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Research article

Numerical solutions of 2D Fredholm integral equation of first kind by discretization technique

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Abstract: A novel numerical technique to solve 2D Fredholm integral equations (2DFIEs) of first kind is proposed in this study. This technique is based on the discretization of 2DFIEs by replacing the unknown function with two-dimensional Bernstein polynomial basis functions. We formulate the convergence analysis which shows the fast converges of this technique to the actual solution. Some problems of 2D linear Fredholm integral equations are illustrated to show the efficiency of the proposed scheme.

Keywords: discretization; numerical solution; 2D Fredholm integral equations ; 2D Bernstein's approximation; convergence analysis
Mathematics Subject Classification: 65A05, 65G99, 65R20

1. Introduction

The mathematical modeling of physical phenomena mostly leads towards integral equations. There are two major types of integral equations, Fredholm integral equation and Volterra integral equation. Further these equations are subdivided into two categories of first and second kind. There are many works on developing and analyzing numerical methods for solving one-dimensional integral equations [1–3]. But little work has been done for two and higher dimensional integral equations [4–8]. Here we are dealing with two-dimensional linear Fredholm integral equations (2DFIEs) of first kind. These equations occurs in various physical and engineering models such as chemical kinetics, fluid dynamics, image processing, electromagnetic, signal processing of radar and many more. Tari and Shahmorad [9] presented a computational method for solving two-dimensional linear Fredholm integral equations of the second kind which based on Legendre or any orthogonal polynomial. Xie and Lin [10] solve the 2DFIEs of second kind by using matrix-vector multiplication algorithms and efficient preconditioners. Guoqiang and Jiong [11] introduced the extrapolation of nystrom solution for 2D non-linear Fredholm integral equations. One of the well-known numerical methods is finite difference method for the solution of integral and differential equations [12–14]. The main disadvantage of this method is to generate computational mesh for any solution which takes time even bigger than meshless methods. The literature on this subject is very dense and it is still expanding as several authors discussed many analytical and numerical technique to solve differential and integral equations [15–20]. The use of Berstein polynomial to solve differential and integral equations has been recently increased because of the fast convergence and less computational cost. This paper develops a numerical technique to solve 2DFIE of first kind by using Bernstein approximation. It gives good accuracy even for lower degree Bernstein polynomial. As degree of the Bernstein polynomial is increased, the convergence of approximate solution to the exact solution is also increased. So, this technique is faster, simple and effectual. This paper is divided into five sections. Second section deals with the basic concepts. In section 3, numerical technique based on 2D Bernstein basis functions is given. In Section 4, some results about convergence analysis are provided. In the last section some numerical problems are carried out. All the computations are performed using MATLAB.

2. Basic concepts

The Bernstein approximation of a function $u : I_1 \times I_2 \to R$ is defined as

$$B_{m,n}(u(x,y)) = \sum_{i=0}^{m}\sum_{j=0}^{n} B_{j,n}^{i,m}(x,y)u\left(a + \frac{\alpha}{m}i, c + \frac{\beta}{n}j\right), \tag{2.1}$$

where $B_{j,n}^{i,m}(x,y) = \eta_{ij}\mu_{ij}(x,y)$ is known as the 2D Bernstein polynomial basis with $x \in I_1$ and $y \in I_2$.

Here

$$\eta_{ij} = \frac{C_i^m C_j^n}{\alpha^m \beta^n},\ \alpha = b-a,\ \beta = d-c,\ \mu_{ij}(x,y) = (x-a)^i(b-x)^{m-i}(c-y)^j(d-y)^{n-j}, \tag{2.2}$$

$$i = 0,\ldots,m, \quad j = 0,\ldots,n,$$

for $I_1 = [a,b]$, $I_2 = [c,d]$ and m, n are arbitrary positive integers.

Theorem 2.1. *(Uniformly Convergence) [5] Let $u \in C^2[I_1 \times I_2]$ and $X \in I_1 \times I_2$. Then the $B_{m,n}u(X)$ converges uniformly to function u for $m,n \to \infty$.*

Now consider the asymptotic formula for 2D Bernstein polynomial approximation for the case $m = n$.

Theorem 2.2. *(Asymptotic Formula) [5] Let $u \in C^2[I_1 \times I_2]$ and $X \in I_1 \times I_2$ then*

$$\lim_{m\to\infty} m((B_{m,m}u(X)) - u(X)) = \sum_{i=1}^{2} \frac{(x_i - a_i)(b_i - x_i)}{2}\frac{\partial^2 u(X)}{\partial x_i^2}$$

$$\leq \frac{1}{8} \sum_{i=1}^{2} ((a_i - b_i)^2) \frac{\partial^2 u(X)}{\partial x_i^2}. \tag{2.3}$$

Proof. See in [5]. □

3. The proposed numerical technique

Consider the 2D Fredholm integral equation of first kind

$$g(x_1, y_1) = \lambda \int_a^b \int_c^d \mathcal{K}(x, x_1, y, y_1) u(x, y) dx dy, \tag{3.1}$$

where $g(x_1, y_1)$ and $\mathcal{K}(x, x_1, y, y_1)$ are analytic functions, $u(x, y)$ is the unknown function and $x, x_1 \in I_1$; $y, y_1 \in I_2$.

To find the numerical solution of (3.1), unknown function is approximated with the help of Bernstein's approximation given in (2.1).

Equation (3.1) can be written as

$$g(x_{1,k}, y_{1,l}) = \sum_{i=0}^{m} \sum_{j=0}^{n} u\left(a + \frac{\alpha}{m} i, c + \frac{\beta}{n} j\right) \eta_{ij} \left[\lambda \int_a^b \int_c^d \mathcal{K}(x, x_{1,k}, y_{1,l}, y) \mu_{ij}(x, y) dx dy\right] \tag{3.2}$$

where η_{ij}, $\mu_{ij}(x, y)$ are defined in (2.2). To find the values of $u\left(a + \frac{\alpha}{m} i, c + \frac{\beta}{n} j\right)$, equation (3.2) is converted into a set of algebraic equations by substituting x_1 as $x_{1,k} = a + \frac{\alpha}{m} k + \epsilon$, $k = 0, \ldots, m-1$ and $x_{1,m} = b - \epsilon$, and y_1 as $y_{1,l} = c + \frac{\beta}{n} l + \epsilon$, $l = 0, \ldots, n-1$ and $y_{1,n} = d - \epsilon$, where ϵ is arbitrary small positive number.

The following matrices A, B and U represent the system of algebraic equations produced by (3.2).

$$A = \eta_{ij} \left[\lambda \int_a^b \int_c^d \mathcal{K}(x, x_{1,k}, y_{1,l}, y) \mu_{ij}(x, y) dx dy\right] = [A_{ijkl}]_{(m+1)\times(n+1)} \tag{3.3}$$

$$= \begin{bmatrix} A_{ij00} & A_{ij01} & \cdots & A_{ij0n} \\ A_{ij10} & A_{ij11} & & A_{ij1n} \\ \vdots & \vdots & \ddots & \\ & & & A_{ij(m-1)n} \\ A_{ijm0} & A_{ijm1} & & A_{ijmn} \end{bmatrix}$$

where

$$A_{ij\nu\omega} = \begin{bmatrix} \eta_{00}\left[\lambda \int_a^b \int_c^d \mathcal{K}(x, x_{1,\nu}, y_{1,\omega}, y) \mu_{00}(x,y) dx dy\right] & \cdots & \eta_{0n}\left[\lambda \int_a^b \int_c^d \mathcal{K}(x, x_{1,\nu}, y_{1,\omega}, y) \mu_{0n}(x,y) dx dy\right] \\ \eta_{10}\left[\lambda \int_a^b \int_c^d \mathcal{K}(x, x_{1,\nu}, y_{1,\omega}, y) \mu_{10}(x,y) dx dy\right] & & \eta_{1n}\left[\lambda \int_a^b \int_c^d \mathcal{K}(x, x_{1,\nu}, y_{1,\omega}, y) \mu_{1n}(x,y) dx dy\right] \\ \vdots & \ddots & \\ \eta_{m0}\left[\lambda \int_a^b \int_c^d \mathcal{K}(x, x_{1,\nu}, y_{1,\omega}, y) \mu_{m0}(x,y) dx dy\right] & & \eta_{mn}\left[\lambda \int_a^b \int_c^d \mathcal{K}(x, x_{1,\nu}, y_{1,\omega}, y) \mu_{mn}(x,y) dx dy\right] \end{bmatrix}$$

for $\nu = 0, 1, \ldots, m$, $\omega = 0, 1, \ldots, n$ respectively. Meanwhile matrix U is given by

$$U = \left[u\left(a + \frac{\alpha}{m}i, c + \frac{\beta}{n}j\right)\right]^t = [U_{ij}]^t_{(n+1)\times(1)} = \begin{bmatrix} U_{i0} \\ U_{i1} \\ \\ \vdots \\ \\ U_{in} \end{bmatrix}$$

where

$$U_{ij} = \left[u\left(a + \frac{\alpha}{m}i, c + \frac{\beta}{n}j\right)\right]^t = \begin{bmatrix} u\left(a, c + \frac{\beta}{n}j\right) \\ u\left(a + \frac{\alpha}{m}, c + \frac{\beta}{n}j\right) \\ \\ \vdots \\ \\ u\left(b, c + \frac{\beta}{n}j\right) \end{bmatrix}$$

for $j = 0, 1, 2, \ldots, n$ and

$$B = [g(x_{1,k}, y_{1,l})]^t = [B_{kl}]^t_{(n+1)\times(1)} = \begin{bmatrix} B_{k0} \\ B_{k1} \\ \\ \vdots \\ \\ B_{kn} \end{bmatrix}$$

where

$$B_{kl} = [g(x_{1,k}, y_{1,l})]^t = \begin{bmatrix} g(x_{1,1}, y_{1,l}) \\ g(x_{1,2}, y_{1,l}) \\ \\ \vdots \\ \\ g(x_{1,m}, y_{1,l}) \end{bmatrix}.$$

Here $u\left(a + \frac{\alpha}{m}i, c + \frac{\beta}{n}j\right)$ are our solutions at nodes $\left(a + \frac{\alpha}{m}i, c + \frac{\beta}{n}j\right)$ for $i = 0, \ldots, m$; $j = 0, \ldots, n$ and by imposing it in (2.1), we obtain $B_{m,n}(u(x_{1,k}, y_{1,l}))$, the approximate solution of (3.1).

Error bound

Theorem 3.1. *Suppose that $\mathcal{K}(x, x_1, y, y_1)$ and $g(x_1, y_1)$ are analytical functions of 2DFIE* (3.1) *on $I_1^2 \times I_2^2$ and $I_1 \times I_2$ respectively. If A defined in* (3.3) *is invertible then*

$$\begin{aligned} &\sup_{x_{1,k}\in I_1, y_{1,l}\in I_2} |u(x_{1,k}, y_{1,l}) - B_{m,n}(u_{m,n}(x_{1,k}, y_{1,l}))| \\ \leq\ & (1 + \alpha\beta M\|A^{-1}\|)\left[\frac{\alpha^2}{8m}\|u_{x_1x_1}\| + \frac{\beta^2}{8n}\|u_{y_1y_1}\|\right] \end{aligned}$$

where $x_{1,k} = a + k\frac{\alpha}{m}$, $y_{1,l} = c + l\frac{\beta}{n}$, $k = 0, \ldots, m$; $l = 0, \ldots, n$, $u(x_1, y_1)$ *is the actual solution,*

$$M = \sup_{x,x_1 \in I_1, y, y_1 \in I_2} |\lambda \mathcal{K}(x, x_1, y, y_1)|$$

and $B_{m,n}(u_{m,n}(x_{1,k}, y_{1,l})$ *is proposed solution of* (3.1).

Proof. Consider

$$\begin{aligned} &\sup_{x_{1,k} \in I_1, y_{1,l} \in I_2} |u(x_{1,k}, y_{1,l}) - B_{m,n}(u_{m,n}(x_{1,k}, y_{1,l}))| \\ &\leq \sup_{x_{1,k} \in I_1, y_{1,l} \in I_2} |u(x_{1,k}, y_{1,l}) - u_{m,n}(x_{1,k}, y_{1,l})| + \\ &\sup_{x_{1,k} \in I_1, y_{1,l} \in I_2} |u_{m,n}(x_{1,k}, y_{1,l}) - B_{m,n}(u_{m,n}(x_{1,k}, y_{1,l}))| \end{aligned} \tag{3.4}$$

From Theorem 2.2 the following bound is obtained:

$$\sup_{x_{1,k} \in I_1, y_{1,l} \in I_2} |u_{m,n}(x_{1,k}, y_{1,l}) - B_{m,n}(u_{m,n}(x_{1,k}, y_{1,l}))| \leq \frac{\alpha^2}{8m} \|u_{x_1 x_1}\| + \frac{\beta^2}{8n} \|u_{y_1 y_1}\|. \tag{3.5}$$

To obtain a bound of

$$\sup_{x_{1,k} \in I_1, y_{1,l} \in I_2} |u(x_{1,k}, y_{1,l}) - u_{m,n}(x_{1,k}, y_{1,l})|$$

we have $AU = B$, $A\hat{U} = \hat{B}$ where

$$B = g(x_1, y_1), \hat{B} = \hat{g}(x_1, y_1), U = B_{m,n}(u(x_1, y_1)) \text{ and } \hat{U} = B_{m,n}(u_{m,n}(x_1, y_1))$$

and we obtain $\hat{g}$ by replacing $u(x_1, y_1)$ with $u_{m,n}(x_1, y_1)$ defined in (3.1). Now by replacing x_1 with $a + \alpha\frac{k}{m}$ and y_1 with $c + \beta\frac{k}{n}$ lead us to:

$$g(x_{1,k}, y_{1,l}) = u(x_{1,k}, y_{1,l})A, \qquad \hat{g}(x_{1,k}, y_{1,l}) = u_{m,n}(x_{1,k}, y_{1,l})A \tag{3.6}$$

and consequently,

$$\sup_{x_{1,k} \in I_1, y_{1,l} \in I_2} |(u(x_{1,k}, y_{1,l}) - u_{m,n}(x_{1,k}, y_{1,l}))| = |g(x_{1,k}, y_{1,l}) - \hat{g}(x_{1,k}, y_{1,l})| \|A^{-1}\|. \tag{3.7}$$

Now consider

$$g(x_1, y_1) - \hat{g}(x_1, y_1) = \lambda \int_a^b \int_c^d \mathcal{K}(x, x_1, y, y_1)((u(x, y) - B_{m,n}(u(x, y)))dxdy.$$

This implies that

$$\sup_{x_1 \in I_1, y_1 \in I_2} |g(x_1, y_1) - \hat{g}(x_1, y_1)| \leq \sup_{x_1, x \in I_1, y_1, y \in I_2} |\lambda \int_a^b \int_c^d \mathcal{K}(x, x_1, y, y_1)(u(x, y) - B_{m,n}(u(x, y)))dxdy|$$

$$\leq \left(\frac{\alpha^2}{8m}\|u_{x_1x_1}\| + \frac{\beta^2}{8n}\|u_{y_1y_1}\|\right)(\alpha\beta M)$$

where

$$M = \sup_{x,x_1 \epsilon I_1, y, y_1 \epsilon I_2} |\lambda\mathcal{K}(x, x_1, y, y_1)|.$$

Now (3.7) becomes

$$\sup_{x_{1,k}\in I_1, y_{1,l}\in I_2} |u(x_{1,k}, y_{1,l}) - u_{m,n}(x_{1,k}, y_{1,l})| \leq \|A^{-1}\| \left[\left(\frac{\alpha^2}{8m}\|u_{x_1x_1}\| + \frac{\beta^2}{8n}\|u_{y_1y_1}\|\right)(\alpha\beta M)\right], \tag{3.8}$$

using (3.5) and (3.8), Inequality (3.4) becomes

$$\begin{aligned}
&\sup_{x_{1,k}\epsilon I_1, y_{1,l}\epsilon I_2} |u(x_{1,k}, y_{1,l}) - B_{m,n}(u_{m,n}(x_{1,k}, y_{1,l}))| \\
\leq\ & \left(\frac{\alpha^2}{8m}\|u_{x_1x_1}\| + \frac{\beta^2}{8n}\|u_{y_1y_1}\|\right) + (\alpha\beta M)\left(\frac{\alpha^2}{8m}\|u_{x_1x_1}\| + \frac{\beta^2}{8n}\|u_{y_1y_1}\|\right)\|A^{-1}\| \\
\leq\ & [1 + \alpha\beta M\|A^{-1}\|]\left[\frac{\alpha^2}{8m}\|u_{x_1x_1}\| + \frac{\beta^2}{8n}\|u_{y_1y_1}\|\right].
\end{aligned}$$

That completes the proof.

□

Lemma 3.2. *Suppose that $\|A - I\| = r_2 < 1$, I is the identity matrix of same order as A, $\|.\|$ is the maximum norm of rows. Then*

$$\|(A^{-1}\| \leq \frac{1}{1 - r_2}, \quad Cond(A) \leq \frac{r_1\alpha\beta}{1 - r_2},$$

where

$$\max_{k,l} |\lambda\mathcal{K}(x_{1,k}, y_{1,l}, x, y)| = r_1 \quad \textit{and} \quad \max_{k,l} |\sum_{i=0}^{m}\sum_{j=0}^{n} \eta_{ij}\mu_{ij}(x_{1,k}, y_{1,l})| = 1.$$

Proof. Let $Cond(A) = \|A\|.\|A^{-1}\|$, to get a bound of $\|A\|$, consider (3.3)

$$\|A\| = \max_{k,l}\left|\sum_{i=0}^{m}\sum_{j=0}^{n}\eta_{ij}\left[\lambda\int_a^b\int_c^d \mathcal{K}(x_{1,k}, y_{1,l}, x, y)\mu_{ij}(x, y)dxdy\right]\right|$$

$$= \max_{k,l}\left|\lambda\int_a^b\int_c^d\sum_{i=0}^{m}\sum_{j=0}^{n}\mathcal{K}(x_{1,k}, y_{1,l}, x, y)\eta_{ij}\mu_{ij}(x, y)dxdy\right|$$

$$\leq r_1\int_a^b\int_c^d dxdy = r_1\alpha\beta, \tag{3.9}$$

where η_{ij}, α, β, μ_{ij} are defined in (2.2) and

$$\max_{k,l} |\lambda\mathcal{K}(x, x_{1,k}, y, y_{1,l})| = r_1.$$

Let $D = A - I$, which implies $||D|| = ||A - I|| = r_2 \leq 1$. Now, to find the bound for $||A^{-1}||$, then by applying geometric series sum on $||A^{-1}|| = ||(I + D)^{-1}||$ we get

$$||A^{-1}|| = \frac{1}{1 - r_2}. \tag{3.10}$$

Hence from (3.9) and (3.10)

$$Cond(A) \leq \frac{r_1\alpha\beta}{1 - r_2}.$$

This completes the proof. □

4. Numerical applications

In this section, precision of proposed technique is presently endorsed by considering some examples. The numerical results of these examples are shown with the help of tables and figures. It is also easy to see from Tables 1, 2 and 3 that the presented technique is very effectual and simple. Absolute error of actual and numerical solution is measured as follows,

$$|e_{m,n}(x_{1,k}, y_{1,l})| = |u(x_{1,k}, y_{1,l}) - B_{m,n}(u(x_{1,k}, y_{1,l}))|,$$

where a, b, c, d are limits of Fredholm integral equation defined in (3.1) for $x_{1,k} = a + \alpha\frac{k}{m} + \epsilon$, $k = 0, \ldots, m - 1$, $x_{1,m} = b - \epsilon$ and $y_{1,l} = c + \beta\frac{l}{n} + \epsilon$, $l = 0, \ldots, n - 1$, $y_{1,n} = d - \epsilon$.

Example 1. *Consider 2DFIE of first kind*

$$0.77364 \sin(x_1 + y_1) = \int_0^1 \int_0^1 \sin(x_1 + y_1)u(x, y)dxdy, \tag{4.1}$$

with exact solution $u(x, y) = \sin(x + y)$, *where* $x, x_1 \in [0, 1]$ *and* $y, y_1 \in [0, 1]$. *Table* 1 *shows absolute error at* $x = y = 0.5$ *on various degree of Bernstein polynomial. The graphical representation of true and numerical solution is illustrated in Figure* 1.

Table 1. Comparison of true and numerical solutions of Example 1 at node (0.5,0.5). The absolute error shows error decreases with the increase of the degree of Bernstein polynomial.

m=n	True Solution	Numerical Solution	Absolute Error
2	0.841500000	0.843547682	$2.076697312E^{-3}$
3	0.841500000	0.841899649	$4.286648348E^{-4}$
4	0.841500000	0.841467886	$3.098606273E^{-6}$
5	0.841500000	0.841468738	$2.246771569E^{-6}$
6	0.841500000	0.841470988	$3.921261258E^{-9}$

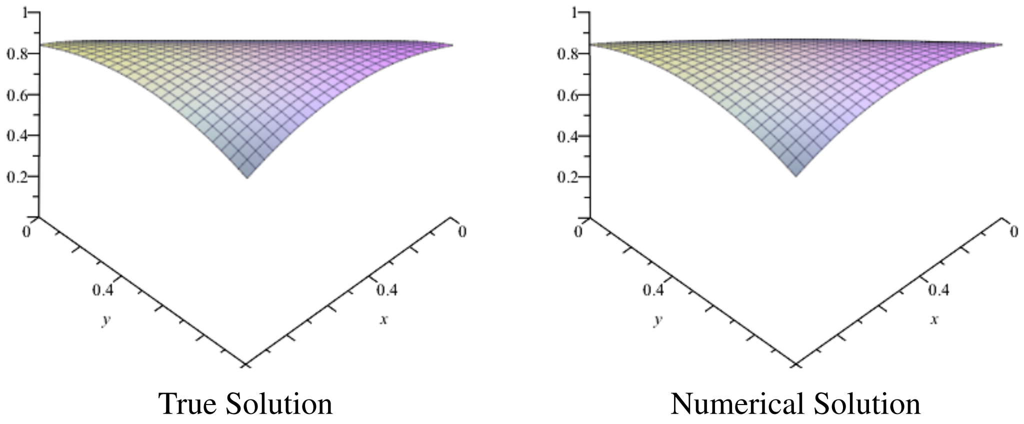


Figure 1. Comparison between true and numerical solutions of Example 1 at m=n=2 obtained by proposed technique.

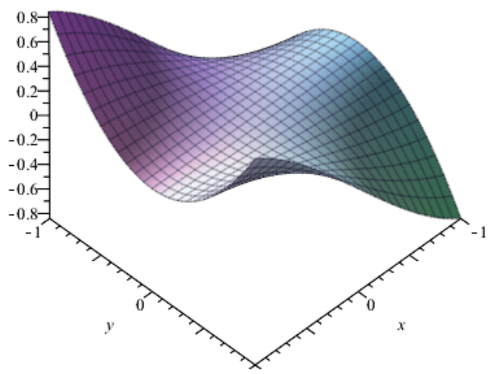
Example 2. *Consider the following 2DFIE*

$$0.803116 = \int_{-1}^{1}\int_{-1}^{1} x_1 + y_1 + 2yu(x,y)dxdy, \tag{4.2}$$

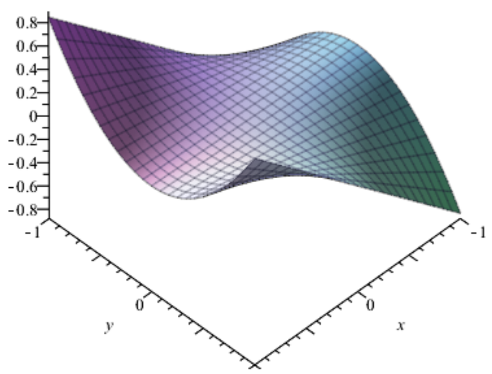
with exact solution $u(x,y) = x^2 \sin y$*, where* $x, x_1 \in [-1,1]$ *and* $y, y_1 \in [-1,1]$*. The approximate solutions of the integral equation is obtained by using Bernstein basis function technique. Table* 2 *shows true and numerical solutions at* $(0.0, 0.5)$ *and Figure* 2 *is graphical representation of numerical and true solutions.*

Table 2. Comparison of true and numerical solutions of Example 2 at node (0,0.5). The absolute error shows error decreases with the increase of the degree of Bernstein polynomial.

m=n	True Solution	Numerical Solution	Absolute Error
2	0.000000000	0.017104157	$1.710415766E^{-2}$
3	0.000000000	0.000280012	$2.800121081E^{-4}$
4	0.000000000	0.000156312	$1.563121317E^{-4}$
5	0.000000000	0.000001719	$1.719066267E^{-6}$
6	0.000000000	0.000001043	$1.043939595E^{-6}$



True Solution



Numerical Solution

Figure 2. Comparison between true and numerical solutions of Example 2 at m=n=2 obtained by proposed technique.

Example 3. *Consider the 2D FIE*

$$2-(x_1-y_1)(e^2+1)-2e(x_1+y_1+1)=\int_0^1\int_0^1(x+y+x_1+y_1)u(x,y)dxdy, \tag{4.3}$$

with true solution $u(x,y)=e^xe^y$, *where* $x,x_1\in[0,1]$ *and* $y,y_1\in[0,1]$. *Table* 3 *shows comparison between the exact and the numerical solutions by the proposed technique for* $m=n=2,3,4,5,6$ *at* $x=y=0.6$. *Figure* 3 *is graphical representation of true and numerical solutions.*

Table 3. Comparison of true and numerical solutions of Example 3 at node (0.6,0.6). The absolute error shows error decreases with the increase of the degree of Bernstein polynomial.

m=n	**True Solution**	**Numerical Solution**	**Absolute Error**
2	3.320100000	3.332679029	$1.256210720E^{-2}$
3	3.320100000	3.317111309	$3.005613457E^{-3}$
4	3.320100000	3.320028028	$8.889379901E^{-5}$
5	3.320100000	3.320113369	$3.553719226E^{-6}$
6	3.320100000	3.320110937	$1.515554617E^{-8}$

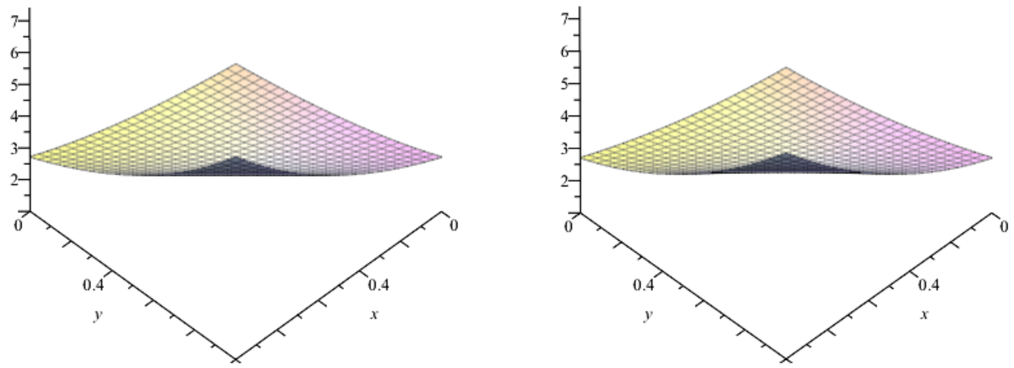


Figure 3. Comparison between true and numerical solutions of Example 3 at m=n=2 obtained by proposed technique.

5. Conclusions

In this paper, two-dimensional Bernstein polynomial approximation is used to solve $2DFIEs$ of first kind. This technique gives a good accuracy at relatively small values of m and n. It is also observed that when degree of Bernstein polynomial is increased, it raised the accuracy of technique. The required accuracy can be obtained by using lower degree Bernstein polynomials, so it is concluded that technique gave excellent approximate solution with low computational cost. In future, the technique can be extended to solve singular and non-linear 2D integral equations.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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