Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(3): 2369–2375.
DOI:10.3934/math.2020156
Received: 29 October 2019
Accepted: 25 February 2020
Published: 04 March 2020

Research article

A note on the space of delta m-subharmonic functions

Van Thien Nguyen[1], **Samsul Ariffin Abdul Karim**[2,*] **and Dinh Dat Truong**[3]

[1] FPT University, Education zone, Hoa Lac high tech park, Km29 Thang Long highway, Thach That ward, Hanoi, Vietnam

[2] Fundamental and Applied Sciences Department and Centre for Smart Grid Energy Research (CSMER), Institute of Autonomous System, Universiti Teknologi PETRONAS, Bandar Seri Iskandar, 32610 Seri Iskandar, Perak Darul Ridzuan, Malaysia

[3] Hanoi National University of Education, 136 Xuan Thuy Street, Cau Giay District, Hanoi, Vietnam

* **Correspondence:** Email: samsul_ariffin@utp.edu.my; Tel: +60192183908.

Abstract: In this note, we present some properties of a certain space of delta m-subharmonic functions. We prove that the convergence in this space implies the convergence in m-capacity.

Keywords: m-subharmonic functions; complex Hessian equations; quasi-Banach space; convergence in capacity
Mathematics Subject Classification: 32U15, 32U20

1. Introduction

Theory of m-subharmonic functions was recently developed by many mathematicians such as Li [20], Błocki [9], Dinew and Kołodziej [14, 15], Lu [21, 22], Sadullaev and Abdullaev [30], Nguyen [23, 24], Åhag, Czyż and Hed [3, 4] and many others. The notion of m-subharmonicity appears naturally in generalization of subharmonicity and plurisubharmonicity. For the similarities and the differences between these notions, we refer the readers to the paper [15].

A bounded domain $\Omega \subset \mathbb{C}^n$ is called m-hyperconvex if there exists an m-subharmonic function $\rho\colon \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega\colon \rho(z) < c\}$ is compact in Ω for every $c \in (-\infty, 0)$. In what follows we will always assume that Ω is an m-hyperconvex domain. Denote by $SH_m(\Omega)$ the set of all m-subharmonic functions in Ω. Let the cones $\mathcal{E}_{0,m}, \mathcal{E}_{p,m}, \mathcal{F}_m$ be defined in the similar way as in [21, 25]:

$$\mathcal{E}_{0,m} = \left\{ u \in SH_m(\Omega) \cap L^\infty(\Omega)\colon \lim_{z\to\partial\Omega} u(z) = 0 \text{ and } \int_\Omega H_m(u) < \infty \right\},$$

$$\mathcal{E}_{p,m} = \left\{ u \in SH_m(\Omega) \colon \exists \{u_j\} \subset \mathcal{E}_{0,m}, u_j \downarrow u, \sup_j \int_\Omega (-u_j)^p H_m(u_j) < \infty \right\},$$

$$\mathcal{F}_m = \left\{ u \in SH_m(\Omega) \colon \exists \{u_j\} \subset \mathcal{E}_{0,m}, u_j \downarrow u \text{ and } \sup_j \int_\Omega H_m(u_j) < \infty \right\}.$$

For the properties and applications of these classes, see [1, 21, 22, 25, 26, 27].

We use the notation $\delta\mathcal{K} = \mathcal{K} = \mathcal{K}$ for $\mathcal{K}$ be one of the classes $\mathcal{E}_{0,m}, \mathcal{E}_{p,m}, \mathcal{F}_m$. Define

$$\|u\|_{p,m} = \inf_{\substack{u=u_1-u_2 \\ u_1,u_2\in\mathcal{E}_{p,m}}} \left\{ \left(\int_\Omega (-u_1 - u_2)^p H_m(u_1 + u_2) \right)^{\frac{1}{m+p}} \right\}, \tag{1.1}$$

with the convention that $(-u_1 - u_2)^p = 1$ if $p = 0$. For the reason why this quasi-norm is effective, please see [2, 13, 16, 22, 29]. It was proved in [25] that $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$ is a quasi-Banach space for $p > 0, p \neq 1$ and it is a Banach space if $p = 1$. Moreover in [17] it was proved that $(\delta\mathcal{F}_m, \|\cdot\|_{0,m})$ is a Banach space. The authors in [12] show that $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$ can not be a Banach space. These facts are counterparts of [5, 6, 10, 18] in m-subharmonic setting.

In Section 2, we shall show that $\mathcal{E}_{0,m}$ and $\delta\mathcal{E}_{0,m}$ are closed neither in $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$ nor in $(\delta\mathcal{F}_m, \|\cdot\|_{0,m})$. Moreover we prove that the inclusions $\mathcal{E}_{0,m} \subsetneq \mathcal{F}_m, \delta\mathcal{E}_{0,m} \subsetneq \delta\mathcal{F}_m$ are proper in the space $(\delta\mathcal{F}_m, \|\cdot\|_{0,m})$.

In Section 3, we prove that the convergence in $\delta\mathcal{E}_{p,m}$ implies the convergence in m-capacity (Theorem 3). But the convergence in m-capacity is not a sufficient condition for the convergence in $\delta\mathcal{E}_{p,m}$ (Example 3). Similar results in plurisubharmonic setting have been proved by Czyż in [11].

2. Preliminaries

In plurisubharmonic case, the following proposition was proved in (see [11]). Let $\mathbb{B} = \mathbb{B}(0,1) \subset \mathbb{C}^n$ be the unit ball in $\mathbb{C}^n$. Then the cones $\mathcal{E}_{0,m}(\mathbb{B})$ and $\delta\mathcal{E}_{0,m}(\mathbb{B})$ are not closed respectively in $(\delta\mathcal{F}_m(\mathbb{B}), \|\cdot\|_{0,m})$ and $(\delta\mathcal{E}_{p,m}(\mathbb{B}), \|\cdot\|_{p,m})$.

Proof. We define

$$v(z) = \begin{cases} \ln|z| & \text{if } m = n, \\ 1 - |z|^{2-\frac{2n}{m}} & \text{if } 1 \le m < n. \end{cases}$$

We obtain that $H_m(v) := dd^c(v) \wedge \beta^{n-m} = c(n,m)\delta_0$, where $c(n,m)$ is a constant depending only on n and m, δ_0 is the Dirac measure at the origin 0 (see [28]). For each $j \in \mathbb{N}$, define the function $v_j \colon \mathbb{B} \to \mathbb{R} \cup \{-\infty\}$ by

$$v_j(z) = \max(a_j v(z), -b_j),$$

where $a_j = \frac{1}{2^j}, b_j = \frac{1}{j}$.

We can see that $v_j \in \mathcal{E}_{0,m}(\mathbb{B})$, for each j. Therefore, the function $u_k := \sum_{j=1}^k v_j$ belongs to $\mathcal{E}_{0,m}(\mathbb{B})$. For $k > l$ we can compute

$$\|u_k - u_l\|_{0,m}^m = \| \sum_{j=l+1}^k v_j \|^m = \int_\mathbb{B} H_m\left(\sum_{j=l+1}^k v_j \right)$$

$$=c(n,m)\left(\sum_{j=l+1}^{k} a_j\right)^m, \tag{2.1}$$

and

$$\begin{aligned}
\|u_k - u_l\|_{p,m}^{p+m} &= \|\sum_{j=l+1}^{k} v_j\|_{p+m}^{p+m} = e_{p,m}\left(\sum_{j=l+1}^{k} v_j\right)\\
&= \int_{\mathbb{B}}\left(-\sum_{j=l+1}^{k} v_j\right)^p H_m\left(\sum_{j=l+1}^{k} v_j\right)\\
&= c(n,m)\sum_{j_1,\cdots,j_m=l+1}^{k}\left[-\sum_{r=l+1}^{k} v_r(\max(t_{j_1},\cdots,t_{j_m}))\right]^p a_{j_1}\cdots a_{j_m}\\
&\leq c(n,m)\sum_{j_1,\cdots,j_m=l+1}^{k}\left[-u_k(\max(t_{j_1},\cdots,t_{j_m}))\right]^p a_{j_1}\cdots a_{j_m}\\
&\leq c(n,m)\left[\sum_{j=l+1}^{k}(-u_k(t_j))^{\frac{p}{m}} a_j\right]^m,
\end{aligned}$$

where

$$t_j = \begin{cases} \left(1+\frac{b_j}{a_j}\right)^{\frac{m}{2(m-n)}}, & \text{if } 1 \leq m < n,\\ e^{-\frac{b_j}{a_j}}, & \text{if } m = n. \end{cases}$$

The last inequality is a consequence of the fact that v_j is increasing function for each j. Since

$$v_l(t_j) = \begin{cases} -\frac{1}{l}, & \text{if } 1 \leq l \leq j,\\ -\frac{2^j}{j2^l}, & \text{if } l > j, \end{cases}$$

we have

$$-u_k(t_j) = \sum_{l=1}^{j}\frac{1}{l} + \frac{2^j}{j}\sum_{l=j+1}^{k}\frac{1}{2^l} \leq j+1.$$

Hence

$$\|u_k - u_l\|_{p,m}^{p+m} \leq c(n,m)\left(\sum_{j=l+1}^{k}\frac{(j+1)^{\frac{p}{m}}}{2^j}\right)^m. \tag{2.2}$$

Let $u\colon \mathbb{B} \to \mathbb{R}\cup\{-\infty\}$ be defined by $u = \lim_{k\to\infty} u_k$. Observe that u is the limit of a decreasing sequence of m-subharmonic functions and $u(z) > -\infty$ on the boundary of the ball $B(0,\frac{1}{2})$. Hence u is m-subharmonic. Moreover $u \notin \mathcal{E}_{0,m}(\mathbb{B})$ since it is not bounded on $\mathbb{B}$, its value is not bounded below at the origin. Equality (2.1) shows that $\{u_k\}$ is a Cauchy sequence in the space $\delta\mathcal{F}_m(\mathbb{B})$. Thus the cone $\mathcal{E}_{0,m}(\mathbb{B})$ and the space $\delta\mathcal{E}_{0,m}(\mathbb{B})$ are not closed in $(\delta\mathcal{F}_m(\mathbb{B}), \|\cdot\|_{0,m})$.

The series $\sum_{j=1}^{\infty}\frac{(j+1)^{\frac{p}{m}}}{2^j}$ is convergent by the ratio test. Therefore $\{u_k\}$ is a Cauchy sequence in $\delta\mathcal{E}_{p,m}$ by (2.2). We have proved that the cone $\mathcal{E}_{0,m}(\mathbb{B})$ and the space $\delta\mathcal{E}_{0,m}(\mathbb{B})$ are not closed in $(\delta\mathcal{E}_{p,m}(\mathbb{B}), \|\cdot\|_{p,m})$. □

The following proposition shows that the closure of the cone $\mathcal{E}_{0,m}$ (resp. $\delta\mathcal{E}_{0,m}$) is strictly smaller than $\mathcal{F}_m$ (resp. $\delta\mathcal{F}_m$) in the space $(\delta\mathcal{F}_m, \|\cdot\|_{0,m})$. We have $\overline{\mathcal{E}_{0,m}} \subsetneq \mathcal{F}_m$ and $\overline{\delta\mathcal{E}_{0,m}} \subsetneq \delta\mathcal{F}_m$ in the space $(\delta\mathcal{F}_m, \|\cdot\|_{0,m})$.

Proof. The definition of the m-Lelong number of a function $v \in SH_m(\Omega)$ at $a \in \Omega$ is the following

$$\nu_{m,a}(v) = \lim_{r\to 0^+} \int_{|z-a|\le r} dd^c v \wedge \left[dd^c(-|z-a|^{2-\frac{2n}{m}})\right]^{m-1} \wedge \beta^{n-m}$$

It is easy to see that m-Lelong number is a linear functional on $\delta\mathcal{F}_m$. Moreover, as in [7, Remark 1], for a function $\varphi \in \mathcal{F}_m$ then

$$\nu_{m,a}(\varphi) \le (H_m(\varphi)(\{a\}))^{\frac{1}{m}} \le (H_m(\varphi)(\Omega))^{\frac{1}{m}}.$$

Hence, for any representation $u = u_1 - u_2$ of $u \in \delta\mathcal{F}_m$ we have

$$|\nu_{m,a}(u)| \le (H_m(u_1 + u_2)(\Omega))^{\frac{1}{m}}.$$

This implies that m-Lelong number is a bounded functional on the space $\delta\mathcal{F}_m$. We have shown that m-Lelong number is continuous on the Banach space $(\delta\mathcal{F}_m, \|\cdot\|_{0,m})$. We recall the definition of m-Green function with pole at a

$$g_{m,\Omega,a}(z) = \sup\{v \in SH_m^-(\Omega) \colon u(z) + |z-a|^{2-\frac{2n}{m}} \le O(1) \text{ as } z \to a\}.$$

The readers can find more properties of m-Green function in [31]. Assume that $\overline{\mathcal{E}_{0,m}} = \mathcal{F}_m$. Then there exists a sequence $\{u_j\}$ in $\mathcal{E}_{0,m}$ that converges to $g_{m,\Omega,a}$ in the space $\delta\mathcal{F}_m$ as $j \to \infty$. The m-Lelong number of all u_j at a vanishes since u_j is bounded, but the m-Lelong number of $g_{m,\Omega,a}$ at a is 1. Hence we get a contradiction. Thus, $\overline{\mathcal{E}_{0,m}} \subsetneq \mathcal{F}_m$. By the same argument, if $\overline{\delta\mathcal{E}_{0,m}} = \delta\mathcal{F}_m$, then there exists a sequence $\{u_j\}$ in $\mathcal{E}_{0,m}$ that converges to $g_{m,\Omega,a}$ in the space $\delta\mathcal{F}_m$ as $j \to \infty$, but this is impossible since $\nu_{m,a}(u_j) = 0$. □

3. The convergence in $\delta\mathcal{E}_{p,m}$

We are going to recall a Błocki type inequality (see [8]) for the class $\mathcal{E}_{p,m}$. Similar results for the class $\mathcal{F}_m$ were proved by Hung and Phu in [19, Proposition 5.3] (see also [1]) and for locally bounded functions were proved by Wan and Wang [31]. Assume that $v \in \mathcal{E}_{p,m}$ and $h \in SH_m$ is such that $-1 \le h \le 0$. Then

$$\int_\Omega (-v)^{m+p} H_m(h) \le m! \int_\Omega (-v)^p H_m(v).$$

Proof. See the proof of [19, Proposition 5.3]. □

Recall that the relative m-capacity of a Borel set $E \subset \Omega$ with respect to Ω is defined by

$$cap_{m,\Omega}(E) = \sup\{\int_E H_m(u) \colon u \in SH_m(\Omega), -1 \le u \le 0\}.$$

We are going to recall the convergence in m-capacity. We say that a sequence $\{u_j\} \subset SH_m(\Omega)$ converges to $u \in SH_m(\Omega)$ in m-capacity if for any $\epsilon > 0$ and $K \Subset \Omega$ then we have

$$\lim_{j\to\infty} cap_{m,\Omega}(K \cap \{|u_j - u| > \epsilon\}) = 0.$$

Let $\{u_j\} \subset \delta\mathcal{E}_{p,m}$ be a sequence that converges to a function $u \in \delta\mathcal{E}_{p,m}$ as j tends to ∞. Then $\{u_j\}$ converges to u in m-capacity.

Proof. Replacing u_j by $u_j - u$, we can assume that $u = 0$. By the definition of $\delta\mathcal{E}_{p,m}$, there exist functions $v_j, w_j \in \mathcal{E}_{p,m}$ such that $u_j = v_j - w_j$ and $e_p(v_j + w_j) \to 0$ as $j \to \infty$. By [25],

$$\max(e_{p,m}(v_i), e_{p,m}(w_j)) \leq e_{p,m}(v_j + w_j),$$

which implies that $e_{p,m}(v_j), e_{p,m}(w_j)$ tend to 0 as $j \to \infty$. Given $\epsilon > 0$ and $K \Subset \Omega$. For a function $\varphi \in SH_m(\Omega)$, $-1 \leq \varphi \leq 0$, we have

$$\int_{\{|v_j|>\epsilon\}\cap K} H_m(\varphi) \leq \frac{1}{\epsilon^{p+m}} \int_{\Omega} (-v_j)^{p+m} H_m(\varphi) \leq \frac{m!}{\epsilon^{p+m}} e_{p,m}(v_j). \tag{3.1}$$

The last inequality comes from Lemma 3. Hence, by taking the supremum over all functions φ in inequality (3.1), we get

$$cap_{m,\Omega}(\{|v_j| > \epsilon\} \cap K) \leq \frac{m!}{\epsilon^{m+p}} e_{p,m}(v_j). \tag{3.2}$$

Similarly,

$$cap_{m,\Omega}(\{|w_j| > \epsilon\} \cap K) \leq \frac{m!}{\epsilon^{m+p}} e_{p,m}(w_j). \tag{3.3}$$

From (3.2), (3.3) we obtain

$$\begin{aligned} & cap_{m,\Omega}(\{|u_j| > \epsilon\} \cap K) \\ \leq & cap_{m,\Omega}(\{|v_j| > \frac{\epsilon}{2}\} \cap K) + cap_{m,\Omega}(\{|w_j| > \frac{\epsilon}{2}\} \cap K) \\ \leq & \frac{m! 2^{m+p}}{\epsilon^{m+p}} (e_{p,m}(v_j) + e_{p,m}(w_j)) \to 0 \text{ as } j \to \infty. \end{aligned}$$

Hence the sequence $\{u_j\}$ tends to 0 in m-capacity and the proof is finished. □

A similar result for the space $\delta\mathcal{F}_m$ is proved in [17]. But the convergence in m-capacity is not a sufficient condition for the convergence in the space $\delta\mathcal{E}_{p,m}$. The following example shows that convergence in m-capacity is strictly weaker than convergences in both $\delta\mathcal{E}_{p,m}$ and $\delta\mathcal{F}_m$. The case $m = n$ has been showed in [11, Example 3.3]. Let $v(z)$ be the function defined in the unit ball in $\mathbb{C}^n$ as in the proof of Proposition 2. We define

$$u_j(z) = \max(j^{\frac{p}{m}} v(z), -\frac{1}{j}), \; v_j(z) = \max(v(z), -\frac{1}{j})$$

Then we have $u_j, v_j \in \mathcal{E}_{0,m}(\mathbb{B})$ for every j, and $e_{p,m}(u_j) = c(n,m), e_{0,m}(v_j) = 1$. These show that the sequence $\{u_j\}$ and $\{v_j\}$ do not converge to 0 in $\delta\mathcal{E}_{p,m}(\mathbb{B})$ and $\delta\mathcal{F}_m(\mathbb{B})$ respectively as $j \to \infty$. Moreover, for fixed $\epsilon > 0$ and $K \Subset \mathbb{B}$ there exists j_0 such that for all $j \geq j_0$ we have

$$u_j = v_j = -\frac{1}{j} \text{ on } K.$$

This infers that both sets $K \cap \{u_j < -\epsilon\}$ and $K \cap \{v_j < -\epsilon\}$ are empty. Hence u_j and v_j tend to 0 in m-capacity.

Acknowledgements

The authors would like to thank Rafał Czyż for many valuable comments and suggestions for this manuscript. We are grateful to the referee whose remarks and comments helped to improve the paper.

Conflict of interest

The authors declare no conflict of interest.

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