, where $\eta_1(A_k)$, $k = 1, 2$ is the largest eigenvalue of the respective matrix.*

**Theorem 4.3** (Hong [6])**.** *Let $G_n^m$ be a connected graph with A-eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then*

$$\lambda_1 \leq \sqrt{2m - n + 1},$$

*where the equality holds if and only if $G_n^m \cong S_n$ or $G_n^m \cong K_n$.*

**Theorem 4.4** (Cao [31])**.** *Let $\mathcal{G} = G(n, m, \Delta_G, \delta_G)$ be a graph with A-eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $\delta_{\mathcal{G}} \geq 1$. Then*

$$\lambda_1 \leq \sqrt{2m - \delta_{\mathcal{G}}(n-1) + (\delta_{\mathcal{G}} - 1)\Delta_{\mathcal{G}}}.$$

**Lemma 4.5** (Zhang [32]). *If $C$ is a symmetric matrix of order $n$ with eigenvalues $\eta_1 \geq \cdots \geq \eta_n$, then for any $y \in \mathbb{R}^n$ with $y \neq 0$,*

$$y^T C y \leq \eta_1 y^T y,$$

*where $y^T$ is the transpose of $y$. Equality holds if and only if $y$ is an eigenvector of $C$ corresponding to the eigenvalue $\eta_1$.*

Now we give bounds on largest $A_{\alpha,\beta}$-eigenvalue of a graph.

**Theorem 4.6.** *Let $n \geq 2$. Also let $\mathcal{G} = G(n, m, \Delta_G, \delta_G)$ be a connected graph with $A_{\alpha,\beta}$-eigenvalues $\sigma_1 \geq \cdots \geq \sigma_n$ and $\alpha, \beta \in \mathbb{R}$.*

(1). *If $\alpha,\ \beta \geq 0$ then*

$$\frac{\mathrm{R}_\alpha(\mathcal{G})}{n\, 2^\beta\, \Delta_{\mathcal{G}}^\beta} \leq \sigma_1 \leq \frac{(n-1)^{2\alpha} \sqrt{2m - n + 1}}{2^\beta}.$$

(2). *If $\alpha,\ \beta \leq 0$ then*

$$\frac{\mathrm{R}_\alpha(\mathcal{G})}{n\, 2^\beta\, \delta_{\mathcal{G}}^\beta} \leq \sigma_1 \leq \frac{\sqrt{2m - n + 1}}{2^\beta\, (n-1)^\beta}.$$

(3). *If $\alpha \geq 0$ and $\beta \leq 0$ then*

$$\frac{\mathrm{R}_\alpha(\mathcal{G})}{n\, 2^\beta\, \delta_{\mathcal{G}}^\beta} \leq \sigma_1 \leq \frac{(n-1)^{2\alpha-\beta} \sqrt{2m - n + 1}}{2^\beta}.$$

(4). *If $\alpha \leq 0$ and $\beta \geq 0$ then*

$$\frac{\mathrm{R}_\alpha(\mathcal{G})}{n\, 2^\beta\, \Delta_{\mathcal{G}}^\beta} \leq \sigma_1 \leq \frac{\sqrt{2m - n + 1}}{2^\beta}.$$

*Proof.* (1). Let $y \in \mathbb{R}^n$ such that $y = (y_1, y_2, \ldots, y_n)^T$. Then

$$y^T \mathcal{A}_{\alpha,\beta}(\mathcal{G})\, y = \sum_{v_i v_j \in E(\mathcal{G})} \frac{\Big(d_{\mathcal{G}}(v_i)\, d_{\mathcal{G}}(v_j)\Big)^\alpha}{\Big(d_{\mathcal{G}}(v_i) + d_{\mathcal{G}}(v_j)\Big)^\beta}\, y_i y_j \geq \sum_{v_i v_j \in E(\mathcal{G})} \frac{\Big(d_{\mathcal{G}}(v_i)\, d_{\mathcal{G}}(v_j)\Big)^\alpha}{2^\beta\, \Delta_{\mathcal{G}}^\beta}\, y_i y_j.$$

Taking $y = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})^T$, we get $\frac{1}{2^\beta \Delta_{\mathcal{G}}^\beta} \sum_{v_i v_j \in E(\mathcal{G})} \Big(d_{\mathcal{G}}(v_i)\, d_{\mathcal{G}}(v_j)\Big)^\alpha\, y_i y_j = \frac{R_\alpha(\mathcal{G})}{n\, 2^\beta\, \Delta_{\mathcal{G}}^\beta}$. Therefore by Lemma 4.5, $\sigma_1 \geq \frac{\mathrm{R}_\alpha(\mathcal{G})}{n\, 2^\beta\, \Delta_{\mathcal{G}}^\beta}$.

Now for any vertex $v_i \in V(\mathcal{G})$, $i = 1, \ldots, n$, we have $1 \leq \delta_{\mathcal{G}} \leq d_{\mathcal{G}}(v_i) \leq \Delta_{\mathcal{G}} \leq (n-1)$. Therefore

$$\frac{\Big(d_{\mathcal{G}}(v_i)\, d_{\mathcal{G}}(v_i)\Big)^\alpha}{\Big(d_{\mathcal{G}}(v_j) + d_{\mathcal{G}}(v_j)\Big)^\beta} \leq \frac{\Delta_{\mathcal{G}}^{2\alpha}}{2^\beta\, \delta_{\mathcal{G}}^\beta} \leq \frac{(n-1)^{2\alpha}}{2^\beta}.$$

If $\eta_1$ is the spectral radius of a matrix $\frac{(n-1)^{2\alpha}}{2^\beta} A(\mathcal{G})$, then by Lemma 4.2 and Theorem 4.3, we obtain

$$\sigma_1 \leq \eta_1 = \frac{(n-1)^{2\alpha} \lambda_1}{2^\beta} \leq \frac{(n-1)^{2\alpha} \sqrt{2m - n + 1}}{2^\beta},$$

where $\lambda_1$ is the spectral radius of $\mathcal{A}(\mathcal{H})$.

(2). Let $y \in \mathbb{R}^n$ such that $y = (y_1, y_2, \ldots, y_n)^T$. Then

$$y^T \mathcal{A}_{\alpha,\beta}(\mathcal{G})\, y = \sum_{v_i v_j \in E(\mathcal{G})} \frac{\Big(d_{\mathcal{G}}(v_i)\, d_{\mathcal{G}}(v_j)\Big)^{\alpha}}{\Big(d_{\mathcal{G}}(v_i) + d_{\mathcal{G}}(v_j)\Big)^{\beta}}\, y_i y_j \geq \sum_{v_i v_j \in E(\mathcal{G})} \frac{\Big(d_{\mathcal{G}}(v_i)\, d_{\mathcal{G}}(v_j)\Big)^{\alpha}}{2^{\beta}\, \delta_{\mathcal{G}}^{\beta}}\, y_i y_j.$$

Taking $y = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})^T$, we get $\frac{1}{2^{\beta}\delta_{\mathcal{G}}^{\beta}} \sum_{v_i v_j \in E(\mathcal{G})} \Big(d_{\mathcal{G}}(v_i) d_{\mathcal{G}}(v_j)\Big)^{\alpha} y_i y_j = \frac{R_{\alpha}(\mathcal{G})}{n\, 2^{\beta}\, \delta_{\mathcal{G}}^{\beta}}$. Therefore by Lemma 4.5, $\sigma_1 \geq \frac{\mathrm{R}_{\alpha}(\mathcal{G})}{n\, 2^{\beta}\, \delta_{\mathcal{G}}^{\beta}}$.

Now for any vertex $v_i \in V(\mathcal{G})$, $i = 1, \ldots, n$, we have $1 \leq \delta_{\mathcal{G}} \leq d_{\mathcal{G}}(v_i) \leq \Delta_{\mathcal{G}} \leq (n-1)$. Since $\alpha, \beta \leq 0$, therefore $\delta_{\mathcal{G}}^{2\alpha} \leq 1$ and $\Delta_{\mathcal{G}}^{\beta} \geq (n-1)^{\beta}$. Now

$$\frac{\Big(d_{\mathcal{G}}(v_i)\, d_{\mathcal{G}}(v_j)\Big)^{\alpha}}{\Big(d_{\mathcal{G}}(v_i) + d_{\mathcal{G}}(v_j)\Big)^{\beta}} \leq \frac{\delta_{\mathcal{G}}^{2\alpha}}{2^{\beta}\, \Delta_{\mathcal{G}}^{\beta}} \leq \frac{1}{2^{\beta}\, (n-1)^{\beta}}.$$

If $\eta_1$ is the spectral radius of a matrix $\frac{1}{2^{\beta}\,(n-1)^{\beta}} A(\mathcal{G})$, then by Lemma 4.2 and Theorem 4.3, we obtain

$$\sigma_1 \leq \eta_1 = \frac{\lambda_1}{2^{\beta}\, (n-1)^{\beta}} \leq \frac{\sqrt{2m - n + 1}}{2^{\beta}\, (n-1)^{\beta}},$$

where $\lambda_1$ is the spectral radius of $\mathcal{A}(\mathcal{G})$.

Parts (3) and (4) can be proved analogously. □

Next theorem gives bounds on the smallest $A_{\alpha,\beta}$-eigenvalue of a graph.

**Theorem 4.7.** *Let $\mathcal{G} = G(n, m, \Delta_G, \delta_G)$ be a graph with $A_{\alpha,\beta}$ -eigenvalues $\sigma_1 \geq \cdots \geq \sigma_n$. Then*

$$\sqrt{\frac{2Q + (n-1)\,(n-2)\,\Omega^{2/n-1}}{2}} \leq \sigma_n \leq \sqrt{\frac{2(n-1)Q}{n}},$$

*where $\alpha, \beta \in \mathbb{R}$.*

*Proof.* By Part (1) of Lemma 4.1, we get

$$\sigma_n^2 = \Big(-\sum_{i=1}^{n-1} \sigma_i\Big)^2 = \sum_{i=1}^{n-1} \sigma_i^2 + 2 \sum_{1 \leq i < j \leq n-1} \sigma_i \sigma_j.$$

Since arithmetic mean is always greater than geometric mean, therefore

$$\frac{2}{(n-1)(n-2)} \sum_{1 \leq i < j \leq n-1} \sigma_i \sigma_j \;\geq\; (\sigma_1^{n-2}\, \sigma_2^{n-2} \ldots \sigma_n^{n-2})^{2/(n-1)(n-2)} = \Big(\det(A_{\alpha,\beta}(\mathcal{G}))\Big)^{2/n-1} = \Omega^{2/n-1}.$$

Hence $\sigma_n^2 \geq (2Q - \sigma_n^2) + (n-1)(n-2)\,\Omega^{2/n-1}$ and $\sigma_n \geq \sqrt{\frac{2Q + (n-1)\,(n-2)\,\Omega^{2/n-1}}{2}}$.

Again using Part (1) of Lemma 4.1 and Cauchy-Schwartz inequality, we have

$$\sigma_n^2 \leq (n-1)\sum_{i=1}^{n-1}\sigma_i^2 = (n-1)\,(2Q-\sigma_n^2).$$

Hence $\sigma_n \leq \sqrt{\frac{2(n-1)Q}{n}}$. □

In the following theorem, we give bounds on spread of the generalized ISI matrix of a graph.

**Theorem 4.8.** *Let $\mathcal{G} = G(n,m,\Delta_G,\delta_G)$ be a connected graph with $A_{\alpha,\beta}$ -eigenvalues $\sigma_1 \geq \cdots \geq \sigma_n$. Then*

(1). *If $\alpha,\beta \geq 0$ then*

$$\begin{aligned} s(A_{\alpha,\beta}(\mathcal{G})) &\geq \frac{\mathrm{R}_\alpha(\mathcal{G})}{n\,2^\beta\,\Delta_{\mathcal{G}}^\beta} - \frac{(n-1)^{2\alpha}}{2^\beta}\sqrt{\frac{2m\,(n-1)}{n}}, \\ s(A_{\alpha,\beta}(\mathcal{G})) &\leq \frac{(n-1)^{2\alpha}\sqrt{2m-n+1}}{2^\beta} - \frac{\sqrt{2m+2^{2\beta}\,(n-1)^{2\beta+1}\,(n-2)\,\Omega^{2/n-1}}}{2^{\frac{1}{2}+\beta}\,(n-1)^\beta}. \end{aligned}$$

(2). *If $\alpha,\beta \leq 0$ then*

$$\begin{aligned} s(A_{\alpha,\beta}(\mathcal{G})) &\geq \frac{\mathrm{R}_\alpha(\mathcal{G})}{n\,2^\beta\,\delta_{\mathcal{G}}^\beta} - \frac{1}{2^\beta\,(n-1)^\beta}\sqrt{\frac{2m\,(n-1)}{n}}, \\ s(A_{\alpha,\beta}(\mathcal{G}) &\leq \frac{\sqrt{2m-n+1}}{2^\beta\,(n-1)^\beta} - \frac{\sqrt{2m\,(n-1)^{4\alpha}+2^{2\beta}\,(n-1)\,(n-2)\,\Omega^{2/n-1}}}{2^{\frac{1}{2}+\beta}}. \end{aligned}$$

(3). *If $\alpha \geq 0$ and $\beta \leq 0$ then*

$$\begin{aligned} s(A_{\alpha,\beta}(\mathcal{G})) &\geq \frac{\mathrm{R}_\alpha(\mathcal{G})}{n\,2^\beta\,\delta_{\mathcal{G}}^\beta} - \frac{(n-1)^{2\alpha-\beta}}{2^\beta}\sqrt{\frac{2m\,(n-1)}{n}}, \\ s(A_{\alpha,\beta}(\mathcal{G})) &\leq \frac{(n-1)^{2\alpha-\beta}\sqrt{2m-n+1}}{2^\beta} - \frac{\sqrt{2m+2^{2\beta}\,(n-1)\,(n-2)\,\Omega^{2/n-1}}}{2^{\frac{1}{2}+\beta}}. \end{aligned}$$

(4). *If $\alpha \leq 0$ and $\beta \geq 0$ then*

$$\begin{aligned} s(A_{\alpha,\beta}(\mathcal{G})) &\geq \frac{\mathrm{R}_\alpha(\mathcal{G})}{n\,2^\beta\,\Delta_{\mathcal{G}}^\beta} - \frac{1}{2^\beta}\sqrt{\frac{2m\,(n-1)}{n}}, \\ s(A_{\alpha,\beta}(\mathcal{G})) &\leq \frac{\sqrt{2m-n+1}}{2^\beta} - \frac{\sqrt{2m\,(n-1)^{4\alpha}+2^{2\beta}\,(n-1)^{1+2\beta}\,(n-2)\,\Omega^{2/n-1}}}{2^{\frac{1}{2}+\beta}(n-1)^\beta}. \end{aligned}$$

*Proof.* (1). We have $1 \leq \delta_{\mathcal{G}} \leq d_{\mathcal{G}}(v_i) \leq \Delta_{\mathcal{G}} \leq (n-1)$ for any vertex $v_i \in V(\mathcal{G})$, $i = 1,\ldots,n$. Therefore

$$2\,Q = 2\sum_{1\leq i<j\leq n}\frac{\Big(d_{\mathcal{G}}(v_i)\,d_{\mathcal{G}}(v_j)\Big)^{2\alpha}}{\Big(d_{\mathcal{G}}(v_i)+d_{\mathcal{G}}(v_j)\Big)^{2\beta}} \geq 2\sum_{1\leq i<j\leq n}\frac{\delta_{\mathcal{G}}^{4\alpha}}{2^{2\beta}\,\Delta_{\mathcal{G}}^{2\beta}} \geq \frac{m\,2^{1-2\beta}}{(n-1)^{2\beta}}. \tag{4.1}$$

Also

$$2\,Q = 2\sum_{1\le i<j\le n}\frac{\Big(d_{\mathcal{G}}(v_i)\,d_{\mathcal{G}}(v_j)\Big)^{2\alpha}}{\Big(d_{\mathcal{G}}(v_i)+d_{\mathcal{G}}(v_j)\Big)^{2\beta}} \le 2\sum_{1\le i<j\le n}\frac{\Delta_{\mathcal{G}}^{4\alpha}}{2^{2\beta}\,\delta_{\mathcal{G}}^{2\beta}} \;\le\; m\,2^{1-2\beta}\,(n-1)^{4\alpha}. \tag{4.2}$$

Hence using Theorem 4.6, Theorem 4.7 and Equations (4.1) and (4.2), we get

$$\begin{aligned} s(A_{\alpha,\beta}(\mathcal{G})) &= \sigma_1-\sigma_n \\ &\le \frac{(n-1)^{2\alpha}\sqrt{2m-n+1}}{2^{\beta}} - \sqrt{\frac{2\,Q+(n-1)\,(n-2)\,\Omega^{2/n-1}}{2}} \\ &\le \frac{(n-1)^{2\alpha}\sqrt{2m-n+1}}{2^{\beta}} - \frac{1}{\sqrt{2}}\sqrt{\frac{2m}{(2(n-1))^{2\beta}}+(n-1)\,(n-2)\,\Omega^{2/n-1}} \\ &= \frac{(n-1)^{2\alpha}\sqrt{2m-n+1}}{2^{\beta}} - \frac{\sqrt{2m+2^{2\beta}\,(n-1)^{2\beta+1}\,(n-2)\,\Omega^{2/n-1}}}{2^{\frac{1}{2}+\beta}\,(n-1)^{\beta}}. \end{aligned}$$

Also

$$s(A_{\alpha,\beta}(\mathcal{G})) = \sigma_1-\sigma_n \;\ge\; \frac{\mathrm{R}_{\alpha}(\mathcal{G})}{n\,2^{\beta}\,\Delta_{\mathcal{G}}^{\beta}} - \sqrt{\frac{2\,(n-1)Q}{n}} \;\ge\; \frac{\mathrm{R}_{\alpha}(\mathcal{G})}{n\,2^{\beta}\,\Delta_{\mathcal{G}}^{\beta}} - \frac{(n-1)^{2\alpha}}{2^{\beta}}\sqrt{\frac{2m\,(n-1)}{n}}.$$

(2). We have $1\le\delta_{\mathcal{G}}\le d_{\mathcal{G}}(v_i)\le\Delta_{\mathcal{G}}\le(n-1)$ for any vertex $v_i\in V(\mathcal{G})$, $i=1,\ldots,n$. Since $\alpha,\beta\le 0$, therefore $\Delta_{\mathcal{G}}^{\alpha}\ge(n-1)^{\alpha}$ and $\delta_{\mathcal{G}}^{\beta}\le 1$. Now

$$2\,Q = 2\sum_{1\le i<j\le n}\frac{\Big(d_{\mathcal{G}}(v_i)\,d_{\mathcal{G}}(v_j)\Big)^{2\alpha}}{\Big(d_{\mathcal{G}}(v_i)+d_{\mathcal{G}}(v_j)\Big)^{2\beta}} \ge 2\sum_{1\le i<j\le n}\frac{\Delta_{\mathcal{G}}^{4\alpha}}{2^{2\beta}\,\delta_{\mathcal{G}}^{2\beta}} \;\ge\; m\,2^{1-2\beta}\,(n-1)^{4\alpha}. \tag{4.3}$$

Also

$$2\,Q = 2\sum_{1\le i<j\le n}\frac{\Big(d_{\mathcal{G}}(v_i)\,d_{\mathcal{G}}(v_j)\Big)^{2\alpha}}{\Big(d_{\mathcal{G}}(v_i)+d_{\mathcal{G}}(v_j)\Big)^{2\beta}} \le 2\sum_{1\le i<j\le n}\frac{\delta_{\mathcal{G}}^{4\alpha}}{2^{2\beta}\,\Delta_{\mathcal{G}}^{2\beta}} \;=\; \frac{m\,2^{1-2\beta}}{(n-1)^{2\beta}}. \tag{4.4}$$

Hence using Theorem 4.6, Theorem 4.7 and Equations (4.3) and (4.4), we get

$$\begin{aligned} s(A_{\alpha,\beta}(\mathcal{G})) &= \sigma_1-\sigma_n \\ &\le \frac{\sqrt{2m-n+1}}{2^{\beta}\,(n-1)^{\beta}} - \sqrt{\frac{2\,Q+(n-1)\,(n-2)\,\Omega^{2/n-1}}{2}} \\ &\le \frac{\sqrt{2m-n+1}}{2^{\beta}\,(n-1)^{\beta}} - \frac{1}{\sqrt{2}}\sqrt{\frac{2m\,(n-1)^{4\alpha}}{2^{2\beta}}+(n-1)\,(n-2)\,\Omega^{2/n-1}} \\ &= \frac{\sqrt{2m-n+1}}{2^{\beta}\,(n-1)^{\beta}} - \frac{\sqrt{2m\,(n-1)^{4\alpha}+2^{2\beta}\,(n-1)\,(n-2)\,\Omega^{2/n-1}}}{2^{\frac{1}{2}+\beta}}. \end{aligned}$$

Also

$$s(A_{\alpha,\beta}(\mathcal{G})) = \sigma_1 - \sigma_n \;\geq\; \frac{\mathrm{R}_\alpha(\mathcal{G})}{n\,2^\beta\,\delta_{\mathcal{G}}^\beta} - \sqrt{\frac{2\,(n-1)Q}{n}} \;\geq\; \frac{\mathrm{R}_\alpha(\mathcal{G})}{n\,2^\beta\,\delta_{\mathcal{G}}^\beta} - \frac{1}{2^\beta\,(n-1)^\beta}\sqrt{\frac{2m\,(n-1)}{n}}.$$

Similarly, one can prove Parts (3) and (4). The proof is complete. □

## 5. Bounds on generalized ISI energy

In this section, we give some bounds for the generalized ISI energy of graphs. We would like to mention that the idea of proof of next theorem is taken from the proof of Theorem 13 [8].

**Theorem 5.1.** *Let $\mathcal{G} = G(n,m,\Delta_G,\delta_G)$ be a connected graph having $A_{\alpha,\beta}$-eigenvalues $\sigma_1 \geq \cdots \geq \sigma_n$ and $\alpha,\beta \in \mathbb{R}$.*

(1). *If $\alpha,\beta \geq 0$ then*

$$\frac{2^{1-\beta}\,\mathrm{R}_\alpha(\mathcal{G})}{n\,(n-1)^\beta} \leq E_{\alpha,\beta}(\mathcal{G}) \leq \frac{(n-1)^{2\alpha}}{2^\beta}\sqrt{2nm}.$$

(2). *If $\alpha,\beta \leq 0$ then*

$$\frac{2^{1-\beta}\,\mathrm{R}_\alpha(\mathcal{G})}{n} \leq E_{\alpha,\beta}(\mathcal{G}) \leq \frac{1}{2^\beta\,(n-1)^\beta}\sqrt{2nm}.$$

(3). *If $\alpha \geq 0$ and $\beta \leq 0$ then*

$$\frac{2^{1-\beta}\,\mathrm{R}_\alpha(\mathcal{G})}{n} \leq E_{\alpha,\beta}(\mathcal{G}) \leq \frac{(n-1)^{2\alpha-\beta}}{2^\beta}\sqrt{2nm}.$$

(4). *If $\alpha \leq 0$ and $\beta \geq 0$ then*

$$\frac{2^{1-\beta}\,\mathrm{R}_\alpha(\mathcal{G})}{n\,(n-1)^\beta} \leq E_{\alpha,\beta}(\mathcal{G}) \leq \frac{1}{2^\beta}\sqrt{2nm}.$$

*Proof.* (1). With no loss of generality, suppose that $\sigma_1,\ldots,\sigma_t$ are positive and $\sigma_{t+1},\ldots,\sigma_n$ are negative. Using Theorem 4.6, we get

$$E_{\alpha,\beta}(\mathcal{G}) = \sum_{i=1}^{n}|\sigma_i| \;=\; 2\sum_{i=1}^{t}\sigma_i \;\geq\; 2\,\sigma_1 \;\geq\; \frac{2\,\mathrm{R}_\alpha(\mathcal{G})}{n\,2^\beta\,\Delta_{\mathcal{G}}^\beta} \;\geq\; \frac{2^{1-\beta}\,\mathrm{R}_\alpha(\mathcal{G})}{n\,(n-1)^\beta}.$$

Now applying Cauchy-Schwartz inequality, Part (2) of Lemma 4.1 and Equation (4.2), we have

$$E_{\alpha,\beta}(\mathcal{G}) = \sum_{i=1}^{n}|\sigma_i| \;\leq\; \sqrt{n\sum_{i=1}^{n}\sigma_i^2} \;=\; \sqrt{2n\,Q} \;\leq\; \sqrt{\frac{2nm\,(n-1)^{4\alpha}}{2^{2\beta}}} \;=\; \frac{(n-1)^{2\alpha}}{2^\beta}\sqrt{2nm}.$$

(2). With no loss of generality, suppose that $\sigma_1,\ldots,\sigma_t$ are positive and $\sigma_{t+1},\ldots,\sigma_n$ are negative. Since $\alpha,\beta \leq 0$ therefore $\delta_{\mathcal{G}}^\beta \leq 1$. Now using Theorem 4.6 (2), we get

$$E_{\alpha,\beta}(\mathcal{G}) = \sum_{i=1}^{n}|\sigma_i| \;=\; 2\sum_{i=1}^{t}\sigma_i \;\geq\; 2\,\sigma_1 \;\geq\; \frac{2\,\mathrm{R}_\alpha(\mathcal{G})}{n\,2^\beta\,\delta_{\mathcal{G}}^\beta} \;\geq\; \frac{2^{1-\beta}\,\mathrm{R}_\alpha(\mathcal{G})}{n}.$$

Now applying Cauchy-Schwartz inequality, Part (2) of Lemma 4.1 and Equation (4.4), we have

$$E_{\alpha,\beta}(\mathcal{G}) = \sum_{i=1}^{n} |\sigma_i| \leq \sqrt{n \sum_{i=1}^{n} \sigma_i^2} = \sqrt{2n\, Q} \leq \sqrt{\frac{2nm}{(n-1)^{2\beta}\, 2^{2\beta}}} = \frac{1}{2^{\beta}\, (n-1)^{\beta}} \sqrt{2nm}.$$

One can prove Parts (3) and (4) in a similar manner. The result is proved. □

**Theorem 5.2.** *Let $\mathcal{G} = G(n, m, \Delta_G, \delta_G)$ be a connected graph with $A_{\alpha,\beta}$-eigenvalues $\sigma_1 \geq \cdots \geq \sigma_n$. Then*

(1). *If $\alpha$, $\beta \geq 0$ then*

$$\frac{2^{1-\beta}\sqrt{m}}{(n-1)^{\beta}} \leq E_{\alpha,\beta}(\mathcal{G}) \leq \frac{(n-1)^{2\alpha}}{2^{\beta}} \left[ \sqrt{2m-n+1} + \sqrt{2m\,(n-1) - \frac{\mathrm{R}_{\alpha}{}^{2}(\mathcal{G})\,(n-1)^{1-2\beta-4\alpha}}{n^2}} \right].$$

(2). *If $\alpha$, $\beta \leq 0$ then*

$$2^{1-\beta}\,(n-1)^{2\alpha}\sqrt{m} \leq E_{\alpha,\beta}(\mathcal{G}) \leq \frac{1}{2^{\beta}\,(n-1)^{\beta}} \left[ \sqrt{2m-n+1} + \sqrt{2m\,(n-1) - \frac{\mathrm{R}_{\alpha}{}^{2}(\mathcal{G})\,(n-1)^{1+2\beta}}{n^2\,\delta_{\mathcal{G}}^{2\beta}}} \right].$$

(3). *If $\alpha \geq 0$ and $\beta \leq 0$ then*

$$2^{1-\beta}\sqrt{m} \leq E_{\alpha,\beta}(\mathcal{G}) \leq \frac{(n-1)^{2\alpha-\beta}}{2^{\beta}} \left[ \sqrt{2m-n+1} + \sqrt{2m\,(n-1) - \frac{\mathrm{R}_{\alpha}{}^{2}(\mathcal{G})\,(n-1)^{1+2\beta-4\alpha}}{n^2\,\delta_{\mathcal{G}}^{2\beta}}} \right].$$

(4). *If $\alpha \leq 0$ and $\beta \geq 0$ then*

$$2^{1-\beta}\,(n-1)^{2\alpha-\beta}\sqrt{m} \leq E_{\alpha,\beta}(\mathcal{G}) \leq \frac{1}{2^{\beta}} \left[ \sqrt{2m-n+1} + \sqrt{2m\,(n-1) - \frac{\mathrm{R}_{\alpha}{}^{2}(\mathcal{G})\,(n-1)}{n^2\,\Delta_{\mathcal{G}}^{2\beta}}} \right].$$

*Proof.* (1). By Part (1) of Lemma 4.1, we have $\sum_{i=1}^{n} \sigma_i^2 = -2 \sum_{1 \leq i < j \leq n} \sigma_i\, \sigma_j$. Using Part (2) of Lemma 4.1, we obtain

$$(E_{\alpha,\beta}(\mathcal{G}))^2 = \left( \sum_{i=1}^{n} |\sigma_i| \right)^2 = \sum_{i=1}^{n} \sigma_i^2 + 2 \sum_{1 \leq i < j \leq n} |\sigma_i\, \sigma_j| \geq 2\, Q + 2\, |\sum_{1 \leq i < j \leq n} \sigma_i\, \sigma_j| = 4\, Q.$$

Now

$$4\, Q = 4 \sum_{1 \leq i < j \leq n} \frac{\Big(d_{\mathcal{G}}(v_i)\, d_{\mathcal{G}}(v_j)\Big)^{2\alpha}}{\Big(d_{\mathcal{G}}(v_i) + d_{\mathcal{G}}(v_j)\Big)^{2\beta}} \geq \sum_{1 \leq i < j \leq n} \frac{4\delta_{\mathcal{G}}^{4\alpha}}{2^{2\beta}\, \Delta_{\mathcal{G}}^{2\beta}} \geq \frac{m\, 2^{2-2\beta}}{(n-1)^{2\beta}}.$$

Hence $E_{\alpha,\beta}(\mathcal{G}) \geq \frac{2^{1-\beta}\sqrt{m}}{(n-1)^{\beta}}$.

To prove inequality on the right side, we apply Cauchy-Schwartz inequality to obtain $(\sum_{i=2}^{n} |\sigma_i|)^2 \leq (n-1)\sum_{i=2}^{n} \sigma_i^2$. Therefore using Part (2) of Lemma 4.1, $(E_{\alpha,\beta}(\mathcal{G}) - \sigma_1)^2 \leq (n-1)\,(2Q - \sigma_1^2)$. Hence by Theorem 4.6 (1), we get

$$\begin{aligned} E_{\alpha,\beta}(\mathcal{G}) &\leq \sigma_1 + \sqrt{(n-1)\,(2Q-\sigma_1^2)} \\ &\leq \frac{(n-1)^{2\alpha}\sqrt{2m-n+1}}{2^{\beta}} + \sqrt{(n-1)\left[m\,2^{1-2\beta}\,(n-1)^{4\alpha} - \frac{{\mathrm{R}_{\alpha}}^2(\mathcal{G})}{n^2\,2^{2\beta}\,(n-1)^{2\beta}}\right]} \\ &= \frac{(n-1)^{2\alpha}}{2^{\beta}}\left[\sqrt{2m-n+1} + \sqrt{2m\,(n-1) - \frac{{\mathrm{R}_{\alpha}}^2(\mathcal{G})(n-1)^{1-2\beta-4\alpha}}{n^2}}\,\right]. \end{aligned}$$

(2). Using Eq (4.3), we get

$$4Q = 2(2Q) \geq 2^{2-2\beta}\,m\,(n-1)^{4\alpha}.$$

Hence $E_{\alpha,\beta}(\mathcal{G}) \geq 2^{1-\beta}\,(n-1)^{2\alpha}\,\sqrt{m}$.

Now using Theorem 4.6 (2) and Eq (4.4), we get

$$\begin{aligned} E_{\alpha,\beta}(\mathcal{G}) &\leq \sigma_1 + \sqrt{(n-1)\,(2Q-\sigma_1^2)} \\ &\leq \frac{\sqrt{2m-n+1}}{2^{\beta}\,(n-1)^{\beta}} + \sqrt{(n-1)\left[\frac{m\,2^{1-2\beta}}{(n-1)^{2\beta}} - \frac{{\mathrm{R}_{\alpha}}^2(\mathcal{G})}{n^2\,2^{2\beta}\,\delta_{\mathcal{G}}^{2\beta}}\right]} \\ &= \frac{1}{2^{\beta}\,(n-1)^{\beta}}\left[\sqrt{2m-n+1} + \sqrt{2m\,(n-1) - \frac{{\mathrm{R}_{\alpha}}^2(\mathcal{G})(n-1)^{1+2\beta}}{n^2\,\delta_{\mathcal{G}}^{2\beta}}}\,\right]. \end{aligned}$$

This gives the required result.

Analogously, one can prove Parts (3) and (4). □

## 6. Nordhaus-Gaddum-type results for generalized ISI spectral radius and energy

The compliment of a simple graph $G$ is a graph represented by $\overline{G}$ with the property that $V(G) = V(\overline{G})$ and $wz \in E(G)$ if and only if $wz \notin E(\overline{G})$. Therefore $n(G) = n(\overline{G})$, $e(\overline{G}) = \frac{n^2(G)-n(G)}{2} - e(G)$, $\Delta_{\overline{G}} = n(G) - 1 - \delta_G$ and $\delta_{\overline{G}} = n(G) - 1 - \Delta_G$. The $A_{\alpha,\beta}$-eigenvalues of $\overline{G}$ are $\overline{\sigma}_i$, $i = 1, 2, \ldots, n$. A maximal connected subgraph of $G$ is called a connected component of $G$.

We first present bounds on $\sigma_1 + \overline{\sigma}_1$.

**Theorem 6.1.** *Let $\mathcal{G} = G(n, m, \Delta_G, \delta_G)$ be a connected graph and $\alpha, \beta \in \mathbb{R}$.*

(1). *If $\beta \geq 0$ then*

$$\sigma_1 + \overline{\sigma}_1 \geq \frac{1}{n\,2^{\beta}}\left[\frac{\mathrm{R}_{\alpha}(\mathcal{G})}{(n-1)^{\beta}} + \frac{\mathrm{R}_{\alpha}(\overline{\mathcal{G}})}{(n-1-\delta_{\mathcal{G}})^{\beta}}\right].$$

(2). *If* $\beta \le 0$ *then*

$$\sigma_1 + \overline{\sigma}_1 \ge \frac{1}{n\,2^{\beta}}\left[\frac{\mathrm{R}_\alpha(\mathcal{G})}{\delta_{\mathcal{G}}^{\beta}} + \frac{\mathrm{R}_\alpha(\overline{\mathcal{G}})}{(n-1-\nabla_{\mathcal{G}})^{\beta}}\right].$$

*Proof.* (1). Let $y \in \mathbb{R}^n$ such that $y = (y_1, y_2, \ldots, y_n)^T$. Then

$$\begin{aligned} y^T\left[A_{\alpha,\beta}(\mathcal{G}) + A_{\alpha,\beta}(\overline{\mathcal{G}})\right]y &= \sum_{v_iv_j\in E(\mathcal{G})} \frac{(d_{\mathcal{G}}(v_i)\,d_{\mathcal{G}}(v_j))^{\alpha}}{(d_{\mathcal{G}}(v_i)+d_{\mathcal{G}}(v_j))^{\beta}}\,y_iy_j + \sum_{v_iv_j\in E(\overline{\mathcal{G}})} \frac{(d_{\overline{\mathcal{G}}}(v_i)\,d_{\overline{\mathcal{G}}}(v_j))^{\alpha}}{(d_{\overline{\mathcal{G}}}(v_i)+d_{\overline{\mathcal{G}}}(v_j))^{\beta}}\,y_iy_j \\ &\ge \sum_{v_iv_j\in E(\mathcal{G})} \frac{(d_{\mathcal{G}}(v_i)\,d_{\mathcal{G}}(v_j))^{\alpha}}{2^{\beta}\Delta_{\mathcal{G}}^{\beta}}\,y_iy_j + \sum_{v_iv_j\in E(\overline{\mathcal{G}})} \frac{(d_{\overline{\mathcal{G}}}(v_i)\,d_{\overline{\mathcal{G}}}(v_j))^{\alpha}}{2^{\beta}\Delta_{\overline{\mathcal{G}}}^{\beta}}\,y_iy_j. \end{aligned}$$

Since $\Delta_{\overline{\mathcal{G}}} = n-1-\delta_{\mathcal{G}}$ and $\Delta_{\mathcal{G}} \le n-1$ therefore taking $y = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})^T$ and using Lemma 4.5, we obtain

$$\sigma_1 + \overline{\sigma}_1 \ge \frac{1}{n2^{\beta}}\left[\frac{R_\alpha(\mathcal{G})}{(n-1)^{\beta}} + \frac{R_\alpha(\overline{\mathcal{G}})}{(n-1-\delta_{\mathcal{G}})^{\beta}}\right].$$

Part (2) can be proved similarly. □

**Theorem 6.2.** *Let* $\mathcal{G} = G(n, m, \delta_G, \Delta_G)$ *be a connected graph and* $\mathcal{G}_1$ *is a connected component of* $\overline{\mathcal{G}}$ *with* $\overline{\sigma}_1 = \sigma_1(\mathcal{G}_1)$.

(1). *Let* $\alpha,\ \beta \ge 0$ .

($a$). *If* $\Delta_{\mathcal{G}} = n-1$ *or* $\Delta_{\overline{\mathcal{G}}} = n-1$, *then*

$$\sigma_1 + \overline{\sigma}_1 \le \frac{1}{2^{\beta}}\left[(n-1)^{2\alpha}\sqrt{2m-n+1} + (n(\mathcal{G}_1)-1)^{2\alpha}\sqrt{2e(\mathcal{G}_1)-n(\mathcal{G}_1)+1}\right],$$

($b$). *If* $\Delta_{\mathcal{G}} \le n-2$ *and* $\Delta_{\overline{\mathcal{G}}} \le n-2$, *then*

$$\sigma_1 + \overline{\sigma}_1 \le \frac{(n-2)^{2\alpha}}{2^{\beta}}\left[\sqrt{2m-n+1} + \sqrt{(n^2-2n-2m+1)+\delta_{\mathcal{G}}(2+\Delta_{\mathcal{G}}-n)}\right].$$

(2) *Let* $\alpha,\ \beta \le 0$.

($a$). *If* $\Delta_{\mathcal{G}} = n-1$ *or* $\Delta_{\overline{\mathcal{G}}} = n-1$, *then*

$$\sigma_1 + \overline{\sigma}_1 \le \frac{1}{2^{\beta}}\left[\frac{\sqrt{2m-n+1}}{(n-1)^{\beta}} + \frac{\sqrt{2e(\mathcal{G}_1)-n(\mathcal{G}_1)+1}}{(n(\mathcal{G}_1)-1)^{\beta}}\right],$$

($b$). *If* $\Delta_{\mathcal{G}} \le n-2$ *and* $\Delta_{\overline{\mathcal{G}}} \le n-2$, *then*

$$\sigma_1 + \overline{\sigma}_1 \le \frac{1}{2^{\beta}\,(n-2)^{\beta}}\left[\sqrt{2m-n+1} + \sqrt{(n^2-2n-2m+1)+\delta_{\mathcal{G}}(2+\Delta_{\mathcal{G}}-n)}\right].$$

(3). *Let* $\alpha \ge 0$ *and* $\beta \le 0$ .

($a$). *If* $\Delta_{\mathcal{G}} = n-1$ *or* $\Delta_{\overline{\mathcal{G}}} = n-1$, *then*

$$\sigma_1 + \overline{\sigma}_1 \le \frac{1}{2^{\beta}}\left[(n-1)^{2\alpha-\beta}\sqrt{2m-n+1} + (n(\mathcal{G}_1)-1)^{2\alpha-\beta}\sqrt{2e(\mathcal{G}_1)-n(\mathcal{G}_1)+1}\right],$$

(*b*). *If* $\Delta_{\mathcal{G}} \leq n-2$ *and* $\Delta_{\overline{\mathcal{G}}} \leq n-2$*, then*

$$\sigma_1 + \overline{\sigma}_1 \leq \frac{(n-2)^{2\alpha-\beta}}{2^\beta}\left[\sqrt{2m-n+1} + \sqrt{(n^2-2n-2m+1)+\delta_{\mathcal{G}}(2+\Delta_{\mathcal{G}}-n)}\right].$$

(4) *Let* $\alpha \leq 0$ *and* $\beta \geq 0$.

(*a*). *If* $\Delta_{\mathcal{G}} = n-1$ *or* $\Delta_{\overline{\mathcal{G}}} = n-1$*, then*

$$\sigma_1 + \overline{\sigma}_1 \leq \frac{1}{2^\beta}\left[\sqrt{2m-n+1} + \sqrt{2e(\mathcal{G}_1)-n(\mathcal{G}_1)+1}\right],$$

(*b*). *If* $\Delta_{\mathcal{G}} \leq n-2$ *and* $\Delta_{\overline{\mathcal{G}}} \leq n-2$*, then*

$$\sigma_1 + \overline{\sigma}_1 \leq \frac{1}{2^\beta}\left[\sqrt{2m-n+1} + \sqrt{(n^2-2n-2m+1)+\delta_{\mathcal{G}}(2+\Delta_{\mathcal{G}}-n)}\right].$$

*Proof.* (1).

(*a*). Assume that $\Delta_{\mathcal{G}} = n-1$. From Theorem 4.6, we have

$$\sigma_1 \leq \frac{(n-1)^{2\alpha}\sqrt{2m-n+1}}{2^\beta}. \tag{6.1}$$

Let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_s$ be connected components of $\overline{\mathcal{G}}$. With no loss of generality, assume that $\sigma_1(\mathcal{G}_1) \geq \sigma_1(\mathcal{G}_2) \geq \cdots \geq \sigma_1(\mathcal{G}_s)$. Also note that $\overline{\sigma}_1 = \sigma_1(\mathcal{G}_1)$. Therefore using Theorem 4.6, we get

$$\overline{\sigma}_1 \leq \frac{(n(\mathcal{G}_1)-1)^{2\alpha}\sqrt{2e(\mathcal{G}_1)-n(\mathcal{G}_1)+1}}{2^\beta}. \tag{6.2}$$

The desired result is obtained by adding Equations (6.1) and (6.2).

(*b*). If $\Delta_{\mathcal{G}} \leq n-2$ and $\Delta_{\overline{\mathcal{G}}} \leq n-2$, then $\delta_{\overline{\mathcal{G}}} \geq 1$. From Theorem 4.6, we have

$$\sigma_1 \leq \frac{(n-2)^{2\alpha}\sqrt{2m-n+1}}{2^\beta}. \tag{6.3}$$

Now using Inequalities $\delta_{\overline{\mathcal{G}}} = n-1-\Delta_{\mathcal{G}}$ and $\Delta_{\overline{\mathcal{G}}} \leq n-2$, Theorem 4.4 and proof of Theorem 4.6, we obtain

$$\begin{aligned}
\overline{\sigma}_1 &\leq \frac{(n-2)^{2\alpha}\sqrt{2\binom{n}{2}-2m-\delta_{\overline{\mathcal{G}}}(n-1)+(\delta_{\overline{\mathcal{G}}}-1)\Delta_{\overline{\mathcal{G}}}}}{2^\beta\delta_{\overline{\mathcal{G}}}^\beta}\\
&= \frac{(n-2)^{2\alpha}}{2^\beta\delta_{\overline{\mathcal{G}}}^\beta}\sqrt{(n^2-2n-2m+1)+\delta_{\mathcal{G}}(2+\Delta_{\mathcal{G}}-n)}\\
&\leq \frac{(n-2)^{2\alpha}}{2^\beta}\sqrt{(n^2-2n-2m+1)+\delta_{\mathcal{G}}(2+\Delta_{\mathcal{G}}-n)}.
\end{aligned} \tag{6.4}$$

By adding Eqs (6.3) and (6.4), we get the result.

Now similarly using Theorem 4.4 and Theorem 4.6, one can prove Parts (2) ~ (4). □

We now give bounds on $E_{\alpha,\beta}(G(n,m,\Delta_G,\delta_G)) + E_{\alpha,\beta}(\overline{G}(n,m,\Delta_{\overline{G}},\delta_{\overline{G}}))$.

**Theorem 6.3.** *Let $\mathcal{G} = G(n,m,\Delta,\delta)$ be a connected graph and $\mathcal{G}_1$ is a connected component of $\overline{\mathcal{G}}$ with $\overline{\sigma}_1 = \sigma_1(\mathcal{G}_1)$.*

(1). *Let $\alpha$, $\beta \geq 0$.*

($a$). *If $\Delta_{\mathcal{G}} = n-1$ or $\Delta_{\overline{\mathcal{G}}} = n-1$, then*

$$E_{\alpha,\beta}(\mathcal{G}) + E_{\alpha,\beta}(\overline{\mathcal{G}}) \leq \frac{(n-1)^{2\alpha}}{2^\beta}\,U + \frac{1}{2^\beta}\Big[(n(\mathcal{G}_1)-1)^{2\alpha}\sqrt{2e(\mathcal{G}_1)-n(\mathcal{G}_1)+1} + W_1\Big],$$

($b$). *If $\Delta_{\mathcal{G}} \leq n-2$ and $\Delta_{\overline{\mathcal{G}}} \leq n-2$, then*

$$E_{\alpha,\beta}(\mathcal{G}) + E_{\alpha,\beta}(\overline{\mathcal{G}}) \leq \frac{(n-2)^{2\alpha}}{2^\beta}\,U' + \frac{(n-1-\delta_{\mathcal{G}})^{2\alpha}\sqrt{n^2-n-2m}}{2^\beta\,(n-1-\Delta_{\mathcal{G}})^\beta}\,W_2.$$

(2). *Let $\alpha$, $\beta \leq 0$.*

($a$). *If $\Delta_{\mathcal{G}} = n-1$ or $\Delta_{\overline{\mathcal{G}}} = n-1$, then*

$$E_{\alpha,\beta}(\mathcal{G}) + E_{\alpha,\beta}(\overline{\mathcal{G}}) \leq \frac{1}{2^\beta\,(n-1)^\beta}U_1 + \frac{1}{2^\beta}\left[\frac{\sqrt{2e(\mathcal{G}_1)-n(\mathcal{G}_1)+1}}{(n(\mathcal{G}_1)-1)^\beta} + \frac{W_3\sqrt{n^2-n-2m}}{(n-1-\delta_{\mathcal{G}})^\beta\,(n-1-\Delta_{\mathcal{G}})^\beta}\right],$$

($b$). *If $\Delta_{\mathcal{G}} \leq n-2$ and $\Delta_{\overline{\mathcal{G}}} \leq n-2$, then*

$$E_{\alpha,\beta}(\mathcal{G}) + E_{\alpha,\beta}(\overline{\mathcal{G}}) \leq \frac{1}{2^\beta\,(n-2)^\beta}\,U_1' + \frac{(n-1-\Delta_{\mathcal{G}})^{2\alpha}\sqrt{n^2-n-2m}}{2^\beta\,(n-1-\delta_{\mathcal{G}})^\beta}\,W_4.$$

(3). *Let $\alpha \geq 0$ and $\beta \leq 0$.*

($a$). *If $\Delta_{\mathcal{G}} = n-1$ or $\Delta_{\overline{\mathcal{G}}} = n-1$, then*

$$\begin{aligned} E_{\alpha,\beta}(\mathcal{G}) + E_{\alpha,\beta}(\overline{\mathcal{G}}) \quad \leq \quad & \frac{(n-1)^{2\alpha-\beta}}{2^\beta}\,U_2 + \frac{1}{2^\beta}\sqrt{2e(\mathcal{G}_1)-n(\mathcal{G}_1)+1}(n(\mathcal{G}_1)-1)^{2\alpha-\beta} \\ & + W_5\frac{\sqrt{n^2-n-2m}}{2^\beta}, \end{aligned}$$

($b$). *If $\Delta_{\mathcal{G}} \leq n-2$ and $\Delta_{\overline{\mathcal{G}}} \leq n-2$, then*

$$E_{\alpha,\beta}(\mathcal{G}) + E_{\alpha,\beta}(\overline{\mathcal{G}}) \leq \frac{(n-2)^{2\alpha-\beta}}{2^\beta}\,U_2' + \frac{(n-1-\delta_{\mathcal{G}})^{2\alpha-\beta}\sqrt{n^2-n-2m}}{2^\beta}\,W_6.$$

(4). *Let $\alpha \leq 0$ and $\beta \geq 0$.*

($a$). *If $\Delta_{\mathcal{G}} = n-1$ or $\Delta_{\overline{\mathcal{G}}} = n-1$, then*

$$E_{\alpha,\beta}(\mathcal{G}) + E_{\alpha,\beta}(\overline{\mathcal{G}}) \leq \frac{1}{2^\beta}\,U_3 + \frac{1}{2^\beta}\Big[\sqrt{2e(\mathcal{G}_1)-n(\mathcal{G}_1)+1} + W_7\sqrt{n^2-n-2m}\,\Big],$$

*(b). If $\Delta_{\mathcal{G}} \leq n-2$ and $\Delta_{\overline{\mathcal{G}}} \leq n-2$, then*

$$E_{\alpha,\beta}(\mathcal{G}) + E_{\alpha,\beta}(\overline{\mathcal{G}}) \leq \frac{1}{2^{\beta}}\, U_3 + \frac{(n-1-\Delta_{\mathcal{G}})^{2\alpha-\beta}\sqrt{n^2-n-2m}}{2^{\beta}}\, W_8,$$

*where*

$$U = \sqrt{2m-n+1} + \sqrt{2m\,(n-1) - \frac{\mathrm{R}_{\alpha}{}^{2}\,(\mathcal{G})\,(n-1)^{1-4\alpha-2\beta}}{n^2}},$$

$$U' = \sqrt{2m-n+1} + \sqrt{2m\,(n-1) - \frac{\mathrm{R}_{\alpha}{}^{2}\,(\mathcal{G})\,(n-1)}{n^2\,(n-2)^{4\alpha+2\beta}}},$$

$$U_1 = \sqrt{2m-n+1} + \sqrt{2m\,(n-1) - \frac{\mathrm{R}_{\alpha}{}^{2}\,(\mathcal{G})\,(n-1)^{1+2\beta}}{n^2\,\delta_{\mathcal{G}}^{2\beta}}},$$

$$U_1' = \sqrt{2m-n+1} + \sqrt{2m\,(n-1) - \frac{\mathrm{R}_{\alpha}{}^{2}\,(\mathcal{G})\,(n-1)\,(n-2)^{2\beta}}{n^2\,\delta_{\mathcal{G}}^{2\beta}}},$$

$$U_2 = \sqrt{2m-n+1} + \sqrt{2m\,(n-1) - \frac{\mathrm{R}_{\alpha}{}^{2}\,(\mathcal{G})\,(n-1)^{1-4\alpha+2\beta}}{n^2\,\delta_{\mathcal{G}}^{2\beta}}},$$

$$U_2' = \sqrt{2m-n+1} + \sqrt{2m\,(n-1) - \frac{\mathrm{R}_{\alpha}{}^{2}\,(\mathcal{G})\,(n-1)}{n^2\,\,(n-2)^{4\alpha-2\beta}\,\delta_{\mathcal{G}}^{2\beta}}},$$

$$U_3 = \sqrt{2m-n+1} + \sqrt{2m\,(n-1) - \frac{\mathrm{R}_{\alpha}{}^{2}\,(\mathcal{G})\,(n-1)}{n^2\,\Delta_{\mathcal{G}}^{2\beta}}},$$

$$W_1 = \sqrt{(n-1)\left[\frac{(n-1-\delta_{\mathcal{G}})^{4\alpha}\,(n^2-n-2m)}{(n-1-\Delta_{\mathcal{G}})^{2\beta}} - \frac{(n-1-\Delta_{\mathcal{G}})^{4\alpha}\,(n^2-n-2m)^2}{4n^2\,(n-1-\delta_{\mathcal{G}})^{2\beta}}\right]},$$

$$W_2 = \sqrt{1+\frac{1-n+\delta_{\mathcal{G}}\,(2+\Delta_{\mathcal{G}}-n)}{n^2-n-2m}} + \sqrt{(n-1) - \frac{(n-1)\,(n-1-\Delta_{\mathcal{G}})^{4\alpha+2\beta}\,(n^2-n-2m)}{4n^2\,(n-1-\delta_{\mathcal{G}})^{2\beta+4\alpha}}},$$

$$W_3 = \sqrt{(n-1)\left[(n-1-\Delta_{\mathcal{G}})^{4\alpha+2\beta} - \frac{(n-1-\delta_{\mathcal{G}})^{4\alpha+2\beta}\,(n^2-n-2m)}{4n^2}\right]},$$

$$W_4 = \sqrt{1+\frac{1-n+\delta_{\mathcal{G}}\,(2+\Delta_{\mathcal{G}}-n)}{n^2-n-2m}} + \sqrt{(n-1) - \frac{(n-1)\,(n-1-\delta_{\mathcal{G}})^{4\alpha+2\beta}\,(n^2-n-2m)}{4n^2\,(n-1-\Delta_{\mathcal{G}})^{2\beta+4\alpha}}},$$

$$W_5 = \sqrt{(n-1)\left[(n-1-\delta_{\mathcal{G}})^{4\alpha-2\beta} - \frac{(n-1-\Delta_{\mathcal{G}})^{4\alpha-2\beta}\,(n^2-n-2m)}{4n^2}\right]},$$

$$W_6 = \sqrt{1+\frac{1-n+\delta_{\mathcal{G}}\,(2+\Delta_{\mathcal{G}}-n)}{n^2-n-2m}} + \sqrt{(n-1) - \frac{(n-1)\,(n-1-\Delta_{\mathcal{G}})^{4\alpha-2\beta}\,(n^2-n-2m)}{4n^2\,(n-1-\delta_{\mathcal{G}})^{4\alpha-2\beta}}},$$

$$W_7 = \sqrt{(n-1)\left[(n-1-\Delta_{\mathcal{G}})^{4\alpha-2\beta} - \frac{(n-1-\delta_{\mathcal{G}})^{4\alpha-2\beta}\,(n^2-n-2m)}{4n^2}\right]},$$

$$W_8 = \sqrt{1+\frac{1-n+\delta_{\mathcal{G}}\,(2+\Delta_{\mathcal{G}}-n)}{n^2-n-2m}} + \sqrt{(n-1)-\frac{(n-1)\,(n-1-\delta_{\mathcal{G}})^{4\alpha-2\beta}\,(n^2-n-2m)}{4n^2\,(n-1-\Delta_{\mathcal{G}})^{4\alpha-2\beta}}}.$$

*Proof.* (1). Note that $\Delta_{\overline{\mathcal{G}}} = n-1-\delta_{\mathcal{G}}$ and $\delta_{\overline{\mathcal{G}}} = n-1-\Delta_{\mathcal{G}}$. Using Part (2) of Lemma 4.1 on compliment of a graph $G$, we see that

$$\begin{aligned} 2\,Q &= 2\sum_{1\le i<j\le n}\frac{(d_{\overline{\mathcal{G}}}(v_i)\,d_{\overline{\mathcal{G}}}(v_j))^{2\alpha}}{(d_{\overline{\mathcal{G}}}(v_i)+d_{\overline{\mathcal{G}}}(v_j))^{2\beta}} \le 2\sum_{wz\in E(\overline{\mathcal{G}})}\frac{(\Delta_{\overline{\mathcal{G}}})^{4\alpha}}{2^{2\beta}(\delta_{\overline{\mathcal{G}}})^{2\beta}} \\ &= 2\left[\left(\binom{n}{2}-m\right)\frac{(n-1-\delta_{\mathcal{G}})^{4\alpha}}{2^{2\beta}(n-1-\Delta_{\mathcal{G}})^{2\beta}}\right] = \frac{(n-1-\delta_{\mathcal{G}})^{4\alpha}\,(n^2-n-2m)}{2^{2\beta}(n-1-\Delta_{\mathcal{G}})^{2\beta}}. \end{aligned}$$

Similar to the proof of Theorem 4.6, we get

$$\overline{\sigma}_1 \ge \frac{(\delta_{\overline{\mathcal{G}}})^{2\alpha}(n^2-n-2m)}{n2^{\beta+1}(\Delta_{\overline{\mathcal{G}}})^{\beta}} = \frac{(n-1-\Delta_{\mathcal{G}})^{2\alpha}\,(n^2-n-2m)}{n\,2^{\beta+1}\,(n-1-\delta_{\mathcal{G}})^{\beta}}. \tag{6.5}$$

Applying Cauchy-Schwartz inequality to obtain $(\sum_{i=2}^{n}|\overline{\sigma}_i|)^2 \le (n-1)\sum_{i=2}^{n}\overline{\sigma}_i^2$. Therefore using Part (2) of Lemma 4.1, $(E_{\alpha,\beta}(\overline{\mathcal{G}})-\overline{\sigma}_1)^2 \le (n-1)\,(2Q-\overline{\sigma}_1^2)$.

($a$). From Theorem 5.2, we see that

$$E_{\alpha,\beta}(\mathcal{G}) \le \frac{(n-1)^{2\alpha}}{2^{\beta}}\left[\sqrt{2m-n+1}+\sqrt{2m(n-1)-\frac{R_{\alpha}^2(\mathcal{G})(n-1)^{1-2\beta-4\alpha}}{n^2}}\right]. \tag{6.6}$$

If $\Delta_{\mathcal{G}} = n-1$ or $\Delta_{\overline{\mathcal{G}}} = n-1$, then by using Inequality (6.2), we obtain

$$\begin{aligned} E_{\alpha,\beta}(\overline{\mathcal{G}}) &\le \overline{\sigma}_1 + \sqrt{(n-1)\,(2Q-\overline{\sigma}_1^2)} \\ &\le \frac{(n(\mathcal{G}_1)-1)^{2\alpha}\,\sqrt{2e(\mathcal{G}_1)-n(\mathcal{G}_1)+1}}{2^{\beta}} \\ &+ \sqrt{(n-1)\left[\frac{(n-1-\delta_{\mathcal{G}})^{4\alpha}\,(n^2-n-2m)}{2^{2\beta}(n-1-\Delta_{\mathcal{G}})^{2\beta}} - \frac{(n-1-\Delta_{\mathcal{G}})^{4\alpha}\,(n^2-n-2m)^2}{n^2\,2^{2\beta+2}\,(n-1-\delta_{\mathcal{G}})^{2\beta}}\right]} \\ &= \frac{1}{2^{\beta}}\left[(n(\mathcal{G}_1)-1)^{2\alpha}\,\sqrt{2e(\mathcal{G}_1)-n(\mathcal{G}_1)+1}+W_1\right]. \end{aligned} \tag{6.7}$$

By adding Eqs. (6.6) and (6.7), we get the desired result.

($b$). From proof of Theorem 5.2 (1), we see that

$$E_{\alpha,\beta}(\mathcal{G}) \le \frac{(n-2)^{2\alpha}}{2^{\beta}}\left[\sqrt{2m-n+1}+\sqrt{2m\,(n-1)-\frac{\mathrm{R}_{\alpha}{}^2(\mathcal{G})\,(n-1)}{n^2\,(n-2)^{2\beta+4\alpha}}}\right]. \tag{6.8}$$

If $\Delta_{\mathcal{G}} \le n-2$ and $\Delta_{\overline{\mathcal{G}}} \le n-2$, then using Lemma 4.4 and proof of Theorem 4.6, we get

$$\begin{aligned} E_{\alpha,\beta}(\overline{\mathcal{G}}) &\le \overline{\sigma}_1 + \sqrt{(n-1)\,(2Q-\overline{\sigma}_1^2)} \\ &\le \frac{(n-1-\delta_{\mathcal{G}})^{2\alpha}}{2^{\beta}(n-1-\Delta_{\mathcal{G}})^{\beta}} \sqrt{(n^2-2n-2m+1)+\delta_{\mathcal{G}}(2+\Delta_{\mathcal{G}}-n)} \\ &+ \sqrt{(n-1)\left[\frac{(n-1-\delta_{\mathcal{G}})^{4\alpha}\,(n^2-n-2m)}{2^{2\beta}(n-1-\Delta_{\mathcal{G}})^{2\beta}} - \frac{(n-1-\Delta_{\mathcal{G}})^{4\alpha}\,(n^2-n-2m)^2}{n^2\,2^{2\beta+2}\,(n-1-\delta_{\mathcal{G}})^{2\beta}}\right]} \\ &= \frac{(n-1-\delta_{\mathcal{G}})^{2\alpha}\sqrt{n^2-n-2m}}{2^{\beta}(n-1-\Delta_{\mathcal{G}})^{\beta}}\,W_2 \end{aligned} \tag{6.9}$$

The desired result is obtained by adding Eqs (6.8) and (6.9).
One can prove Parts (2) ~ (4) similarly. □

**Theorem 6.4.** *Let $\mathcal{G} = G(n, m, \Delta_G, \delta_G)$ be a connected graph and $\alpha, \beta$ are real numbers.*

(1). *If $\alpha,\ \beta \ge 0$ then*

$$E_{\alpha,\beta}(\mathcal{G}) + E_{\alpha,\beta}(\overline{\mathcal{G}}) \ge \frac{2^{1-\beta}\ \sqrt{m}}{(n-1)^{\beta}} + \frac{\sqrt{2\,(n^2-n-2m)}\,(n-1-\Delta_{\mathcal{G}})^{2\alpha}}{2^{\beta}\,(n-1-\delta_{\mathcal{G}})^{\beta}}.$$

(2). *If $\alpha,\ \beta \le 0$ then*

$$E_{\alpha,\beta}(\mathcal{G}) + E_{\alpha,\beta}(\overline{\mathcal{G}}) \ge 2^{1-\beta}\,(n-1)^{2\alpha}\ \sqrt{m} + \frac{\sqrt{2\,(n^2-n-2m)}\,(n-1-\delta_{\mathcal{G}})^{2\alpha}}{2^{\beta}\,(n-1-\Delta_{\mathcal{G}})^{\beta}}.$$

(3). *If $\alpha \ge 0$ and $\beta \le 0$ then*

$$E_{\alpha,\beta}(\mathcal{G}) + E_{\alpha,\beta}(\overline{\mathcal{G}}) \ge 2^{1-\beta}\ \sqrt{m} + \frac{\sqrt{2\,(n^2-n-2m)}\,(n-1-\nabla_{\mathcal{G}})^{2\alpha-\beta}}{2^{\beta}}.$$

(4). *If $\alpha \le 0$ and $\beta \ge 0$ then*

$$E_{\alpha,\beta}(\mathcal{G}) + E_{\alpha,\beta}(\overline{\mathcal{G}}) \ge 2^{1-\beta}\,(n-1)^{2\alpha-\beta}\ \sqrt{m} + \frac{\sqrt{2\,(n^2-n-2m)}\,(n-1-\delta_{\mathcal{G}})^{2\alpha-\beta}}{2^{\beta}}.$$

*Proof.* (1). From Theorem 5.2, we see that

$$E_{\alpha,\beta}(\mathcal{G}) \ge \frac{2^{1-\beta}\ \sqrt{m}}{(n-1)^{\beta}}. \tag{6.10}$$

By Part (1) of Lemma 4.1, we have $\sum\limits_{i=1}^{n} \overline{\sigma}_i^2 = -2 \sum\limits_{1\le i<j\le n} \overline{\sigma}_i\overline{\sigma}_j$. Using Part (2) of Lemma 4.1, we obtain

$$(E_{\alpha,\beta}(\overline{\mathcal{G}}))^2 = \Big(\sum_{i=1}^{n} |\overline{\sigma}_i|\Big)^2 = \sum_{i=1}^{n} \overline{\sigma}_i^2 + 2 \sum_{1\le i<j\le n} |\overline{\sigma}_i\,\overline{\sigma}_j| \ \ge\ 2\,Q + 2\,\Big|\sum_{1\le i<j\le n} \overline{\sigma}_i\,\overline{\sigma}_j\Big| \ =\ 4\,Q.$$

We know that $\Delta_{\overline{\mathcal{G}}} = n - 1 - \delta_{\mathcal{G}}$ and $\delta_{\overline{\mathcal{G}}} = n - 1 - \Delta_{\mathcal{G}}$. Now

$$4\,Q = 4 \sum_{1\le i<j\le n} \frac{(d_{\overline{\mathcal{G}}}(v_i)\, d_{\overline{\mathcal{G}}}(v_j))^{2\alpha}}{(d_{\overline{\mathcal{G}}}(v_i) + d_{\overline{\mathcal{G}}}(v_j))^{2\beta}} \ge \sum_{1\le i<j\le n} \frac{4(\delta_{\overline{\mathcal{G}}})^{4\alpha}}{2^{2\beta}(\Delta_{\overline{\mathcal{G}}})^{2\beta}} = \frac{2^{1-2\beta}\,(n^2 - n - 2m)\,(n - 1 - \Delta_{\mathcal{G}})^{4\alpha}}{(n - 1 - \delta_{\mathcal{G}})^{2\beta}}.$$

Hence

$$E_{\alpha,\beta}(\overline{\mathcal{G}}) \ge \frac{(n - 1 - \Delta_{\mathcal{G}})^{2\alpha}\,\sqrt{2\,(n^2 - n - 2m)}}{2^{\beta}\,(n - 1 - \delta_{\mathcal{G}})^{\beta}}. \tag{6.11}$$

The result is obtained by adding Eqs (6.10) and (6.11).

Now using Theorem 5.2, one can prove Parts (2) ∼ (4) in a similar manner. □

## 7. Conclusions

We introduce generalized inverse sum indeg index and energy of graphs. Under certain conditions, we discuss the monotonicity of generalized ISI index by adding edges to a graph. We find extremal graphs with respect to generalized ISI index in class of trees, a class of connected graphs with smallest degree 2 and a class of graphs with given independence number. Bounds on spectral radius and spread of generalized ISI matrix are determined. We also find bounds on generalized ISI energy and Nordhaus-Gaddum-type results for generalized inverse sum indeg index spectral radius and energy. In future, one can find the extremal graphs with respect to generalized ISI index in class of trees, chemical trees, unicyclic graphs, bicyclic graphs for general values of parameters $\alpha$ and $\beta$. One can also study the spectral properties of graph operations with respect to generalized ISI matrix. Extremal graphs with respect to generalized ISI energy in class of trees, chemical trees, unicyclic graphs and bicyclic graphs can also be determined.

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## Conflict of interest

The authors declare no conflict of interest.

## References

1. I. Gutman, O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer, Berlin 1986.
2. I. Gutman, B. Ruščič, N. Trinajstič, et al., *Graph theory and molecular orbitals. XII Acyclic polyenes*, J. Chem. Phys., **62** (1975), 3399–3405.
3. I. Gutman, N. Trinajstič, *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett., **17** (1972), 535–538.
4. S. Hafeez, R. Farooq, *Inverse sum indeg energy of graphs*, IEEE Access, **7** (2019), 100860–100866.

5. R. A. Horn, C. R. Johnson, *Matrix Analysis*, New York, Cambridge Univ. Press, 1990.
6. Y. Hong, *Bounds of eigenvalues of graphs*, Discret. Math., **123** (1993), 65–74.
7. S. M. Hosamani, B. B. Kulkarni, R. G. Boli, et al., *QSPR analysis of certain graph theoretical matrices and their corresponding energy*, Appl. Math. Nonlinear Sci., **2** (2017), 131–150.
8. K. C. Das, I. Gutman, B. Furtula, *On spectral radius and energy of extended adjacency matrix of graphs*, Appl. Math. Comput., **296** (2017), 116–123.
9. S. Fajtlowicz, *On conjectures of Graffiti II*, Congr. Numer., **60** (1987), 187–197.
10. L. Mirsky, *The spread of a matrix*, Mathematika., **3** (1956), 127–130.
11. B. Zhou, N. Trinajstiȼ, *On general sum-connectivity index*, J. Math. Chem., **47** (2010), 210–218.
12. N. J. Rad, A. Jahanbani, I. Gutman, *Zagreb energy and Zagreb Estrada index of graphs*, MATCH Commun. Math. Comput. Chem., **79** (2018), 371–386.
13. M. Randiȼ, *Characterization of molecular branching*, J. Amer. Chem. Soc., **97** (1975), 6609–6615.
14. D. Vukičević, B. Furtula, *Topological index based on the ratios of geometrical and arithmetical means of end-vertex degrees of edges*, J. Math. Chem., **46** (2000), 1369–1376.
15. J. M. Rodriguez, J. M. Sigarreta, *Spectral properties of geometricarithmetic index*, Appl. Math. Comput., **277** (2016), 142–153.
16. J. Sedlar, D. Stevanović, A. Vasilyev, *On the inverse sum indeg index*, Discrete Appl. Math., **184** (2015), 202–212.
17. V. S. Shegehall, R. Kanabur, *Arithmeticgeometric indices of path graph*, J. Math. Comput. Sci., **16** (2015), 19–24.
18. R. Todeschini, V. Consonni, *Molecular Descriptors for Chemoinformatics*, Wiley-VCH, Weinheim, 2009.
19. D. Vukičeviȼ, M. Gaš̌perov, *Bond additive modelling 1. Adriatic indices*, Croat. Chem. Acta., **83** (2010), 261–273.
20. I. Gutman, *The energy of a graph*, Ber. Math. statist. Sekt. Forschungszentrum. Graz., **103** (1978), 1–22.
21. S. Zangi, M. Ghorbani, M. Eslampour, *On the eigenvalues of some matrices based on vertex Degree*, Iranian J. Math. Chem., **9** (2018), 149–156.
22. B. Zhou, N. Trinajstiȼ, *On a novel connectivity index*, J. Math. Chem., **46** (2009), 1252–1270.
23. B. Zhou, N. Trinajsti, *On the sum-connectivity matrix and sum-connectivity energy of (molecular) graphs*, Acta Chim. Slov., **57** (2010), 518–523.
24. H. Deng, H. Huang, J. Zhang, *On the eigenvalues of general sum-connectivity laplacian matrix*, J. Oper. Res. Soc. Chi., **1** (2013), 347–358.
25. Å. B. Bozkurt, A. D. Güngör, I. Gutman, *Randiȼ spectral radius and Randiȼ energy*, MATCH Commun. Math. Comput. Chem., **64** (2010), 321–334.
26. Å. B. Bozkurt, A. D. Güngör, I. Gutman, et al., *Randiȼ matrix and Randiȼ energy*, MATCH Commun. Math. Comput. Chem., **64** (2010), 239–250.
27. B. Bollobás, P. Erdös, *Graphs of extremal weights*, Ars. Combin., **50** (1998), 225–233.

28. R. Gu, F. Huang, X. Li, *General Randiċ matrix and general Randiċ energy*, Trans. Comb., **3** (2014), 21–33.

29. I. Gutman, *Degree-based topological indices*, Croat. Chem. Acta., **86** (2013), 351–361.

30. M. An, L. Xiong, *Some results on the inverse sum indeg index of a graph*, Inf. Proc. Lett., **134** (2018), 42–46.

31. D. Cao, *Bounds on eigenvalues and chromatic numbers*. Linear Algebra Appl., **270** (1998), 1–13.

32. F. Zhang, *Matrix Theory: Basic Results and Techniques*, New York, Springer, 1999.


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